TAMKANG JOURNAL OF MATHEMATICS
Volume vol, Number i, xxx-xxx, 20xx
DOI:10.5556/j.tkjm.57.2026.5867

Parabolic frequency on Ricci-Bourguignon flow and Yamabe flow

Shahroud Azami and Abimbola Abolarinwa

Abstract. We analyze the evolution of certain parabolic-type energy expressions associated with linear heat-type equations in the setting of compact Riemannian manifolds undergoing the Yamabe and Ricci–Bourguignon flows. By formulating appropriate frequency-type quantities tailored to the underlying geometric dynamics, we demonstrate their monotonic nature under specific curvature assumptions and flow-adapted conditions. This study contributes to the understanding of the interaction between geometric evolution and analytic invariants.

Keywords. Parabolic frequency, Ricci-Bourguignon flow, heat equation

1 Introduction

During the late 1970s, Almgren [1] devised a now-fundamental tool in the study of harmonic functions, known as the *elliptic frequency function*. Let $w : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function, and fix a reference point $p \in \mathbb{R}^n$. The frequency associated with w at p over a Euclidean radius $s > 0$ is expressed by

$$\mathcal{N}(s) = \frac{s \int_{B(p,s)} \|\nabla w\|^2 \, dx}{\int_{\partial B(p,s)} w^2 \, d\sigma},$$

where $B(p, s)$ stands for the standard open ball in $\mathbb{R}^n$ centered at p, and $d\sigma$ is the surface measure induced on its boundary. The quantity $\mathcal{N}(s)$ effectively encodes the degree to which the function w "vanishes" near the point p. Almgren established that $\mathcal{N}(s)$ is monotonically non-decreasing as s increases, a fact that has become instrumental in investigations surrounding the fine structure and regularity theory of both harmonic functions and area-minimizing hypersurfaces [1]. Expanding upon Almgren's foundational framework, the non-decreasing behavior of the frequency functional has proven to be a powerful analytical tool within the realm of geometric analysis and the theory of PDEs. In a pioneering series of works, Garofalo and Lin [6, 7] harnessed this monotonicity to investigate strong unique continuation properties for second-order elliptic operators on Riemannian manifolds. Parallel to this, Lin [13] implemented frequency-based strategies to derive bounds on the Hausdorff measure of nodal sets associated with solutions to both elliptic

received date: August 7, 2025; Published online: June 24, 2026.
2010 *Mathematics Subject Classification*. 58C40, 53E20, 53C21.
Corresponding author: Shahroud Azami .

and parabolic PDEs. Comprehensive accounts of subsequent refinements and diverse applications can be found in [5, 12].

Motivated by the elliptic case, Poon [15] introduced a parabolic variant of the frequency functional tailored for the heat equation in the Euclidean setting. This formulation was aimed at addressing issues related to backward uniqueness in parabolic dynamics. Later, Ni [14] extended this perspective to Riemannian and Kähler geometries, demonstrating the monotonicity of these parabolic frequencies under geometric curvature constraints such as nonnegative sectional or bisectional curvature and the presence of parallel Ricci tensors. A broader extension of the parabolic frequency concept was developed by Baldauf and Kim [9], where they analyzed a generalized heat equation incorporating a time-evolving coefficient $\theta(t)$, a backward time scaling function $\sigma(t)$, and a weighted measure $d\nu$ on the underlying manifold. For a time-dependent solution $w(t)$, they introduced the corresponding *generalized parabolic frequency functional* as

$$Q(t) = -\frac{\sigma(t)\|\nabla w\|^2_{L^2(d\nu)}}{\|w\|^2_{L^2(d\nu)}} \exp\left(-\int \frac{1-\theta(t)}{\sigma(t)}\,dt\right),$$

This modified functional was shown to possess monotonic growth along the Ricci flow trajectory, contingent on the condition that the Bakry–Émery Ricci curvature remains bounded throughout the evolution.

