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New results on Einstein and η–Einstein tangent sphere bundles

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Abstract. We complete the results from [14] concerning tangent sphere bundles of constant radius r, previously valid for nonflat base manifolds of dimension $n > 2$, by taking as base manifold an arbitrary 2–dimensional space form (M, g). We derive expressions of the metric ${}^rG$ induced on T_rM by a natural diagonal metric from TM, such that $(T_rM, {}^rG)$ is an Einstein manifold, i.e. its Ricci tensor field is $\mathrm{Ric} = \rho \cdot {}^rG$. In particular, if the metric is induced by the Sasaki metric, T_rM is an Einstein manifold if and only if (M, g) has positive constant sectional curvature c, $r = \frac{\sqrt{c}}{c}$, $\rho = \frac{c}{2}$, or (M, g) is flat and then T_rM is Ricci flat. Moreover, we show that for a flat n–dimensional Riemannian manifold (M, g), T_rM is Einstein with respect to the metric induced by the natural diagonal metric from TM if and only if $n = 2$, $(T_rM, {}^rG)$ is Ricci flat, and a certain function coefficient of the metric vanishes. Finally, we consider the Sasakian manifold $(T_rM, \varphi, \xi, \eta, G)$ from [15] and provide a simpler characterization of the η–Einstein Sasakian tangent sphere bundles in terms of rational expressions involving the coefficients of the metric, the dimension of the base manifold, and the radius r.

Keywords. Tangent bundle, tangent sphere bundle of constant radius, natural lift, Einstein manifold, almost contact structure, Sasakian manifold, η–Einstein manifold

1 Introduction

Tangent sphere bundles of constant radius have been the subject of research over the past few decades, for example in the papers [1]–[15], [19]–[21], [24], [26]–[28].

The tangent sphere bundle T_rM of constant radius $r > 0$ over a differentiable manifold M is defined as the set of all tangent vectors with norm r. This bundle naturally appears as a hypersurface within the total space TM of the tangent bundle. If TM carries an almost Hermitian structure, this induces an almost contact structure on T_rM. The first metric used on TM in this setting was the Sasaki metric. Later, Boeckx and Vanhecke showed in [8] that the unit tangent bundle T_1M endowed with the metric induced by the Cheeger–Gromoll metric on TM is isometric to $T_{\frac{1}{\sqrt{2}}}M$ with Sasaki-induced metric.

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Then, in [18], Kowalski and Sekizawa investigated the geometric properties of tangent sphere bundles of constant radii, equipped with the Sasaki-induced metrics, highlighting the dependence of these properties on the radius. Later, in 2008, the same authors published the survey [20] on the Riemannian geometry of tangent sphere bundles with arbitrary constant radii.

In [21] Munteanu proved that the unit tangent bundle T_1M, endowed with a metric G_A induced by a metric g_A on TM which generalizes both the Sasaki and Cheeger–Gromoll metrics, is isometric to (T_rM, G_r) when $r = \sqrt{a}$, where G_r is the metric induced on T_rM by the Sasaki metric, and a is a constant appearing in the definition of G_A. He also showed that the contact structures on T_1M and T_rM induced by the almost Hermitian structure of TM are preserved. Lastly, he demonstrated that the contact metric structure on T_1M is K-contact if and only if (M, g) has constant sectional curvature $\frac{1}{a} > 0$, in which case T_1M becomes a Sasakian manifold.

The K-contact, Sasakian, strongly pseudo-convex CR-structures on T_1M endowed with metrics induced by g-natural metrics from TM were studied by Abbassi and Calvaruso in [1], extending the results from [28] and [11].

Park and Sekigawa proved in [26] that the geometry of the tangent sphere bundle of constant radius r reduces to the geometry of the unit tangent bundle with Sasaki-induced metric. Furthermore, they showed in [27] that T_rM equipped with the induced contact metric structure is η-Einstein if and only if the n-dimensional base manifold M has constant sectional curvature $\frac{1}{r^2}$ or $\frac{n-2}{r^2}$. In [15] we extended this result to the background of the tangent sphere bundle T_rM endowed with the almost contact metric structure (φ, ξ, η, G) obtained from an almost Hermitian structure of natural diagonal lift type on the total space TM of the tangent bundle over a Riemannian manifold (M, g).

In the present paper we consider the metric ${}^rG$ from [14], induced on T_rM by a metric ${}^DG$ of natural diagonal lift type on the tangent bundle of a Riemannian manifold (M, g). We complete the characterization of the Einstein tangent sphere bundles $(T_rM, {}^rG)$, which was given in [14] for tangent sphere bundles over nonflat space forms of dimension $n > 2$, by considering now as base manifold an arbitrary 2–dimensional Riemannian manifold (M, g) of constant sectional curvature c. In this case we prove that T_rM is an Einstein manifold with respect to the Sasaki-induced metric if and only if one of the following instances holds: the base manifold is flat and the tangent sphere bundle is Ricci flat, or $c > 0$, $r = \frac{\sqrt{c}}{c}$, $\rho = \frac{c}{2}$, where ρ is the constant from the Einstein condition. We also show that the tangent sphere bundle T_rM over a flat n–dimensional Riemannian manifold (M, g) is Einstein with respect to the metric ${}^rG$ mentioned above if and only if (M, g) is 2–dimensional, a certain coefficient involved in the expression of ${}^rG$ is zero and $(T_rM, {}^rG)$ is Ricci flat. The final section of the paper is devoted to the η–Einstein Sasakian tangent sphere bundles $(T_rM, \varphi, \xi, \eta, G)$, characterized in a simpler way than in [15].

2 Preliminary results

According to [29], if (M, g) is a smooth Riemannian manifold of dimension n, then the total space TM of the tangent bundle $\tau : TM \to M$ possesses a structure of $2n-$dimensional smooth manifold, since every local chart $(U, x^i)_{i=1}^n$ on M gives rise to a local chart $(\tau^{-1}(U), x^i, y^j)_{i,j=1}^n$ on TM, where $x^i := x^i \circ \tau$ denote the base point coordinates of a tangent vector $y \in \tau^{-1}(U)$ and y^j are its vector space coordinates with respect to the natural local basis in $T_{\tau(y)}M$. The tangent space of TM in y decomposes as

$$T_yTM = V_yTM \oplus H_yTM,$$

for all $y \in TM$. The vertical space is $V_yTM = \text{Ker } \tau_{*,y}$ and the horizontal space H_yTM is defined by the Levi–Civita connection of g. If the Christoffel symbols of the metric g are denoted by $\Gamma^h_{ki}(x)$, then the horizontal generators are $\{\frac{\delta}{\delta x^i}|_y\}_{i=\overline{1,n}}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^l\Gamma^h_{li}(x)\frac{\partial}{\partial y^h}$, and the vertical generators are $\{\frac{\partial}{\partial y^i}|_y\}_{i=\overline{1,n}}$. Then, a local frame field for the vertical distribution VTM and a local frame field for the horizontal distribution HTM are given by the sets of vector fields $\{\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$, $\{\frac{\delta}{\delta x^i}\}_{i=\overline{1,n}}$, respectively. The module of the smooth vector fields tangent to TM, denoted here by TTM, decomposes into the direct sum:

$$TTM = VTM \oplus HTM,$$

for which an local adapted frame on TM is $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\delta_i, \partial_j\}_{i,j=\overline{1,n}}$.

