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On the uniform boundedness of a class of hypersingular integral operators on the Hardy space

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Abstract. For a class of hypersingular integral operators, we establish optimal uniform bounds for their norms on the Hardy space $H^1(\mathbb{R})$. Our results extend the classical result of Fefferman-Stein for the phase function $1/y$ to phase functions of the form $1/P(y)$ where P is an arbitrary real polynomial. It is revealed that the presence and absence of a constant term in P play a crucial role in the outcome.

Keywords. Oscillatory integrals, singular integrals, Calderón-Zygmund kernels, Hardy space

1 Introduction

We begin with the following classical hypersingular integral:

$$f \to \text{p.v.} \int_{-\infty}^{\infty} f(x-y) e^{i\frac{1}{y}} \frac{dy}{y},$$

which was initially studied by Fefferman in [4] and subsequently shown to be a bounded operator on the Hardy space $H^1(\mathbb{R})$ by Fefferman and Stein in [5]. For other related results see [3] and [7], among others.

To introduce our main results, let us recall the definition of Calderón-Zygmund kernels used in this paper.

Definition 1.1. A function $K \in C^1(\mathbb{R}\backslash\{0\})$ is called a Calderón-Zygmund kernel if it is odd and there exists a positive constant B such that

$$|K(x)| + |x||K'(x)| \leq B|x|^{-1}. \tag{1.1}$$

For every nonzero polynomial P with real coefficients, we define the operator $T_{K,1/P}$ by

$$T_{K,1/P} f(x) = \text{p.v.} \int_{\mathbb{R}} e^{i\frac{1}{P(y)}} K(y) f(x-y) dy.$$

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Let $\mathcal{P}_d$ denote the collection of all polynomials with real coefficients and degree d. For every $\lambda \in \mathbb{R}$, let

$$\mathcal{P}_{d,\lambda} = \{P \in \mathcal{P}_d : P(0) = \lambda\}.$$

We begin with the following uniform (H^1, H^1) estimates for $\lambda \neq 0$:

Theorem 1.1. Let K be a Calderón-Zygmund kernel and $d \in \mathbb{N}$.

(i) There exists a positive constant $C(d, B)$ such that, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\sup_{P \in \mathcal{P}_{d,\lambda}} \|T_{K,1/P}\|_{H^1 \to H^1} \leq C(d, B)\left[1 + \log^+\left(\frac{1}{|\lambda|}\right)\right]. \tag{1.2}$$

The constant $C(d, B)$ may depend on d and the bound B in (1.1), but is independent of λ.

(ii) If, in addition to (1.1), K also satisfies $|K(x)| \geq \tilde{B}|x|^{-1}$ for $x \in \mathbb{R} \setminus \{0\}$ (a prominent example of this being $1/(\pi x)$, the kernel of the Hilbert transform), then there exists a positive constant $\tilde{C}(B, \tilde{B})$ such that, for every $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\sup_{P \in \mathcal{P}_{d,\lambda}} \|T_{K,1/P}\|_{H^1 \to H^1} \geq \tilde{C}(B, \tilde{B})\left[1 + \log^+\left(\frac{1}{|\lambda|}\right)\right]. \tag{1.3}$$

The constant $\tilde{C}(B, \tilde{B})$ may depend on B and $\tilde{B}$, but is independent of λ and d.

By Theorem 1.1, for any fixed d, we have

$$\sup_{P \in \mathcal{P}_{d,\lambda}} \|T_{1/x,1/P}\|_{H^1 \to H^1} \approx 1 + \log^+\left(\frac{1}{|\lambda|}\right).$$

Allowing $\lambda \to 0$ shows that $\{T_{1/x,1/P} : P \in \mathcal{P}_d\}$ are not uniformly bounded on $H^1(\mathbb{R})$.

Interestingly, the operators corresponding to $\lambda = 0$ are themselves uniformly bounded on $H^1(\mathbb{R})$. The aforementioned result by Fefferman and Stein belongs to this case.

Theorem 1.2. Let K be a Calderón-Zygmund kernel and $d \in \mathbb{N}$. Then there exists a positive constant $C(d, B)$ such that

$$\sup_{P \in \mathcal{P}_{d,0}} \|T_{K,1/P}\|_{H^1 \to H^1} \leq C(d, B). \tag{1.4}$$

We will prove Theorem 1.2 in Section 2. The upper bound part of Theorem 1.1 (Part (i)) will be addressed in Section 3, and the lower bound part (Part (ii)) in Section 4.