Motivated by the aforementioned foundational works, this paper focuses on analyzing the monotonicity properties of a rigorously defined parabolic frequency functional (see equation (1.3) below) associated with a certain linear heat equation, specified in equation (1.2), evolving under the Ricci–Bourguignon flow (RBF). This investigation aims, on one hand, to establish a generalized monotonicity formula applicable to evolving Riemannian manifolds, as the RBF serves as a unifying framework encompassing various intrinsic geometric flows including the classical Ricci flow, harmonic-Ricci flow, and Yamabe flow (YF), among others. On the other hand, it provides a pathway to extend the study of parabolic frequency monotonicity to the setting of the backward YF.

We now turn to a precise formulation of the Ricci–Bourguignon evolution process. Consider a smooth, closed n-dimensional Riemannian manifold (M, g) equipped with a one-parameter family of metrics $g(t)$. The metric evolves according to the deformation law proposed by Bourguignon [3], given by

$$\frac{\partial g}{\partial t} = -2\left(\mathrm{S} - \gamma R g\right), \qquad g(0) = g_0, \tag{1.1}$$

where S is the Ricci tensor, R the scalar curvature, and $\gamma \in \mathbb{R}$ is a real constant. The initial value problem for (1.1) admits a unique short-time solution for any smooth initial metric g_0, provided that $\gamma < \frac{1}{2(n-1)}$, as established in the work of Catino and collaborators [4]. Furthermore, their results confirm that under appropriate conditions, certain geometric curvature properties are preserved throughout the evolution dictated by the flow. It is noteworthy that the Ricci–Bourguignon flow encompasses several well-known geometric flows as special cases determined by the choice of γ. Specifically:

- For $\gamma = 0$, equation (1.1) reduces to the classical Ricci flow.
- When $\gamma = \frac{1}{n}$, the flow corresponds to the normalized Ricci flow, since the tensor $\mathrm{S} - \gamma R g$ becomes traceless.

- The value $\gamma = \frac{1}{2(n-1)}$ yields the Schouten tensor flow, where the right-hand side is a scalar multiple of the Schouten tensor.

Furthermore, for nonpositive values of γ, after an appropriate time reparametrization, the flow interpolates continuously between the YF and the Ricci flow.

Let $\phi_t(y,t)$ denote the partial derivative of $\phi(y,t)$ with respect to time, and let $\Delta = \Delta_{g(t)}$ represent the Laplacian corresponding to the evolving metric $g(t)$. We focus on the linear heat equation given by

$$\phi_t(y,t) - \big(1 - 2(n-1)\gamma\big)\Delta\phi(y,t) = \vartheta(t)\phi(y,t), \tag{1.2}$$

here, $\vartheta(t)$ is a smooth function of time, and ϕ is a smooth map defined on the product manifold $M \times [0,T)$. Adopting the approach from [11], the parabolic frequency functional $Q(t)$ associated with solutions to equation (1.2) evolving under the Ricci–Bourguignon flow is defined as

$$Q(t) = \exp\left\{-\int_{t_0}^{t} \frac{\theta(s) + \rho'(s)}{\rho(s)}\, ds\right\} \frac{\rho(t)\int_M |\nabla\phi|^2\, dV}{\int_M \phi^2\, dV}, \tag{1.3}$$

the functions $\theta(t)$ and $\rho(t)$ are assumed to be smooth and vary with time over the interval $[t_0, t] \subset [0,T)$, while the volume element $dV = dV_{g(t)}$ corresponds to the measure induced by the time-dependent metric $g(t)$ (refer to equation (2.4) for the detailed definition). The monotonicity property of the functional $Q(t)$ along the Ricci and Ricci–harmonic flows, established by Li et al. in [11] under appropriate curvature assumptions, serves as a foundation for the current study, which extends these results by formulating a more general monotonicity theorem applicable to a broader category of geometric flows.