If a vector field X on M has on U the local expression $X = X^i\frac{\partial}{\partial x^i}$, then its horizontal and vertical lifts to TM have, respectively, the following local expressions on $\tau^{-1}(U)$:

$$X^H = X^i\frac{\delta}{\delta x^i}, \qquad X^V = X^j\frac{\partial}{\partial y^j}.$$

The energy density t of an arbitrary tangent vector $y \in \tau^{-1}(U)$, has the following expression with respect to the Riemannian metric g of the base manifold:

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^iy^k = \frac{1}{2}g_{0i}(x)y^i \geq 0,$$

where $g_{0i} = g_{ki}y^k$.

In the sequel we work with geometric structures induced on the tangent sphere bundle T_rM by the almost Hermitian structures of natural diagonal lift type on TM, hence we first consider a natural diagonal Riemannian metric on TM constructed by Oproiu in [25] by using the results from [16], [20]. This metric, denoted here by ${}^DG$, is invariantly given by:

$$\begin{cases} {}^DG(X^H_y, Y^H_y) = c_1(t)g_{\tau(y)}(X,Y) + d_1(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ {}^DG(X^V_y, Y^V_y) = c_2(t)g_{\tau(y)}(X,Y) + d_2(t)g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ {}^DG(X^V_y, Y^H_y) = {}^DG(X^H_y, X^V_y) = 0, \end{cases} \tag{2.1}$$

for every $X, Y \in \mathcal{T}^1_0(TM), y \in TM$, where c_1, c_2, d_1, d_2 are smooth functions of the energy density on TM. The metric ${}^DG$ is nondegenerate if

$$c_1c_2 \neq 0,\ (c_1 + 2td_1)(c_2 + 2td_2) \neq 0,\ \forall t \geq 0,$$

and it is positive definite if

$$c_1 > 0,\ c_2 > 0,\ c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0,\ \forall t \geq 0.$$

With respect to the local adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$ the matrix associated to the metric ${}^DG$ is

$$\begin{pmatrix} {}^DG^{(1)}_{ij} & 0 \\ 0 & {}^DG^{(2)}_{ij} \end{pmatrix} = \begin{pmatrix} c_1(t)g_{ij} + d_1(t)g_{0i}g_{0j} & 0 \\ 0 & c_2(t)g_{ij} + d_2(t)g_{0i}g_{0j} \end{pmatrix},$$

and its inverse matrix is

$$\begin{pmatrix} {}^DH^{kl}_{(1)} & 0 \\ 0 & {}^DH^{kl}_{(2)} \end{pmatrix} = \begin{pmatrix} p_1(t)g^{kl} + q_1(t)y^ky^l & 0 \\ 0 & p_2(t)g^{kl} + q_2(t)y^ky^l \end{pmatrix},$$

where g^{kl} are the entries of the inverse matrix of $(g_{ij})_{i,j=\overline{1,n}}$, and $p_1,\ q_1,\ p_2,\ q_2$, are some real smooth functions of the energy density with the expressions

$$p_1=\frac{1}{c_1},\ p_2=\frac{1}{c_2},\ q_1=-\frac{d_1}{c_1(c_1+2td_1)},\ q_2=-\frac{d_2}{c_2(c_2+2td_2)}.$$

Next we recall some technical results from [14].

Proposition 2.1. ([14]) *The Levi–Civita connection ${}^D\nabla$ associated to the Riemannian metric ${}^DG$ given by (2.1) on the total space TM of the tangent bundle of a Riemannian manifold (M,g) has the form*

$$\begin{cases}{}^D\nabla_{X^V}Y^V={}^DQ(X^V,Y^V),\ {}^D\nabla_{X^H}Y^V=(\dot\nabla_XY)^V+{}^DP(Y^V,X^H),\\ {}^D\nabla_{X^V}Y^H={}^DP(X^V,Y^H),\ {}^D\nabla_{X^H}Y^H=(\dot\nabla_XY)^H+{}^DS(X^H,Y^H),\end{cases}$$

for all $X,Y\in\mathcal{T}_0^1(M)$, where the M–tensor fields ${}^DQ,\ {}^DP,\ {}^DS$, have the following components with respect to the adapted local frame $\{\partial_i,\delta_j\}_{i,j=\overline{1,n}}$:

$$\begin{aligned}{}^DQ^h_{ij}&=\tfrac{1}{2}(\partial_i{}^DG^{(2)}_{jk}+\partial_j{}^DG^{(2)}_{ik}-\partial_k{}^DG^{(2)}_{ij})^DH^{kh}_{(2)},\\ {}^DP^h_{ij}&=\tfrac{1}{2}(\partial_i{}^DG^{(1)}_{jk}+R^l_{0jk}{}^DG^{(2)}_{li})^DH^{kh}_{(1)},\\ {}^DS^h_{ij}&=-\tfrac{1}{2}(\partial_k{}^DG^{(2)}_{ij}+R^l_{0ij}{}^DG^{(2)}_{lk})^DH^{kh}_{(2)},\end{aligned}$$

where R^h_{kij} are the components of the curvature tensor field of the Levi–Civita connection $\dot\nabla$ from the base manifold (M,g) and $R^h_{0ij}=y^kR^h_{kij}$. The vector fields ${}^DQ(X^V,Y^V)$ and ${}^DS(X^H,Y^H)$ are vertically valued, while ${}^DP(Y^V,X^H)$ is horizontally valued.

The M–tensor fields ${}^DQ,\ {}^DP,\ {}^DS$ have the following detailed expressions derived in [14]:

$$\begin{aligned}{}^DQ(X^V,Y^V)=&\frac{c_2'}{2c_2}[g(y,X)Y^V+g(y,Y)X^V]-\frac{c_2'-2d_2}{2(c_2+2td_2)}g(X,Y)y^V\\ &+\frac{c_2d_2'-2c_2'd_2}{2c_2(c_2+2td_2)}g(y,X)g(y,Y)y^V,\\ {}^DP(X^V,Y^H)=&\frac{c_1'}{2c_1}g(y,X)Y^H+\frac{d_1}{2c_1}g(y,Y)X^H+\frac{d_1}{2(c_1+2td_1)}g(X,Y)y^H\\ &+\frac{c_1d_1'-c_1'd_1-d_1^2}{2c_1(c_1+2td_1)}g(y,X)g(y,Y)y^H-\frac{c_2}{2c_1}(R(X,y)Y)^H\\ &-\frac{c_2d_1}{2c_1(c_1+2td_1)}g(X,R(Y,y)y)y^H,\\ {}^DS(X^H,Y^H)=&-\frac{d_1}{2c_2}[g(y,X)\ Y^V+g(y,Y)\ X^V]-\frac{c_1'}{2(c_2+2td_2)}g(X,Y)y^V\\ &-\frac{c_2d_1'-2d_1d_2}{2c_2(c_2+2td_2)}g(y,X)g(y,Y)y^V-\frac{1}{2}(R(X,Y)y)^V,\end{aligned}$$

for all $X,Y\in\mathcal{T}_0^1(M),\ y\in TM$.