In the rest of the paper we shall use $A \lesssim B$ ($A \gtrsim B$) to mean that $A \leq cB$ ($A \geq cB$) for a certain constant c whose actual value is not essential for the relevant arguments to work. We shall also use $A \approx B$ to means "$A \lesssim B$ and $B \lesssim A$".

2 Proof of Theorem 1.2

Recall that an H^1 atom is a function $a(\cdot)$ which is supported in an interval I, satisfies $\|a\|_\infty \leq |I|^{-1}$ and

$$\int_I a(x)dx = 0.$$

Lemma 2.1. Let $A > 1$, $d \in \mathbb{N}$, $p_0, p_1, \dots, p_d \in \mathbb{R}$ and $p_d \neq 0$. Then there are m $(1 \leq m \leq d+1)$ disjoint open intervals $G_1 = (L_1,\, R_1), \dots, G_m = (L_m,\, R_m)$ such that

(i) $0 = L_1 < R_1 < L_2 < R_2 < \cdots < L_m < R_m = \infty$;

(ii) For each $j \in \{1, \dots, m\}$, there exists a $l_j \in \{0, 1, \dots, d\}$ such that

$$|p_{l_j} x^{l_j}| > A \cdot \max\{|p_k x^k| : \, k \in \{0, 1, \dots, d\} \backslash \{l_j\}\}$$

for all $x \in G_j$;

(iii) For $1 \leq j \leq m-1$,

$$\frac{L_{j+1}}{R_j} \leq A^{d(d+1)/2}.$$

The above lemma is a slight extension of Lemma 2.1 of [1]. The same arguments work here without any essential changes.

Let $\mathcal{H}$ denote the Hilbert transform. The following can be found in [2], [6] and [8].

Lemma 2.2. For every $f \in H^1(\mathbb{R})$, we have

$$\|\mathcal{H}f\|_{H^1(\mathbb{R})} \lesssim \|f\|_{H^1(\mathbb{R})} \tag{2.1}$$

and

$$\|f\|_{H^1(\mathbb{R})} \approx \|f\|_{L^1(\mathbb{R})} + \|\mathcal{H}f\|_{L^1(\mathbb{R})}. \tag{2.2}$$

We now present the proof of Theorem 1.2. Let $P \in \mathcal{P}_{d,0}$, i.e., $P(x) = \sum_{k=1}^{d} p_k x^k$ where $p_1, \dots, p_d \in \mathbb{R}$ and $p_d \neq 0$.

We will begin by establishing the $H^1 \to L^1$ bound of $T_{K,1/P}$. By the atomic decomposition of $H^1(\mathbb{R})$, it suffices to prove that, for any H^1 atom $a(\,\cdot\,)$,

$$\|T_{K,1/P} a\|_{L^1(\mathbb{R})} \leq C(d, K), \tag{2.3}$$

where $C(d, K)$ is independent of $p_1, \dots, p_d$ and $a(\,\cdot\,)$.

For $\delta > 0$, let θ_δ denote the dilation operator given by

$$(\theta_\delta f)(x) = \delta f(\delta x).$$

Then $\theta_\delta \circ T_{K,1/P} = T_{\theta_\delta K, 1/P_\delta} \circ \theta_\delta$, where $P_\delta(x) = P(\delta x)$ and $\theta_\delta K$ is also a Calderón-Zygmund kernel satisfying (1.1) with the same constant B. The class of polynomials $\mathcal{P}_{d,0}$ is invariant under the mapping $P \to P_\delta$ (as is $\mathcal{P}_{d,\lambda}$ for every $\lambda \in \mathbb{R}\backslash\{0\}$ which we will need later). Additionally, the operator $T_{K,1/P}$ is translation invariant. Thus, in order to prove that (2.3) holds uniformly in the coefficients of P, we may further assume that the atom $a(\,\cdot\,)$ is supported in $[-1,\, 1]$, satisfies $\|a\|_\infty \leq 1$ and $\int_{\mathbb{R}} a(x)dx = 0$.