Let us fix an integer $n \geq 3$. The primary aim here is to verify that the parabolic frequency functional introduced in equation (1.3) exhibits monotone behavior when the metric $g(t)$ evolves in tandem with the heat equation (1.2). Specifically, we focus on the evolution of the coupled system involving the metric-function pair $(g(y,t), \phi(y,t))$ given by:

$$\begin{cases} \dfrac{\partial}{\partial t} g(y,t) + 2\,\mathrm{S}(y,t) = 2\gamma\, R(y,t) g(y,t), \\ \dfrac{\partial}{\partial t}\phi(y,t) - \big(1 - 2(n-1)\gamma\big)\Delta_{g(t)}\phi(y,t) = \vartheta(t)\phi(y,t), \\ (g(y,0), \phi(y,0)) = (g_0, \phi_0). \end{cases}$$

The primary theorem concerning this monotonicity is formulated as Theorem 3.1. By specializing the parameter γ to various values, the results naturally extend to encompass several notable geometric flows. In the special case $n = 2$, we restrict to $\gamma < 0$ and normalize it via the relation $\gamma = \frac{1-\alpha}{2}$ with $\alpha > \bar{\alpha} \geq 1$. Under these assumptions, we study the following coupled system:

$$\begin{cases} \dfrac{\partial}{\partial t} g(y,t) + \bar{\alpha} g(y,t) R(y,t) = 0, \\ \dfrac{\partial}{\partial t}\phi(y,t) - \bar{\alpha}\Delta_{g(t)}\phi(y,t) = \vartheta(t)\phi(y,t), \\ (g(y,0), \phi(y,0)) = (g_0, \phi_0). \end{cases} \tag{1.4}$$

The coupled system (1.4) may also find significant applications in conformal geometry for dimensions $n \geq 3$, where the evolving metric $g(t)$ can be expressed in terms of a conformal factor as

$$g(t) = \phi^{\frac{4}{n-2}} g_0.$$

It is evident that the first equation in (1.4) corresponds to the YF for parameters $\bar{\alpha} \geq 1$. Further details and implications of this connection are elaborated in Section 4.

Prior to the detailed exposition and discussion of our main results, we provide essential preliminaries in the subsequent section that form the foundational framework for our analysis.

2 Preliminaries

We begin by establishing the necessary notations in this section. Consider a manifold M of dimension $n \geq 2$ equipped with a family of metrics $g(t)$. The scalar curvature, Ricci curvature tensor, and volume element associated to $(M, g(t))$ will be denoted by $R = R_g$, $\mathrm{S} = \mathrm{S}_g$, and $d\mu = d\mu_g$, respectively. The Levi-Civita connection corresponding to $g(t)$ is represented by $\nabla = \nabla_g$. We introduce the f-Laplacian, also known as the Witten or weighted Laplacian, defined as

$$\Delta_f := \Delta - \nabla f \cdot \nabla.$$

Under the RBF described in (1.1), let us denote the backward time parameter by

$$\sigma(t) := T - t.$$

We consider the conjugate heat equation given by

$$\frac{\partial}{\partial t}\Theta(t) = -(1 - 2(n-1)\gamma)\Delta\Theta(t) - (n\gamma - 1)R\Theta(t). \tag{2.1}$$

A smooth positive solution $\Theta(t)$ to (2.1) can be expressed in the form

$$\Theta(t) = (4\pi\sigma(t))^{-\frac{n}{2}} e^{-f(t)}, \tag{2.2}$$

where $f(t)$ is a time-dependent smooth function defined on $(M, g(t))$. Utilizing the characterization of $\Theta(t)$ alongside the conjugate heat equation (2.1), it follows that the smooth scalar function $f(t)$ evolves according to the equation associated with the RBF, specifically satisfying:

$$\partial_t f(t) = -(1 - 2(n-1)\gamma)\big(\Delta f - |\nabla f|^2\big) + (n\gamma - 1)R + \frac{n}{2\sigma(t)}. \tag{2.3}$$

The associated weighted volume measure on $(M, g(t))$ is defined by

$$dV := \Theta(t)\, d\mu = (4\pi\sigma(t))^{-\frac{n}{2}} e^{-f(t)} d\mu, \quad \text{with} \quad \int_M dV = 1, \tag{2.4}$$

here, the notation $d\mu$ stands for the volume form determined by the evolving metric $g(t)$. Within the weighted Riemannian manifold (M^n, g, dV), the classical Bochner identity takes the following adjusted form:

$$\Delta_f |\nabla\phi|^2 = 2|\nabla^2\phi|^2 + 2\mathrm{S}_f(\nabla\phi, \nabla\phi) + 2\langle\nabla\phi, \nabla\Delta_f\phi\rangle, \tag{2.5}$$