Since the tangent sphere bundle T_rM consists of spheres of constant radius r, any tangent vector $y\in T_rM$ has the energy density $t=\frac{r^2}{2}$, hence the metric ${}^rG$ obtained on T_rM as the

restriction of the metric ${}^{D}G$ will have constant coefficients. Subsequently, the nondegeneracy conditions of the metric ${}^{r}G$ will be

$$c_1c_2 \neq 0,\ (c_1 + r^2d_1)(c_2 + r^2d_2) \neq 0, \tag{2.2}$$

and the conditions for the positive definiteness will be

$$c_1 > 0,\ c_2 > 0,\ c_1 + r^2d_1,\ c_2 + r^2d_2 > 0. \tag{2.3}$$

Restricted to T_rM, the M–tensor fields involved in the expression of the Levi–Civita connection will take the forms:

$$\begin{aligned}
{}^{r}Q(X^V, Y^V) =& \frac{d_2}{c_2 + r^2d_2} g(X, Y)y^V,\\
{}^{r}P(X^V, Y^H) =& \frac{d_1}{2c_1} g(Y, y)X^H - \frac{c_2d_1}{2c_1(c_1 + r^2d_1)} g(X, R(Y, y)y)y^H\\
&+ \frac{d_1}{2c_1(c_1 + r^2d_1)}[c_1g(X, Y) - d_1g(X, y)g(Y, y)]y^H\\
&- \frac{c_2}{2c_1}(R(X, y)Y)^H,\\
{}^{r}S(X^H, Y^H) =& -\frac{d_1}{2c_2}[g(X, y)Y^V + g(Y, y)X^V]\\
&+ \frac{d_1d_2}{c_2(c_2 + r^2d_2)} g(y, X)g(y, Y)y^V - \frac{1}{2}(R(X, Y)y)^V,
\end{aligned} \tag{2.4}$$

for every $X, Y \in \mathcal{T}^1_0(M),\ y \in T_rM$.

3 Tangent sphere bundles of constant radius r

In this section we recall some basic facts on the background of the present work, mentioned also in [14].

The tangent sphere bundle of constant radius $r > 0$ over an n-dimensional Riemannian manifold (M, g) is the hypersurface T_rM of TM defined as the set of all the tangent vectors of norm equal to r:

$$T_rM = \{y \in TM : g_{\tau(y)}(y, y) = r^2\}.$$

The projection of T_rM onto the base manifold is $\overline{\tau} : T_rM \to M$, $\overline{\tau} = \tau \circ i$, where $i : T_rM \to TM$ is the inclusion map.

Different from the horizontal lift, the vertical lift of any vector field on M is not always tangent to T_rM, but one can define the tangential lift of a vector $X \in T_xM$ to $y \in T_rM$ (see e.g. [14, 18, 19, 24, 27]), given by the relation

$$X^T_y = X^V_y - \frac{1}{r^2} g_{\tau(y)}(X, y)y^V_y.$$

The generators of the total space of the tangent bundle to T_rM are $\delta_i = \frac{\partial}{\partial x^i} - y^k\Gamma^l_{ki}\partial_l$ and $\partial^T_j = \partial_j - \frac{1}{r^2} g_{0j}y^k\partial_k$, $i, j, k, l = \overline{1, n}$. The tangential generators $\{\partial^T_j\}_{j=\overline{1,n}}$ are not independent vector fields. At any point of T_rM they span an $(n-1)$–dimensional subspace of TT_rM, since they verify the relation

$$y^j\partial^T_j = 0. \tag{3.1}$$

The normal vector field to T_rM in TM is

$$N = y^i\partial_i.$$

We denote by ${}^rG$ the metric induced on T_rM by the natural diagonal metric ${}^DG$ on TM given by (2.1). The energy density of any tangent vector $y \in T_rM$ is $t = \frac{r^2}{2}$, hence the function coefficients of the metric ${}^DG$ become constants. Thus, the metric ${}^rG$ is given by the relations

$$\begin{cases} {}^rG(X^H_y, Y^H_y) = c_1 g_{\tau(y)}(X,Y) + d_1 g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y), \\ {}^rG(X^T_y, Y^T_y) = c_2[g_{\tau(y)}(X,Y) - \frac{1}{r^2} g_{\tau(y)}(X,y)g_{\tau(y)}(Y,y)], \\ {}^rG(X^H_y, Y^T_y) = {}^rG(Y^T_y, X^H_y) = 0, \end{cases}$$

for any $X, Y \in \mathcal{T}^1_0(M), y \in T_rM$, where $c_1,\ d_1,\ c_2$ are some real constants such that the conditions (2.2) or (2.3) are satisfied.

With respect to the system of generators $\{\delta_i, \partial^T_j\}_{i,j=\overline{1,n}}$ of T_rM, the metric ${}^rG$ has the expressions

$$\begin{cases} {}^rG(\delta_i, \delta_j) = {}^rG^{(1)}_{ij} = c_1 g_{ij} + d_1 g_{0i}g_{0j}, \\ {}^rG(\partial^T_i, \partial^T_j) = {}^rG^{(2)}_{ij} = c_2(g_{ij} - \frac{1}{r^2} g_{0i}g_{0j}), \\ {}^rG(\delta_i, \partial^T_j) = {}^rG(\partial^T_j, \delta_i) = 0. \end{cases}$$

Proposition 3.1. ([14]) *The Levi–Civita connection ${}^r\nabla$, associated to the Riemannian metric ${}^rG$ on the tangent sphere bundle T_rM of constant radius r has the expression*

$$\begin{cases} {}^r\nabla_{\partial^T_i}\partial^T_j = A^h_{ij}\partial^T_h,\ {}^r\nabla_{\delta_i}\partial^T_j = \Gamma^h_{ij}\partial^T_h + B^h_{ji}\delta_h, \\ {}^r\nabla_{\partial^T_i}\delta_j = B^h_{ij}\delta_h,\ {}^r\nabla_{\delta_i}\delta_j = \Gamma^h_{ij}\delta_h + C^h_{ij}\partial^T_h, \end{cases}$$

where the M–tensor fields involved as coefficients have the expressions:

$$\begin{aligned} A^h_{ij} =& -\frac{1}{r^2} g_{0j}\delta^h_i, \quad C^h_{ij} = -\frac{d_1}{2c_2}(\delta^h_j g_{0i} - \delta^h_i g_{0j}) - \frac{1}{2}R^h_{0ij}, \\ B^h_{ij} =& {}^rP^h_{ij} - \frac{1}{r^2} g_{i0}\,{}^rP^h_{0j} \\ =& \frac{d_1}{2c_1}\delta^h_i g_{0j} + \frac{d_1}{2(c_1 + r^2 d_1)}\left(g_{ij} - \frac{2c_1 + r^2 d_1}{r^2 c_1} g_{0i}g_{0j}\right) y^h \\ & - \frac{c_2}{2c_1} R^h_{jik}y^k - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} R_{ikjl}y^h y^k y^l, \end{aligned} \tag{3.2}$$

where the tensor field ${}^rP$ is the one given in (2.4), $g_{0j} = g_{j0} = y^i g_{ij}$, $P^h_{0j} = y^i P^h_{ij}$.