By the uniform L^2-boundedness of $T_{K,1/P}$,

$$\int_{|x|\leq 2} |(T_{K,1/P} a)(x)| dx \lesssim \|T_{K,1/P} a\|_{L^2(\mathbb{R})} \lesssim \|a\|_{L^2(\mathbb{R})} \lesssim 1. \tag{2.4}$$

For $|x|>2$ and $|y|\le 1$, we have

$$|K(x-y)-K(x)|\lesssim |x|^{-2}.$$

Thus,

$$\int_{|x|>2}\left|(T_{K,1/P}a)(x)-K(x)\int_{\mathbb{R}}e^{i\frac{1}{P(x-y)}}a(y)dy\right|dx\lesssim\left(\int_{|x|>2}|x|^{-2}dx\right)\|a\|_{L^1(\mathbb{R})}\lesssim 1. \tag{2.5}$$

By (2.4)-(2.5) and (1.1), the proof of (2.3) has now been reduced to the verification of the following inequalities:

$$\int_2^{\infty}\left|\int_{\mathbb{R}}e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{x}\lesssim 1; \tag{2.6}$$

$$\int_{-\infty}^{-2}\left|\int_{\mathbb{R}}e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{|x|}\lesssim 1. \tag{2.7}$$

We shall present the proof of (2.6) only as (2.7) can be handled in a similar fashion.

Let $A=2^{9(d+1)}(d+1)^5>1$ and $G_j=(L_j,\,R_j)$, $1\le j\le m$ be the disjoint open intervals given as in Lemma 2.1. By Lemma 2.1(iii),

$$\begin{aligned}&\int_{(2,\,\infty)\setminus\bigcup_{j=1}^m G_j}\left|\int_{\mathbb{R}}e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{x}\\ \lesssim&\|a\|_{L^1(\mathbb{R})}\\ \lesssim&1.\end{aligned} \tag{2.8}$$

Next we shall seek to establish that

$$\int_{(2,\,\infty)\cap G_j}\left|\int_{\mathbb{R}}e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{x}\lesssim 1 \tag{2.9}$$

for $j=1,\dots,m$.

Clearly one only needs to consider those j's for which $(2,\,\infty)\cap G_j\neq\emptyset$. For any such fixed $j\in\{1,\dots,m\}$, $(2,\,\infty)\cap G_j=(\max\{2,L_j\},\,R_j)$ and there exists an $l_j\in\{1,\dots,d\}$ such that the inequality in part (ii) of Lemma 2.1 holds. To simplify our notations, let $\alpha=\max\{2,L_j\}$, $\beta=R_j$ and $l=l_j\ge 1$. Then $\alpha\in\mathbb{R}$, $\alpha\ge 2$, but β could possibly be ∞. Thus, for all $x\in(\alpha,\,\beta)$,

$$|p_l|x^l>A\cdot\max\{|p_k|x^k:\,1\le k\le d\text{ and }k\neq l\}. \tag{2.10}$$

We may further assume that $\beta>64\alpha$, for otherwise (2.9) becomes trivial.

For $\nu\in\mathbb{N}$, define the operator S_ν by

$$S_\nu f(x)=\chi_{[2^\nu,\,2^{\nu+1}]}(x)\int_{-1}^{1}e^{i\frac{1}{P(x-y)}}f(y)dy.$$

Then

$$S_\nu^*S_\nu f(x)=\int_{\mathbb{R}}L_\nu(x,y)f(y)dy$$

where

$$L_\nu(x,y)=\chi_{[-1,\,1]}(x)\chi_{[-1,\,1]}(y)\int_{2^\nu}^{2^{\nu+1}}e^{i\left[\frac{1}{P(z-x)}-\frac{1}{P(z-y)}\right]}dz.$$

For any $x, y \in [-1,\, 1]$ and $z \in [2^\nu,\, 2^{\nu+1}] \subset (2\alpha, \beta/2)$, we have $z-x,\ z-y \in [2^{\nu-1},\, 2^{\nu+2}] \subseteq (\alpha, \beta)$. Thus

$$\begin{aligned}|P(z-x)| \leq & |p_l|(z-x)^l + d\cdot\max\{|p_k|(z-x)^k : 0\leq k\leq d \text{ and } k\neq l\}\\ \leq&(1+d/A)|p_l|(z-x)^l\\ \leq&2^{2d+1}|p_l|2^{\nu l}.\end{aligned} \tag{2.11}$$