for any smooth function $\phi \in C^\infty(M)$. Here, the Bakry–Émery Ricci tensor is defined as

$$\mathrm{S}_f := \mathrm{S} + \nabla^2 f. \tag{2.6}$$

Additionally, we introduce the modified Bakry–Émery Ricci tensor by

$$\overline{\mathrm{S}}_f := \mathrm{S} + (1 - 2(n-1)\gamma)\nabla^2 f.$$

The volume element $d\mu$ evolves according to the differential equation dictated by the RBF, explicitly satisfying:

$$\partial_t d\mu = (n\gamma - 1)R\, d\mu, \tag{2.7}$$

while the weighted volume measure dV evolves according to

$$\partial_t dV = -(1-2(n-1)\gamma)\frac{\Delta\Theta}{\Theta} dV. \tag{2.8}$$

Consider a smooth mapping $\phi : M \times [t_0, t_1] \to \mathbb{R}$ such that for every fixed time $t \in [t_0, t_1] \subset [0, T)$, the functions $\phi(\cdot, t)$ and its time derivative $\partial_t\phi(\cdot, t)$ both lie in the Sobolev space $W_0^{2,2}(dV)$. With this setup, we introduce the following functionals:

$$J(t) := \int_M \phi^2\, dV, \tag{2.9}$$

$$D(t) := \rho(t) \int_M |\nabla\phi|^2\, dV, \tag{2.10}$$

here, $\rho(t)$ denotes a smooth function varying continuously with time.

Accordingly, the parabolic frequency functional $Q(t)$, as defined in equation (1.3), admits the following representation:

$$Q(t) = \exp\left(-\int_{t_0}^{t} \frac{\theta(s) + \rho'(s)}{\rho(s)}\, ds\right) \frac{D(t)}{J(t)}, \tag{2.11}$$

with $\rho(t)$ and $\theta(t)$ smooth functions of time.

3 Monotonicity of the functional under RBF

This section is devoted to presenting and proving the principal results concerning the behavior of the parabolic frequency functional along the RBF.

Theorem 3.1. *Consider the family of metrics $(M^n, g(t))$ defined for $t \in [0, T)$, evolving under the RBF described by (1.1), and assume the curvature condition*

$$\overline{\mathrm{S}}_f \leq \left(\frac{\theta(t)}{2\rho(t)} + \gamma R\right) g,$$

where the parameter satisfies $\gamma < \frac{1}{2(n-1)}$.

- *(i) Provided that $\rho(t)$ remains positive throughout the interval, the parabolic frequency functional $Q(t)$ is non-increasing as the flow progresses.*
- *(ii) Conversely, if $\rho(t)$ is strictly negative for all t, then $Q(t)$ is guaranteed to be non-decreasing along the evolution.*

Proof. Our attention is directed toward establishing part (i), the argument for (ii) follows analogously with minor modifications.

Differentiating $|\nabla\phi|^2$ with respect to time and using the metric evolution (1.1) yields

$$\partial_t|\nabla\phi|^2 = 2\,\mathrm{S}(\nabla\phi, \nabla\phi) - 2\gamma R|\nabla\phi|^2 + 2\langle\nabla\phi, \nabla\partial_t\phi\rangle. \tag{3.1}$$

By integrating this result with the weighted Bochner identity, we deduce that

$$\begin{aligned}&(\partial_t-(1+2(1-n)\gamma)\Delta)|\nabla\phi|^2\\&=-2\gamma R|\nabla\phi|^2+2\langle\nabla\phi,\nabla(\partial_t-(1+2(1-n)\gamma)\Delta)\phi\rangle\\&\quad-2(1-2(n-1)\gamma)|\nabla^2\phi|^2+4(n-1)\gamma\,\mathrm{S}(\nabla\phi,\nabla\phi).\end{aligned}\tag{3.2}$$