The invariant expressions of the Levi–Civita connection ${}^r\nabla$ presented in [14] are

$$\begin{aligned} {}^r\nabla_{X^T}Y^T =& -\frac{1}{r^2} g(Y,y)X^T, \\ {}^r\nabla_{X^T}Y^H =& \frac{d_1}{2c_1} g(Y,y)\ X^H + \frac{d_1}{2(c_1 + d_1 r^2)} g(X,Y)y^H \\ & - \frac{d_1(2c_1 + r^2 d_1)}{2r^2 c_1(c_1 + r^2 d_1)} g(X,y)g(Y,y)y^H - \frac{c_2}{2c_1}(R(X,y)Y)^H \\ & - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} g(X, R(Y,y)y)y^H, \end{aligned}$$

$$
\begin{aligned}
{}^r\nabla_{X^H}Y^T =&(\dot\nabla_X Y)^T+\frac{d_1}{2c_1}g(X,y)\ Y^H+\frac{d_1}{2(c_1+r^2d_1)}g(X,Y)y^H\\
&-\frac{d_1(2c_1+r^2d_1)}{2r^2c_1(c_1+r^2d_1)}g(X,y)g(Y,y)y^H\\
&-\frac{c_2}{2c_1}(R(Y,y)X)^H-\frac{c_2d_1}{2c_1(c_1+r^2d_1)}g(Y,R(X,y)y)y^H,\\
{}^r\nabla_{X^H}Y^H =&(\dot\nabla_X Y)^H-\frac{d_1}{2c_2}[g(X,y)\ Y^T-g(Y,y)\ X^T]-\frac{1}{2}(R(X,Y)y)^T.
\end{aligned}
$$

The curvature tensor field K of the connection ${}^r\nabla$ from T_rM is given by the relation

$$K(\overline{X},\overline{Y})\overline{Z}={}^r\nabla_{\overline{X}}{}^r\nabla_{\overline{Y}}\overline{Z}-{}^r\nabla_{\overline{Y}}{}^r\nabla_{\overline{X}}\overline{Z}-{}^r\nabla_{[\overline{X},\overline{Y}]}\overline{Z},\ \forall\overline{X},\overline{Y},\overline{Z}\in\mathcal{T}_0^1(T_rM).$$

With respect to the system of generators $\{\delta_i,\partial_j^T\}_{i,j=\overline{1,n}}$ of T_rM, the curvature tensor field K of the connection ${}^r\nabla$ from T_rM has the expressions

$$
\begin{aligned}
&K(\delta_i,\delta_j)\delta_k=HHHH^h_{kij}\delta_h+HHHT^h_{kij}\partial_h^T,\\
&K(\delta_i,\delta_j)\partial_k^T=HHTH^h_{kij}\delta_h+HHTT^h_{kij}\partial_h^T,\\
&K(\partial_i^T,\partial_j^T)\delta_k=TTHH^h_{kij}\delta_h,\quad K(\partial_i^T,\partial_j^T)\partial_k^T=TTTT^h_{kij}\partial_h^T,\\
&K(\partial_i^T,\delta_j)\delta_k=THHH^h_{kij}\delta_h+THHT^h_{kij}\partial_h^T,\quad K(\partial_i^T,\delta_j)\partial_k^T=THTH^h_{kij}\delta_h,
\end{aligned}
$$

where the coefficients of horizontal and tangential parts are the non-zero M–tensor fields given in [14] by

$$
\begin{aligned}
&HHHH^h_{kij}=R^h_{kij}+B^h_{li}C^l_{jk}-B^h_{lj}C^l_{ik}+B^h_{lk}R^l_{0ij},\\
&HHHT^h_{kij}=\frac{1}{2}(\dot\nabla_jR^h_{lik}y^l-\dot\nabla_iR^h_{ljk}y^l),\\
&HHTH^h_{kij}=\frac{c_2}{2c_1}(\dot\nabla_jR^h_{ikl}-\dot\nabla_iR^h_{jkl})y^l\\
&\qquad\qquad+\frac{c_2d_1}{2c_1(c_1+r^2d_1)}(\dot\nabla_jR_{ilkm}-\dot\nabla_iR_{jlkm})y^ly^my^h\\
&HHTT^h_{kij}\partial_h^T=(R^h_{kij}+C^h_{il}B^l_{kj}-C^h_{jl}B^l_{ki}+A^h_{lk}R^l_{0ij})\partial_h^T,\\
&TTHH^h_{kij}=\partial_i^TB^h_{jk}-\partial_j^TB^h_{ik}+B^h_{il}B^l_{jk}-B^h_{jl}B^l_{ik}-\frac{1}{r^2}(g_{i0}B^h_{jk}-g_{j0}B^h_{ik}),\\
&TTTT^h_{kij}=\frac{1}{r^2}(g_{jk}\delta^h_i-g_{ik}\delta^h_j)+\frac{1}{r^4}(\delta^h_jg_{0i}g_{0k}-\delta^h_ig_{0j}g_{0k}),\\
&THHH^h_{kij}=\frac{c_2}{2c_1}\dot\nabla_jR^h_{kil}y^l+\frac{c_2d_1}{2c_1(c_1+r^2d_1)}\dot\nabla_jR_{ilkr}y^ly^ry^h,\\
&THHT^h_{kij}\partial_h^T=(\partial_i^TC^h_{jk}+A^h_{il}C^l_{jk}-C^h_{jl}B^l_{ik}-\dot\nabla_jR^h_{lik}y^l)\partial_h^T,\\
&THTH^h_{kij}=\partial_i^TB^h_{kj}+B^l_{kj}B^h_{il}-A^l_{ik}B^h_{lj}.
\end{aligned}
$$

The final expressions of the above M–tensor fields can be obtained by using (3.2). To compute the tangential parts $HHTT^h_{kij}\partial_h^T$ and $THHT^h_{kij}\partial_h^T$ one has to take into account the relation (3.1).

4 Einstein tangent sphere bundles

Here we will recall the results obtained in [14] on Einstein tangent sphere bundles of constant radius r over nonflat base manifolds of dimension $n>2$, then we will complete them by con-

sidering base manifolds which are arbitrary space forms of dimension 2 or flat base manifolds of arbitrary dimension.