Similarly,

$$|P(z-y)| \leq 2^{2d+1}|p_l|2^{\nu l}. \tag{2.12}$$

For the purpose of estimating $\frac{d}{dz}\left(\frac{1}{P(z-x)} - \frac{1}{P(z-y)}\right)$, we also have

$$\begin{aligned}
&|(P(z-x))^2P'(z-y) - (P(z-y))^2P'(z-x)|\\
=&\Big|\sum_{k_1=0}^{d}\sum_{k_2=0}^{d}\sum_{k_3=1}^{d} k_3p_{k_1}p_{k_2}p_{k_3}[(z-x)^{k_1+k_2}(z-y)^{k_3-1} - (z-y)^{k_1+k_2}(z-x)^{k_3-1}]\Big|\\
\geq& l|p_l|^3|(z-x)^{2l}(z-y)^{l-1} - (z-y)^{2l}(z-x)^{l-1}|\\
&-\Big|\sum_{\substack{0\leq k_1,k_2\leq d,\, 1\leq k_3\leq d\\ (k_1,k_2,k_3)\neq(l,l,l)}} k_3p_{k_1}p_{k_2}p_{k_3}[(z-x)^{k_1+k_2}(z-y)^{k_3-1}\\
&-(z-y)^{k_1+k_2}(z-x)^{k_3-1}]\Big|\\
\geq&|p_l|^3(z-x)^{l-1}(z-y)^{l-1}|(z-x)^{l+1} - (z-y)^{l+1}|\\
&-\sum_{\substack{0\leq k_1,k_2\leq d,\, 2\leq k_3\leq d\\ (k_1,k_2,k_3)\neq(l,l,l)}} k_3|p_{k_1}p_{k_2}p_{k_3}|(z-x)^{k_1+k_2}|(z-y)^{k_3-1} - (z-x)^{k_3-1}|\\
&-\sum_{\substack{0\leq k_1,k_2\leq d,\, 1\leq k_3\leq d\\ (k_1,k_2,k_3)\neq(l,l,l)}} k_3|p_{k_1}p_{k_2}p_{k_3}|(z-x)^{k_3-1}|(z-x)^{k_1+k_2} - (z-y)^{k_1+k_2}|\\
=&|x-y|\Bigg[|p_l|^3|(z-x)^{l-1}(z-y)^{l-1}\sum_{s=0}^{l}(z-x)^{l-s}(z-y)^s\\
&-\sum_{\substack{0\leq k_1,k_2\leq d,\, 2\leq k_3\leq d\\ (k_1,k_2,k_3)\neq(l,l,l)}} k_3|p_{k_1}p_{k_2}p_{k_3}|(z-x)^{k_1+k_2}\sum_{s=0}^{k_3-2}(z-y)^{k_3-2-s}(z-x)^s\\
&-\sum_{\substack{0\leq k_1,k_2\leq d,\, 1\leq k_3\leq d\\ k_1+k_2\geq 1,\, (k_1,k_2,k_3)\neq(l,l,l)}} k_3|p_{k_1}p_{k_2}p_{k_3}|(z-x)^{k_3-1}\times\\
&\left(\sum_{s=0}^{k_1+k_2-1}(z-x)^{k_1+k_2-1-s}(z-y)^s\right)\Bigg]\\
\geq&|x-y|\big[(|p_l|2^{(\nu-1)l})^3 2^{-2(\nu+2)} - 2A^{-1}d(d+1)^3(2d+1)(|p_l|2^{(\nu+2)l})^3 2^{-2(\nu-1)}\big]\\
\geq&2^{-(4+3l+2\nu)}(|p_l|2^{\nu l})^3\left[1-\frac{(d+1)^5 2^{9d+8}}{A}\right]|x-y|\\
\geq&2^{-(5+3d+2\nu)}(|p_l|2^{\nu l})^3|x-y|.
\end{aligned} \tag{2.13}$$