Utilizing the linear heat equation (1.2), we can simplify the previous expression to obtain:

$$\begin{aligned}&(\partial_t-(1-2(n-1)\gamma)\Delta)|\nabla\phi|^2\\&=(-2\gamma R+2\vartheta(t))|\nabla\phi|^2\\&\quad-2(1-2(n-1)\gamma)|\nabla^2\phi|^2+4(n-1)\gamma\,\mathrm{S}(\nabla\phi,\nabla\phi).\end{aligned}\tag{3.3}$$

Utilizing equation (2.8) together with integration by parts, the derivative of $J(t)$ with respect to time can be expressed as

$$\begin{aligned}J'(t)&=\int_M 2\phi\phi_t\,dV+\int_M\phi^2\partial_t(dV)\\&=\int_M\left(2\phi\phi_t-(1-2(n-1)\gamma)\phi^2\frac{\Delta\Theta}{\Theta}\right)dV\\&=\int_M\left(2\phi\phi_t-(1-2(n-1)\gamma)\Delta\phi^2\right)dV\\&=\int_M\left(2\phi\phi_t-2(1-2(n-1)\gamma)\phi\Delta\phi-2(1-2(n-1)\gamma)|\nabla\phi|^2\right)dV.\end{aligned}\tag{3.4}$$

Since

$$\int_M|\nabla\phi|^2dV=-\int_M\phi\Delta_f\phi\,dV=\frac{D(t)}{\rho(t)},\tag{3.5}$$

and

$$\int_M(\phi_t-(1-2(n-1)\gamma)\Delta\phi)\phi\,dV=\int_M\vartheta(t)\phi^2dV=\vartheta(t)J(t),\tag{3.6}$$

we deduce

$$J'(t)=2\vartheta(t)J(t)-2(1-2(n-1)\gamma)\frac{D(t)}{\rho(t)}.\tag{3.7}$$

Define

$$\hat{J}(t)=\exp\left\{-\int_{t_0}^t2\vartheta(s)ds\right\}J(t).\tag{3.8}$$

Then from (3.7), we get

$$\hat{J}'(t)=-(1-2(n-1)\gamma)\frac{2}{\rho(t)}\exp\left\{-\int_{t_0}^t2\vartheta(s)ds\right\}D(t).\tag{3.9}$$

Subsequently, we evaluate the time derivative of $D(t)$. Employing equations (2.8) and (2.10) along with integration by parts, we find

$$\begin{aligned}D'(t)&=\rho(t)\int_M\left(\partial_t|\nabla\phi|^2-(1-2(n-1)\gamma)|\nabla\phi|^2\frac{\Delta\Theta}{\Theta}\right)dV+\rho'(t)\int_M|\nabla\phi|^2dV\\&=\rho(t)\int_M(\partial_t-(1-2(n-1)\gamma)\Delta)|\nabla\phi|^2dV+\rho'(t)\int_M|\nabla\phi|^2dV.\end{aligned}\tag{3.10}$$

Substituting (3.2) into (3.10) yields

$$D'(t) = \rho(t)\int_M \big(-2\gamma R|\nabla\phi|^2 + 2\vartheta(t)|\nabla\phi|^2 - 2(1-2(n-1)\gamma)|\nabla^2\phi|^2 \\ + 4(n-1)\gamma \mathrm{S}(\nabla\phi,\nabla\phi)\big)dV + \rho'(t)\int_M |\nabla\phi|^2 dV. \tag{3.11}$$

Applying the weighted Bochner formula (2.5) and using $\int_M \Delta_f(|\nabla\phi|^2)dV = 0$, we rewrite (3.11) as

$$D'(t) = (2\vartheta(t)\rho(t) + \rho'(t))\int_M |\nabla\phi|^2 dV + \rho(t)\int_M \big(-2\gamma R|\nabla\phi|^2 \\ - 2(1-2(n-1)\gamma)\big((\Delta_f\phi)^2 - \mathrm{S}_f(\nabla\phi,\nabla\phi)\big) + 4(n-1)\gamma \mathrm{S}(\nabla\phi,\nabla\phi)\big)dV.$$