We take first an n-dimensional Riemannian manifold (M, g) and the total space TM of the tangent bundle, endowed with the natural diagonal metric ${}^{D}G$ given by (2.1). The Ricci tensor field on TM is a $(0,2)-$tensor field $\widetilde{\mathrm{Ric}}$ defined by the relation

$$\widetilde{\mathrm{Ric}}(\overline{Y},\overline{Z}) = \mathrm{trace}(\overline{X} \to K(\overline{X},\overline{Y})\overline{Z}),\ \forall \overline{X},\overline{Y},\overline{Z} \in \mathcal{T}_0^1(TM).$$

The components of $\widetilde{\mathrm{Ric}}$ in the local adapted frame field $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$ are

$$\begin{aligned}
&\widetilde{\mathrm{Ric}}HH_{jk} = \widetilde{\mathrm{Ric}}(\delta_j,\delta_k) = HHHH^h_{khj} + VHHV^h_{khj},\\
&\widetilde{\mathrm{Ric}}VV_{jk} = \widetilde{\mathrm{Ric}}(\partial_j,\partial_k) = VVVV^h_{khj} - VHVH^h_{kjh},\\
&\widetilde{\mathrm{Ric}}HV_{jk} = \widetilde{\mathrm{Ric}}(\delta_j,\partial_k) = \widetilde{\mathrm{Ric}}VH_{kj} = \widetilde{\mathrm{Ric}}(\partial_k,\delta_j) = VVHV^h_{jhk} + HVHH^h_{jhk},
\end{aligned}$$

where we denoted by sequences of V and H the components of the curvature tensor field, indicating vertical or horizontal argument, respectively, on a certain position.

In the case of a base manifold which is locally symmetric, in particular a space form, the mixed components of the $\widetilde{\mathrm{Ric}}$ vanish: $\widetilde{\mathrm{Ric}}HV_{jk} = \widetilde{\mathrm{Ric}}VH_{jk} = 0$.

In [14] we determined the components of the Ricci tensor field Ric of $(T_rM, {}^{r}G)$, with respect to the system of generators $\{\delta_i, \partial_j^T\}_{i,j=\overline{1,n}}$ of T_rM. By taking into account the expression of the last tangential generator ∂_n^T,

$$\partial_n^T = -\frac{1}{y^n}\sum_{i=1}^{n-1} y^i \partial_i^T,$$

we obtained

$$\begin{aligned}
\mathrm{Ric}HH_{jk} =& \mathrm{Ric}_{jk} - \frac{d_1(2c_1+r^2d_1)}{2c_2(c_1+r^2d_1)}g_{jk} + \frac{d_1(2c_1+r^2d_1)[c_1n+r^2d_1(n-1)]}{2r^2c_1c_2(c_1+r^2d_1)}g_{0j}g_{0k} \qquad (4.1)\\
&- \frac{c_2}{2c_1}R^h{}_{0lk}R^l{}_{jh0} + \frac{c_2d_1}{2c_1(c_1+r^2d_1)}R^h{}_{0k0}R_{h0j0},\\
\mathrm{Ric}TT_{jk} =& \frac{r^4d_1^2 - 2c_1(c_1+r^2d_1)(n-2)}{2c_1r^2(c_1+d_1r^2)}\left(\frac{1}{r^2}g_{0j}g_{0k} - g_{jk}\right) + \frac{c_2^2}{4c_1^2}R_{hik0}R^{hi}{}_{j0} \qquad (4.2)\\
&- \frac{c_2^2d_1}{2c_1^2(c_1+r^2d_1)}R^h{}_{0j0}R_{h0k0},
\end{aligned}$$

and then

$$\begin{aligned}
&\mathrm{Ric}HH_{jk} - \rho\cdot{}^{r}G^{(1)}_{jk} = \mathrm{Ric}_{jk} - \tfrac{d_1(2c_1+r^2d_1)+2\rho c_1c_2(c_1+r^2d_1)}{2c_2(c_1+r^2d_1)}g_{jk}\\
&+\tfrac{d_1(2c_1+r^2d_1)[c_1n+r^2d_1(n-1)]-2\rho r^2c_1c_2d_1(c_1+r^2d_1)}{2c_1c_2r^2(c_1+r^2d_1)}g_{0j}g_{0k}\\
&-\tfrac{c_2}{2c_1}R^h{}_{0lk}R^l{}_{jh0} + \tfrac{c_2d_1}{2c_1(c_1+r^2d_1)}R_{h0j0}R^h{}_{0k0},\\
&\\
&\mathrm{Ric}TT_{jk} - \rho\cdot{}^{r}G^{(2)}_{jk} = \tfrac{c_2^2}{4c_1^2}R_{hik0}R^{hi}{}_{j0} - \tfrac{c_2^2d_1}{2c_1^2(c_1+r^2d_1)}R_{h0k0}R^h{}_{0j0}\\
&+\tfrac{r^4d_1^2+2c_1(c_1+r^2d_1)(2-n+\rho c_2r^2)}{2r^2c_1(c_1+r^2d_1)}\left(\tfrac{1}{r^2}g_{0j}g_{0k} - g_{jk}\right),
\end{aligned}$$

where

$$R^h{}_{0kl} = R^h{}_{ikl}y^i,\ R^l{}_{jh0} = R^l{}_{jhi}y^i,\ R^h{}_{0k0} = R^h{}_{ikj}y^iy^j,\ R_{h0j0} = R_{hijk}y^iy^k,\ R_{hijk} = g_{hl}R^l{}_{ijk}.$$

When the base manifold (M,g) is a space form, i.e. when

$$R^h_{kij}=c(g_{jk}\delta^h_i-g_{ik}\delta^h_j), \tag{4.3}$$

where c is a real constant, the above expressions turn into

$$\begin{aligned}&\mathrm{Ric}HH_{jk}-\rho\cdot{}^rG^{(1)}_{jk}=\left\{\tfrac{cc_2[2(n-1)(c_1+r^2d_1)-cc_2r^2]-d_1(2c_1+r^2d_1)}{2c_2(c_1+r^2d_1)}-\rho c_1\right\}g_{jk}\\&+\left\{\tfrac{d_1[2c_1^2n+r^4(d_1^2-c^2c_2^2)(n-1)]+r^2c_1[d_1^2(3n-2)-c^2c_2^2(n-2)]}{2c_1c_2r^2(c_1+r^2d_1)}-\rho d_1\right\}g_{0j}g_{0k},\\&\mathrm{Ric}TT_{jk}-\rho\cdot{}^rG^{(2)}_{jk}=\tfrac{r^4(d_1^2-c^2c_2^2)+2c_1(c_1+r^2d_1)(2-n+\rho c_2r^2)}{2r^2c_1(c_1+r^2d_1)}\left(\tfrac{1}{r^2}g_{0j}g_{0k}-g_{jk}\right),\end{aligned}$$

hence $\mathrm{Ric}TT_{jk}=\rho\cdot{}^rG^{(2)}_{jk}$ if and only if

$$\rho=\tfrac{2c_1^2(n-2)+r^2[2c_1d_1(n-2)+r^2(c^2c_2^2-d_1^2)]}{2r^2c_1c_2(c_1+r^2d_1)}, \tag{4.4}$$

then the difference $RicHH_{jk}-\rho\cdot{}^rG^{(1)}_{jk}$ turns into

$$\begin{aligned}&\mathrm{Ric}HH_{jk}-\rho\cdot{}^rG^{(1)}_{jk}\\&=\tfrac{1}{2c_1c_2r^2(c_1+r^2d_1)}\cdot\left\{2c_1(cc_2r^2-c_1)[c_1(n-2)-r^2(cc_2-d_1(n-1))]g_{jk}\right.\\&\left.+[4c_1^2d_1+r^2c_1(d_1^2(n+2)-c^2c_2^2(n-2))+r^4d_1(d_1^2-c^2c_2^2)n]g_{0j}g_{0k}\right\}.\end{aligned} \tag{4.5}$$

Taking into account all above, we proved in [14] the following result.