It follows from (2.11)-(2.13) that

$$\begin{aligned}
&\left|\frac{d}{dz}\left(\frac{1}{P(z-x)}-\frac{1}{P(z-y)}\right)\right| \\
=&\left|\frac{(P(z-x))^2P'(z-y)-(P(z-y))^2P'(z-x)}{(P(z-x))^2(P(z-y))^2}\right| \\
\geq&\left(2^{2d+1}|p_l|2^{\nu l}\right)^{-4}2^{-(5+3d+2\nu)}(|p_l|2^{\nu l})^3|x-y| \\
\gtrsim&2^{-2\nu}(|p_l|2^{\nu l})^{-1}|x-y|.
\end{aligned} \tag{2.14}$$

By (2.14) and van der Corput's lemma,

$$|L_\nu(x,y)|\lesssim 2^{2\nu}(|p_l|2^{\nu l})|x-y|^{-1}\chi_{[-1,\,1]}(x)\chi_{[-1,\,1]}(y). \tag{2.15}$$

Trivially we also have

$$|L_\nu(x,y)|\leq 2^{\nu}\chi_{[-1,\,1]}(x)\chi_{[-1,\,1]}(y). \tag{2.16}$$

By interpolating between (2.15) and (2.16),

$$|L_\nu(x,y)|\lesssim\frac{2^{3\nu/2}(|p_l|2^{\nu l})^{1/2}\chi_{[-1,\,1]}(x)\chi_{[-1,\,1]}(y)}{|x-y|^{1/2}}. \tag{2.17}$$

Thus,

$$\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}|L_\nu(x,y)|dy+\sup_{y\in\mathbb{R}}\int_{\mathbb{R}}|L_\nu(x,y)|dx\lesssim 2^{3\nu/2}(|p_l|2^{\nu l})^{1/2}. \tag{2.18}$$

By (2.18) and Schur's test,

$$\|S_\nu^*S_\nu\|_{2,2}\lesssim 2^{3\nu/2}(|p_l|2^{\nu l})^{1/2},$$

which implies that

$$\|S_\nu\|_{2,2}\lesssim 2^{3\nu/4}(|p_l|2^{\nu l})^{1/4}. \tag{2.19}$$

Additionally, for $x\in(2\alpha,\beta/2)$ and $|y|\leq 1$,

$$\begin{aligned}
&\left|\frac{1}{P(x-y)}-\frac{1}{P(x)}\right| \\
\leq&(1-A^{-1}d)^{-2}(|p_l|x^l)^{-1}(|p_l|(x-y)^l)^{-1}\left|\sum_{k=1}^{d}p_ky\left(\sum_{s=0}^{k-1}x^{k-1-s}(x-y)^s\right)\right| \\
\leq&\frac{2^{2d}d^2|y|\max\{|p_k|x^k:\,1\leq k\leq d\}}{(1-A^{-1}d)^2(|p_l|x^l)^2x} \\
\lesssim&\frac{|y|}{|p_l|x^{l+1}}.
\end{aligned} \tag{2.20}$$

Let $u=[\log_2(4\alpha)]$. If $32\alpha<|p_l|^{-1/(l+1)}<\beta/2$, let $\gamma=|p_l|^{-1/(l+1)}$ and $w=[\log_2(\gamma/2)]$. By

Hölder's inequality, (2.19) and (2.20),

$$
\begin{aligned}
&\int_\alpha^\beta \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
=&\int_\alpha^{2^u} + \int_{2^u}^{2^w} + \int_{2^w}^{\gamma} + \int_\gamma^{\beta/2} + \int_{\beta/2}^{\beta} \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
\leq& 4 + \sum_{\nu=u}^{w-1} \int_{2^\nu}^{2^{\nu+1}} \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
&+ \int_\gamma^{\beta/2} \left| \int_{\mathbb{R}} \left(e^{i\frac{1}{P(x-y)}} - e^{i\frac{1}{P(x)}} \right) a(y)dy \right| \frac{dx}{x} \\
\lesssim& 4 + \sum_{\nu=u}^{w-1} \|S_\nu a\|_{L^2(\mathbb{R})} \left(\int_{2^u}^{2^{u+1}} \frac{dx}{x^2} \right)^{1/2} + |p_l|^{-1} \left(\int_\gamma^{\beta/2} \frac{dx}{x^{l+2}} \right) \left(\int_{\mathbb{R}} |y||a(y)|dy \right) \\
\lesssim& 4 + \sum_{\nu=u}^{w-1} 2^{-\nu/2} 2^{3\nu/4} (|p_l| 2^{\nu l})^{1/4} \|a\|_{L^2(\mathbb{R})} + (|p_l|\gamma^{l+1})^{-1} \|a\|_{L^1(\mathbb{R})} \\
\lesssim& 4 + |p_l|^{1/4} 2^{w(l+1)/4} + 1 \\
\lesssim& 1.
\end{aligned} \tag{2.21}
$$