Theorem 4.1. ([14]) *The tangent sphere bundle T_rM of an $(n>2)$-dimensional Riemannian manifold (M,g) of constant sectional curvature $c\neq 0$ is Einstein with respect to the metric ${}^rG$ induced from the natural diagonal lifted metric ${}^DG$ defined on TM by (2.1), i.e. it exists a real constant ρ such that*

$$\mathrm{Ric}(\overline{X},\overline{Y})=\rho\cdot{}^rG(\overline{X},\overline{Y}),\ \forall\overline{X},\overline{Y}\in\mathcal{T}^1_0(T_rM), \tag{4.6}$$

if and only if

$$c_1=\frac{r^2d_1n}{n-2},\ c_2=\frac{d_1n}{c(n-2)},\ \rho=\frac{c(n-1)^2(n-2)}{r^2d_1n^2},$$

where d_1 is an arbitrary nonzero real constant such that the metric ${}^rG$ is non-degenerate.

Since in the case of the Sasaki metric on TM the functions coefficients from relation (2.1) have the particular values $c_1=c_2=1,\ d_1=0$, they do not satisfy the conditions in Theorem 4.1, hence we have the following nonexistence result:

Corollary 4.2. ([14]) *The tangent sphere bundle T_rM of an $(n>2)$–dimensional Riemannian manifold (M,g) of nonzero constant sectional curvature is never Einstein with respect to the metric induced by the Sasaki metric g^s from TM.*

Studying the situation when the base manifold is 2–dimensional, we prove now the following characterization.

Theorem 4.3. *The tangent sphere bundle T_rM of a 2–dimensional Riemannian manifold (M,g) of constant sectional curvature c is an Einstein manifold with respect to the metric ${}^rG$ induced from the natural diagonal lifted metric ${}^DG$ defined on TM by (2.1), i.e. there exists a real constant ρ such that the relation (4.6) is satisfied, if and only if the coefficients of ${}^rG$, c and ρ satisfy one of the following items:*

i) $c_1 = cc_2r^2$, $d_1 = 0$, $\rho = \frac{1}{2c_2r^2}$, *c_2 and c are some arbitrary nonzero real constants of the same sign;*

ii) $c = 0$, *i.e. the base manifold is flat, c_1 and c_2 are arbitrary positive constants, $d_1 = 0$, and the manifold $(T_rM, {}^rG)$ is Ricci flat.*

Proof. When the dimension of the manifold M is $n = 2$, the differences $\mathrm{Ric}HH_{jk} - \rho \cdot {}^rG^{(1)}_{jk}$ and $\mathrm{Ric}TT_{jk} - \rho \cdot {}^rG^{(2)}_{jk}$ become of the forms

$$\begin{aligned}
\mathrm{Ric}HH_{jk} - \rho \cdot {}^rG^{(1)}_{jk} &= \tfrac{cc_2[2(c_1+r^2d_1)-cc_2r^2]-d_1(2c_1+r^2d_1)-2\rho c_1c_2(c_1+r^2d_1)}{2c_2(c_1+r^2d_1)} g_{jk} \\
&\quad + \tfrac{d_1[4c_1^2+r^4(d_1^2-c^2c_2^2)]+4r^2c_1d_1^2-2\rho r^2c_1c_2d_1(c_1+r^2d_1)}{2r^2c_1c_2(c_1+r^2d_1)} g_{0j}g_{0k}, \\
\mathrm{Ric}TT_{jk} - \rho \cdot {}^rG^{(2)}_{jk} &= \tfrac{r^2(d_1^2-c^2c_2^2)+2\rho c_1c_2(c_1+r^2d_1)}{2c_1(c_1+r^2d_1)} \big(\tfrac{1}{r^2} g_{0j}g_{0k} - g_{jk}\big).
\end{aligned}$$

From the condition of vanishing of the last difference above we get

$$\rho = \frac{r^2(c^2c_2^2 - d_1^2)}{2c_1c_2(c_1 + r^2d_1)},$$

then

$$\begin{aligned}
\mathrm{Ric}HH_{jk} - \rho \cdot {}^rG^{(1)}_{jk} = &\frac{1}{c_1c_2r^2(c_1 + r^2d_1)} \{c_1(cc_2r^2 - c_1)(d_1 - cc_2)r^2 g_{jk} \\
&+ d_1[2c_1(c_1 + r^2d_1) + r^4(d_1^2 - c^2c_2^2)] g_{0j}g_{0k}\}.
\end{aligned}$$

This expression vanishes if and only if the coefficients of g_{jk} and $g_{0j}g_{0k}$ vanish simultaneously. The coefficient of g_{jk} vanishes if and only if one of the following situations occurs: $c_1 = cc_2r^2$ or $d_1 = cc_2$.

If $c_1 = cc_2r^2$, then the coefficient of $g_{0j}g_{0k}$ turns into $\frac{d_1(cc_2+d_1)}{r^2cc_2^2}$ and, due to the nondegeneracy conditions (2.2) of the metric ${}^rG$, it vanishes if and only if $d_1 = 0$. In this case $\rho = \frac{1}{2c_2r^2}$, and item *i)* is proved.

If $d_1 = cc_2$, then the coefficient of $g_{0j}g_{0k}$ reduces to $\frac{2c}{r^2}$, hence it vanishes if and only if the base manifold is flat. In this case $c = 0$, $d_1 = 0$ and $\rho = 0$, hence the manifold $(T_rM, {}^rG)$ is Ricci flat. Thus, item *ii)* is also proved. □

From Theorem 4.3 and Corollary 4.2 we conclude the following:

Corollary 4.4. *The tangent sphere bundle T_rM of a n–dimensional Riemannian manifold (M,g) of constant sectional curvature c is an Einstein manifold with respect to the*

metric induced from the Sasaki metric on TM, i.e. there exists a real constant ρ such that the relation (4.6) is satisfied, if and only if $n = 2$ and one of the following instances happens:

i) $c > 0$, $r = \frac{\sqrt{c}}{c}$, $\rho = \frac{c}{2}$;

ii) $c = 0$, *i.e. the base manifold is flat. In this case T_rM endowed with the metric induced from the Sasaki metric is Ricci flat.*

Analysing the Einstein condition for the tangent sphere bundle T_rM over an arbitrary flat manifold of dimension n we prove the next result.

Theorem 4.5. *The tangent sphere bundle T_rM of a flat n–dimensional Riemannian manifold (M, g) is an Einstein manifold with respect to the metric ${}^rG$ induced by the natural diagonal lifted metric ${}^DG$ defined on TM by (2.1) if and only if $n = 2$, $d_1 = 0$ and $(T_rM, {}^rG)$ is Ricci flat.*

Proof. We showed above that when the base manifold (M, g) has constant sectional curvature c, the difference $\mathrm{Ric}TT_{jk} - \rho \cdot {}^rG^{(2)}_{jk}$ vanishes if and only if the constant ρ involved in the Einstein condition has the expression (4.4), and then the difference $\mathrm{Ric}HH_{jk} - \rho \cdot {}^rG^{(1)}_{jk}$ takes the form (4.5). The characterization of the Einstein manifold $(T_rM, {}^rG)$ when the base manifold is 2-dimensional flat was already provided in Theorem 4.3 *ii)*. In the sequel we shall consider $n \neq 2$.

Under the assumption that $c = 0$, the relation (4.5) reduces to

$$\begin{aligned} &\mathrm{Ric}HH_{jk} - \rho \cdot {}^rG^{(1)}_{jk} \\ &= \tfrac{1}{2c_1c_2r^2(c_1+r^2d_1)} \cdot \big\{-2c_1^2[c_1(n-2) + r^2d_1(n-1))]g_{jk} \\ &+[4c_1^2d_1 + r^2c_1d_1^2(n+2) + r^4d_1^3n]g_{0j}g_{0k}\big\}, \end{aligned} \tag{4.7}$$

which vanishes if and only if the coefficients of g_{jk} and $g_{0j}g_{0k}$ vanish simultaneously.

Since for $n = 1$ the coefficient of g_{jk} is $\frac{c_1^2}{c_2r^2(c_1+r^2d_1)} \neq 0$, it follows that a necessary condition for $(T_rM, {}^rG)$ to be an Einstein manifold is that the dimension of the base manifold is $n \neq 1$. Then, since $n \neq 1$ and $n \neq 2$, the coefficient of g_{jk} in (4.7) vanishes if and only if

$$c_1 = -\frac{d_1r^2(n-1)}{n-2}. \tag{4.8}$$

Substituting (4.8) into (4.7) it follows that

$$\mathrm{Ric}HH_{jk} - \rho \cdot {}^rG^{(1)}_{jk} = \tfrac{d_1n^2}{2c_2(n-1)r^2} g_{0j}g_{0k},$$

which is zero if and only if $d_1 = 0$, but taking into account (4.8) we obtain $c_1 = 0$, hence the nondegeneracy condition (2.2) of the metric is not satisfied when $n \neq 2$.

With all above, taking into account Theorem 4.3 *ii)*, the proof is complete. □

5 Natural diagonal η-Einstein tangent sphere bundles

In [15] we determined the almost contact metric structures induced on the tangent sphere bundle T_rM of a Riemannian manifold (M,g) by an almost Hermitian structure of natural diagonal lift type, we provided the conditions under which the obtained almost contact metric manifold is a Sasakian manifold, then the conditions for this Sasakian manifold to be η-Einstein. In this section we will prove a simpler characterization for the Sasakian structures derived from natural diagonal almost Hermitian structures on TM, with respect to which the tangent sphere bundle T_rM is an η-Einstein manifold.

In [25] Oproiu used a $(1,1)-$tensor field on TM, obtained as a diagonal natural first order lift of the metric from the Riemannian manifold (M,g), denoted here by ${}^DJ$ and given by

$$\begin{cases} {}^DJX_y^H = a_1(t)X_y^V + b_1(t)g_{\tau(y)}(X,y)y_y^V, \\ {}^DJX_y^V = -a_2(t)X_y^H - b_2(t)g_{\tau(y)}(X,y)y_y^H, \end{cases} \tag{5.1}$$

for all $X \in \mathcal{T}_0^1(M), y \in TM$, where a_1, a_2, b_1, b_2 are smooth functions of the energy density t of the tangent vector y.

The characterization of the almost complex structures of natural diagonal lift type on TM was provided by Oproiu in [25, Proposition2], recalled in the sequel.

Proposition 5.1. ([25]) The operator ${}^DJ$ given by (5.1) defines an almost complex structure on TM if and only if

$$a_1a_2 = 1,\ (a_1 + 2tb_1)(a_2 + 2tb_2) = 1. \tag{5.2}$$

Remark 1. For an almost complex structure ${}^DJ$ characterized in Proposition 5.1, the functions a_1, a_2, $a_1 + 2tb_1$, $a_2 + 2tb_2$ cannot vanish and the relations (5.2) are equivalent to

$$a_2 = \frac{1}{a_1},\ b_2 = -\frac{b_1}{a_1(a_1 + 2tb_1)}. \tag{5.3}$$

Oproiu characterized the almost Hermitian structures of natural diagonal lift type on TM in [25, Theorem 3], as follows:

Theorem 5.1. ([25]) Let ${}^DJ$ be a diagonal natural almost complex structure on TM defined by g and the parameters a_1, a_2, b_1, b_2. The family of diagonal natural, Riemannian metrics ${}^DG$ on TM defined by g and the parameters c_1, c_2, d_1, d_2, such that $(TM, {}^DG, {}^DJ)$ is an almost Hermitian manifold, is given by (2.1) where

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda,\ \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu, \tag{5.4}$$

where λ, μ are smooth functions depending on t and $\lambda > 0$, $\lambda + 2t\mu > 0$.

Now we recall the characterization of the Sasakian structures derived on T_rM from the almost Hermitian structure $({}^DG, {}^DJ)$ on TM.

Theorem 5.2. ([15]) *The almost contact metric structure (φ, ξ, η, G) on T_rM, with φ, ξ, η, and G given by*

$$\varphi X^H = a_1 X^T, \quad \varphi X^T = -a_2 X^H + \frac{a_2}{r^2} g(X,y) y^H, \tag{5.5}$$

$$\xi = \frac{1}{2\lambda r^2 \alpha} y^H, \ \eta(X^T) = 0, \ \eta(X^H) = 2\alpha\lambda g(X,y), \ G = \alpha \cdot {}^rG, \tag{5.6}$$

for every $X, Y \in \mathcal{T}_0^1(M)$, $y \in T_rM$, where λ is the proportionality factor from (5.4), $\alpha = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}$ and $a_1 a_2 = 1$, is a contact metric structure, and it is Sakakian if and only if the base manifold has constant sectional curvature $c = \frac{a_1^2}{r^2}$.