If $|p_l|^{-1/(l+1)} \geq \beta/2$ (this also implies that $\beta < \infty$), let $w = [\log_2(\beta/4)]$. Then,

$$
\begin{aligned}
&\int_\alpha^\beta \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
=&\int_\alpha^{2^u} + \int_{2^u}^{2^w} + \int_{2^w}^{\beta} \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
\lesssim& 5 + |p_l|^{1/4} 2^{w(l+1)/4} \\
\lesssim& 5 + \frac{2^{w(l+1)/4}}{\beta^{(l+1)/4}} \\
\lesssim& 1.
\end{aligned} \tag{2.22}
$$

If $|p_l|^{-1/(l+1)} \leq 32\alpha$, let $\gamma = 2\alpha$. Then,

$$
\begin{aligned}
&\int_\alpha^\beta \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
=&\int_\alpha^{\gamma} + \int_{\gamma}^{\beta/2} + \int_{\beta/2}^{\beta} \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\
\lesssim& 2 + \frac{1}{|p_l|\gamma^{l+1}} \\
\lesssim& 2 + \frac{1}{|p_l|(2\alpha)^{l+1}} \\
\lesssim& 1.
\end{aligned} \tag{2.23}
$$

By (2.21)-(2.23), we obtain (2.9). It follows that, for all $f \in H^1(\mathbb{R})$,

$$
\|T_{K,1/P} f\|_{L^1(\mathbb{R})} \lesssim \|f\|_{H^1(\mathbb{R})}. \tag{2.24}
$$

By the translation-invariance of $T_{K,1/P}$, (2.24) and (2.1),

$$\begin{aligned}\|\mathcal{H}(T_{K,1/P}f)\|_{L^1(\mathbb{R})} =&\|T_{K,1/P}(\mathcal{H}f)\|_{L^1(\mathbb{R})}\\ \lesssim&\|\mathcal{H}f\|_{H^1(\mathbb{R})}\\ \lesssim&\|f\|_{H^1(\mathbb{R})}.\end{aligned} \tag{2.25}$$

By combining (2.24), (2.25) and (2.2), we obtain (1.4). The proof of Theorem 1.2 is now complete.

3 Proof of Theorem 1.1(i)

Let $\lambda \in \mathbb{R}\backslash\{0\}$ and $P \in \mathcal{P}_{d,\lambda}$, i.e.,

$$P(x) = \lambda + \sum_{k=1}^{d} p_k x^k$$

where $p_1, \ldots, p_d \in \mathbb{R}$ and $p_d \neq 0$. Let $a(\,\cdot\,)$ be an arbitrary function which satisfies $\operatorname{supp}(a) \subseteq [-1,\, 1]$, $\|a\|_\infty \leq 1$ and

$$\int_{\mathbb{R}} a(x)dx = 0. \tag{3.1}$$

By a reduction used in the proof of Theorem 1.2, it suffices to prove that

$$\|T_{K,1/P}a\|_{L^1(\mathbb{R})} \leq C(d,B)\left[1+\log^+\left(\frac{1}{|\lambda|}\right)\right]. \tag{3.2}$$

A quick examination of the proof of Theorem 1.2 reveals that all but one of its steps remain valid when $P(0) = 0$ is replaced by $P(0) = \lambda \neq 0$. The exception is (2.9), which holds for all $1 \leq j \leq m$ under the condition $P(0) = 0$. However, this step is valid only for $2 \leq j \leq m$ when $P(0) = \lambda \neq 0$. The reason for this discrepancy lies in the argument leading to (2.9), which fails for $j = 1$ because the exponent l in (2.10) becomes 0 for G_1. In contrast, the corresponding exponents for $G_2, \ldots, G_m$ remain positive. Thus, we have

$$\begin{aligned}\|T_{K,1/P}a\|_{L^1(\mathbb{R})} \lesssim&1+\int_{(2,\,\infty)\cap G_1}\left|\int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{x}\\ &+\int_{(-\infty,\,-2)\cap(-G_1)}\left|\int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{|x|},\end{aligned} \tag{3.3}$$

where

$$G_1 = \left(0,\ \min\left\{\left|\frac{\lambda}{Ap_k}\right|^{1/k} :\ 1\leq k\leq d\ \text{ and }\ p_k\neq 0\right\}\right),$$

with $A = 2^{9(d+1)}(d+1)^5$. Thus, it suffices to prove that

$$\int_{(2,\,\infty)\cap G_1}\left|\int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{x} \lesssim 1+\log^+\left(\frac{1}{|\lambda|}\right) \tag{3.4}$$

and

$$\int_{(-\infty,\,-2)\cap(-G_1)}\left|\int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}}a(y)dy\right|\frac{dx}{|x|} \lesssim 1+\log^+\left(\frac{1}{|\lambda|}\right). \tag{3.5}$$

We will provide the proof of (3.4) only, as the proof of (3.5) can be obtained in a similar fashion. Let $\alpha = \max\{2, |\lambda|^{-1}\}$ and

$$\beta = \min\left\{ \left|\frac{\lambda}{Ap_k}\right|^{1/k} : 1 \le k \le d \text{ and } p_k \ne 0 \right\}.$$

If $\beta \le 2$, then $(2, \infty) \cap G_1 = \emptyset$, in which case (3.4) holds trivially. If $2 < \beta \le |\lambda|^{-1}$, then (3.4) follows from

$$\begin{aligned}\int_{(2,\infty)\cap G_1} \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \le & \|a\|_1 \left(\int_2^{|\lambda|^{-1}} \frac{dx}{x} \right) \\ \lesssim & 1 + \log^+ \left(\frac{1}{|\lambda|} \right).\end{aligned}$$

In the remaining case we have $\beta > \alpha$. Let $x \in (2, \infty) \cap G_1 = (2, \beta)$ and $y \in [-1, 1]$. Then $|x - y| \le 2|x|$ and $|p_k|x^k \le |\lambda|/A$ for $1 \le k \le d$. Thus,

$$|P(x-y)| \ge |\lambda| - \sum_{k=1}^{d} \left(\frac{2^k |\lambda|}{A} \right) \gtrsim |\lambda|. \tag{3.6}$$

Similarly, we have

$$|P(x)| \gtrsim |\lambda|. \tag{3.7}$$

On the other hand,

$$\begin{aligned}|P(x) - P(x-y)| \le & \sum_{k=1}^{d} |p_k||x^k - (x-y)^k| \\ \lesssim & x^{-1} \left(\sum_{k=1}^{d} |p_k| x^k \right) \\ \lesssim & |\lambda| x^{-1}.\end{aligned} \tag{3.8}$$

By (3.1) and (3.6)-(3.8),

$$\begin{aligned}&\int_{(2,\infty)\cap G_1} \left| \int_{\mathbb{R}} e^{i\frac{1}{P(x-y)}} a(y)dy \right| \frac{dx}{x} \\ \le & \|a\|_1 \left(\int_2^{\alpha} \frac{dx}{x} \right) + \int_{\alpha}^{\beta} \left| \int_{\mathbb{R}} \left(e^{i\frac{1}{P(x-y)}} - e^{i\frac{1}{P(x)}} \right) a(y)dy \right| \frac{dx}{x} \\ \le & \|a\|_1 \log(\alpha/2) + \int_{\alpha}^{\beta} \left(\int_{-1}^{1} \left| \frac{P(x) - P(x-y)}{P(x-y)P(x)} \right| |a(y)|dy \right) \frac{dx}{x} \\ \lesssim & \|a\|_1 \log(\alpha/2) + \frac{\|a\|_1}{|\lambda|} \left(\int_{\alpha}^{\beta} \frac{dx}{x^2} \right) \\ \lesssim & \log(\alpha/2) + (|\lambda|\alpha)^{-1} \\ \lesssim & 1 + \log^+ \left(\frac{1}{|\lambda|} \right),\end{aligned}$$

which completes the proof of Part (i) of Theorem 1.1.

4 Proof of Theorem 1.1(ii)

We now assume that K satisfies (1.1) and

$$|K(x)| \geq \tilde{B}|x|^{-1} \tag{4.1}$$

for $x \in \mathbb{R} \setminus \{0\}$.