We study the conditions under which the Sasakian manifold $(T_rM, \varphi, \xi, \eta, G)$ from Theorem 5.2 is $\eta-$Einstein, i.e. the Ricci tensor has the expression

$$\mathrm{Ric} = \nu G + \sigma \eta \otimes \eta, \tag{5.7}$$

where ν and σ are smooth real functions on T_rM.

By using the relations (4.1), (4.2) and (5.6), we obtain that

$$\begin{aligned} \mathrm{Ric}TT_{jk} - \nu\alpha \cdot {}^rG^{(2)}_{jk} &= -\tfrac{c_2^2}{4c_1^2} R^i{}_{hj0} R^h{}_{ik0} - \tfrac{c_2^2 d_1}{2c_1^2(c_1 + r^2 d_1)} R_{h0j0} R^h{}_{0k0} \\ &+ \tfrac{2c_1^2(n-2) - 2c_1[\nu\alpha c_1 c_2 - d_1(n-2)]r^2 - d_1(2\nu\alpha c_1 c_2 + d_1) r^4}{2c_1 r^2 (c_1 + r^2 d_1)} (g_{jk} - \tfrac{1}{r^2} g_{0j} g_{0k}), \end{aligned} \tag{5.8}$$

$$\begin{aligned} \mathrm{Ric}HH_{jk} - \nu\alpha \cdot {}^rG^{(1)}_{jk} - \sigma\eta(\delta_j)\eta(\delta_k) &= -\tfrac{2c_1(\nu\alpha c_1 c_2 + d_1) + d_1(2\nu\alpha c_1 c_2 + d_1) r^2}{2c_2(c_1 + r^2 d_1)} g_{jk} \\ + \mathrm{Ric}_{jk} + \Big\{ \tfrac{d_1\{2c_1^2 n + c_1[3d_1 n - 2(\nu\alpha c_1 c_2 + d_1)]r^2 + d_1[d_1(n-1) - 2\nu\alpha c_1 c_2] r^4\}}{2c_1 c_2 r^2 (c_1 + r^2 d_1)} \\ - \tfrac{\sigma(c_1 + d_1 r^2)^2}{4\lambda^2 r^4} \Big\} g_{0j} g_{0k} + \tfrac{c_2}{2c_1} R^l{}_{kh0} R^h{}_{0jl} + \tfrac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} R_{h0k0} R^h{}_{0j0}. \end{aligned} \tag{5.9}$$

The η–Einstein condition written with respect to the system of generators $\{\delta_i, \partial_j^T\}_{i,j=\overline{1,n}}$ of T_rM reduces to the conditions of vanishing for the tensor fields expressed by (5.8) and (5.9). By using Theorem 5.1, we provide the characterization of the η–Einstein Sasakian manifold $(T_rM, \varphi, \xi, \eta, G)$, in a simpler way than in [15, Theorem 4.1].

Theorem 5.3. *The Sasakian manifold $(T_rM, \varphi, \xi, \eta, G)$, with φ, ξ, η, and G given in (5.5) and (5.6), is η–Einstein if and only if the functions involved in (5.7) are the constant functions*

$$\nu = \frac{c_1(2n-3) - r^2 d_1}{2r^2 \alpha c_1 c_2}, \ \sigma = \frac{r^2 n d_1 - c_1(n-2)}{(c_1 + r^2 d_1)/2}. \tag{5.10}$$

Proof. If $(T_rM, \varphi, \xi, \eta, G)$ is a Sasakian manifold, then, according to Theorem 5.2, the base manifold (M, g) has constant sectional curvature c, so the relation (5.8) becomes

$$\mathrm{Ric}TT_{jk} - \nu\alpha \cdot {}^rG^{(2)}_{jk}$$

$$= \frac{2c_1(n-2-\alpha\nu r^2c_2)(c_1+r^2d_1)+r^4(c^2c_2^2-d_1^2)}{2c_1r^2(c_1+r^2d_1)}\left(g_{jk}-\frac{1}{r^2}g_{0j}g_{0k}\right)$$

and it vanishes if and only if the function ν has the expression

$$\nu = \frac{2c_1(n-2)(c_1+r^2d_1)+r^4(c^2c_2^2-d_1^2)}{2c_1c_2\alpha r^2(c_1+r^2d_1)}. \tag{5.11}$$

By using (4.3) and (5.11), (5.9) turns into

$$\begin{aligned}\mathrm{Ric}HH_{jk}-\nu\alpha\cdot{}^rG^{(1)}_{jk}-\sigma\eta(\delta_j)\eta(\delta_k) &= -\tfrac{(c_1-cc_2r^2)[c_1(n-2)+d_1r^2(n-1)-cr^2c_2]}{c_2r^2(c_1+r^2d_1)}g_{jk}\\ &+\tfrac{\lambda^2r^2\{4c_1^2d_1-c_1[c^2c_2^2(n-2)-d_1^2(n+2)]r^2-r^4d_1(c^2c_2^2-d_1^2)n\}-\frac{\sigma}{2}c_1c_2(c_1+r^2d_1)^3}{2r^4\lambda^2c_1c_2(c_1+r^2d_1)}g_{0j}g_{0k},\end{aligned} \tag{5.12}$$

hence we obtain that

$$\mathrm{Ric}HH_{jk} = \nu\alpha\cdot{}^rG^{(1)}_{jk}+\sigma\eta(\delta_j)\eta(\delta_k)$$

if and only if the coefficients of g_{jk} and $g_{0j}g_{0k}$ in (5.12) vanish simultaneously.

Due to the relations (5.3), (5.4) and $c=\frac{a_1^2}{r^2}$, it follows that $c_1=cc_2r^2$, then the expression in (5.12) vanishes if and only if

$$-\frac{(cc_2+d_1)\{\lambda^2[cc_2(n-2)-d_1n]+\frac{\sigma}{2}cc_2^2(cc_2+d_1)r^2\}}{2cc_2^2\lambda^2r^2}g_{0j}g_{0k}=0. \tag{5.13}$$

Since $c_1=cc_2r^2$, the nondegeneracy condition (2.2) yields $d_1\neq -cc_2$, hence (5.13) is satisfied if and only if

$$\sigma = \lambda^2\frac{d_1n-cc_2(n-2)}{cc_2^2(cc_2+d_1)r^2/2}. \tag{5.14}$$

Thus, the Sasakian manifold $(T_rM,\varphi,\xi,\eta,G)$ is η–Einstein if and only if ν and σ are given by (5.11) and (5.14), respectively. Taking into account that $c_1=cc_2r^2$, the expression (5.11) turns into

$$\nu = \frac{cc_2(2n-3)-d_1}{2cc_2^2\alpha r^2},$$

which can be written as in (5.10), and (5.14) becomes

$$\sigma = \lambda^2\frac{nr^2d_1-c_1(n-2)}{c_1c_2(c_1+r^2d_1)/2}.$$

Due to (5.4) and (5.3), the above expression of σ coincides with the one in (5.10), hence the proof is complete. □

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