To establish the desired lower bound (1.3), it suffices to consider $|\lambda| \to 0$ only. Without loss of generality we may assume that $0 < \lambda < (16e^{40B/\tilde{B}})^{-1}$.

Let $a_0(y) = (1/2)\operatorname{sgn}(y)\chi_{[-1,1]}(y)$ and $P_\tau(x) = \tau x^d + \lambda^2 x + \lambda$ for $\tau \geq 0$. Then,

$$\begin{aligned}\sup_{P\in\mathcal{P}_{d,\lambda}} \|T_{K,1/P}a_0\|_{H^1(\mathbb{R})} &\geq \sup_{\varepsilon>0} \|T_{K,1/P_\varepsilon}a_0\|_{H^1(\mathbb{R})} \\ &\geq \sup_{\varepsilon>0} \|T_{K,1/P_\varepsilon}a_0\|_{L^1(\mathbb{R})} \\ &\geq \sup_{\varepsilon>0} \int_2^{(2\lambda)^{-1}} \left| \int_{-1}^1 e^{i\frac{1}{P_\varepsilon(x-y)}} K(x-y)a_0(y)dy \right| dx \\ &\geq \int_2^{(2\lambda)^{-1}} \left| \int_{-1}^1 e^{i\frac{1}{P_0(x-y)}} K(x-y)a_0(y)dy \right| dx.\end{aligned} \tag{4.2}$$

For $2 < x < (2\lambda)^{-1}$ and $|y| \leq 1$, let

$$\psi(x,y,\lambda) = \frac{1}{\lambda+\lambda^2 x} + \frac{y}{(1+\lambda x)^2}.$$

Then,

$$\left| \frac{1}{P_0(x-y)} - \psi(x,y,\lambda) \right| = \frac{\lambda y^2}{[1+\lambda(x-y)](1+\lambda x)^2} \leq \lambda$$

for $2 < x < (2\lambda)^{-1}$ and $|y| \leq 1$. Thus,

$$\int_2^{(2\lambda)^{-1}} \left| \int_{-1}^1 \left(e^{i\frac{1}{P_0(x-y)}} - e^{i\psi(x,y,\lambda)} \right) a_0(y)dy \right| \frac{dx}{x} \leq \lambda \log\left(\frac{1}{4\lambda}\right). \tag{4.3}$$

It follows from (1.1) and (4.1)-(4.3) that

$$\begin{aligned}\sup_{P\in\mathcal{P}_{d,\lambda}} \|T_{K,1/P}a_0\|_{H^1(\mathbb{R})} \geq& \int_2^{(2\lambda)^{-1}} \left| \int_{-1}^1 e^{i\frac{1}{P_0(x-y)}} a_0(y)dy \right| |K(x)|dx \\ &- \int_2^{(2\lambda)^{-1}} \int_{-1}^1 |K(x)-K(x-y)||a_0(y)|dydx \\ \geq& \tilde{B} \int_2^{(2\lambda)^{-1}} \left| \int_{-1}^1 e^{i\frac{1}{P_0(x-y)}} a_0(y)dy \right| \frac{dx}{x} - 2B \\ \geq& \tilde{B} \int_2^{(2\lambda)^{-1}} \left| \int_{-1}^1 e^{i\psi(x,y,\lambda)} a_0(y)dy \right| \frac{dx}{x} - \tilde{B}\lambda\log\left(\frac{1}{4\lambda}\right) - 2B \\ \geq& \tilde{B} \int_2^{(2\lambda)^{-1}} \left[\int_0^1 \sin\left(\frac{y}{(1+\lambda x)^2}\right) dy \right] \frac{dx}{x} - \tilde{B}\lambda\log\left(\frac{1}{4\lambda}\right) - 2B \\ \geq& \left(\frac{\tilde{B}\sin(2/9)}{2}\right) \log\left(\frac{1}{4\lambda}\right) - \tilde{B}\lambda\log\left(\frac{1}{4\lambda}\right) - 2B \\ \geq& \left(\frac{\tilde{B}}{20}\right) \log\left(\frac{1}{\lambda}\right).\end{aligned}$$

This completes the proof of Theorem 1.1(ii).

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