Since $\mathrm{S}_f = \mathrm{S} + \nabla^2 f$, and

$$2(1+2(1-n)\gamma)\mathrm{S}_f(\nabla\phi,\nabla\phi) + 4(n-1)\gamma\mathrm{S}(\nabla\phi,\nabla\phi) \\ = 2(1-2(n-1)\gamma)\nabla^2 f(\nabla\phi,\nabla\phi) + 2\mathrm{S}(\nabla\phi,\nabla\phi) = 2\overline{\mathrm{S}}_f(\nabla\phi,\nabla\phi),$$

we arrive at

$$D'(t) = \rho(t)\int_M (-2\gamma R|\nabla\phi|^2 + 2\overline{\mathrm{S}}_f(\nabla\phi,\nabla\phi) - 2(1-2(n-1)\gamma)(\Delta_f\phi)^2)dV \\ + (2\vartheta(t)\rho(t) + \rho'(t))\int_M |\nabla\phi|^2 dV. \tag{3.12}$$

Using the curvature assumption

$$\overline{\mathrm{S}}_f \leq \left(\frac{\theta(t)}{2\rho(t)} + \gamma R\right) g,$$

with $\rho(t) > 0$, we can estimate the first term in (3.12) as follows:

$$\rho(t)\int_M \overline{\mathrm{S}}_f(\nabla\phi,\nabla\phi)dV \leq \rho(t)\int_M \left(\frac{\theta(t)}{2\rho(t)} + \gamma R\right)|\nabla\phi|^2 dV \\ = \frac{\theta(t)}{2}\int_M |\nabla\phi|^2 dV + \gamma\rho(t)\int_M R|\nabla\phi|^2 dV.$$

Substituting this estimate into (3.12) yields:

$$D'(t) \leq (\theta(t) + 2\vartheta(t)\rho(t) + \rho'(t))\int_M |\nabla\phi|^2 dV - 2(1+2(1-n)\gamma)\rho(t)\int_M (\Delta_f\phi)^2 dV \\ = \left(\frac{\theta(t)+\rho'(t)}{\rho(t)} + 2\vartheta(t)\right)D(t) - 2(1-2(n-1)\gamma)\rho(t)\int_M (\Delta_f\phi)^2 dV.$$

Define

$$\hat{D}(t) = \exp\left\{-\int_{t_0}^t \left(\frac{\theta(s)+\rho'(s)}{\rho(s)} + 2\vartheta(s)\right)ds\right\} D(t). \tag{3.13}$$

It follows that

$$\hat{D}'(t) \leq -2(1-2(n-1)\gamma)\rho(t)\exp\left\{-\int_{t_0}^t \left(\frac{\rho'(s)+\theta(s)}{\rho(s)} + 2\vartheta(s)\right)ds\right\}\int_M (\Delta_f\phi)^2 dV. \tag{3.14}$$

It follows from its definition that the parabolic frequency functional can be equivalently formulated as

$$Q(t) = \frac{\hat{D}(t)}{\hat{J}(t)}.$$

Combining (3.9) and (3.14) yields

$$\begin{aligned}\hat{J}^2(t)Q'(t) &= -\hat{J}'(t)\hat{D}(t) + \hat{D}'(t)\hat{J}(t)\\ &\leq -2(1-2(n-1)\gamma)\rho(t)\exp\left\{-\int_{t_0}^{t}\left(\frac{\theta(s)+\rho'(s)}{\rho(s)} + 4\vartheta(s)\right)ds\right\}\\ &\quad\times\left[\int_M (\Delta_f\phi)^2 dV \int_M \phi^2 dV - \left(\int_M |\nabla\phi|^2 dV\right)^2\right]. \end{aligned} \tag{3.15}$$

By the Cauchy–Schwarz inequality,

$$\left(\int_M |\nabla\phi|^2 dV\right)^2 = \left(\int_M \phi\Delta_f\phi\, dV\right)^2 \leq \left(\int_M (\Delta_f\phi)^2 dV\right)\left(\int_M \phi^2 dV\right), \tag{3.16}$$

which implies $Q'(t) \leq 0$. Hence, $Q(t)$ is monotone decreasing along the RBF. □

Corollary 3.2. *Let $(M^n, g(t))$ be a time-dependent Riemannian manifold evolving under the RBF(1.1) for $t \in [0,T)$, and assume the metric satisfies the curvature constraint*

$$\overline{\mathrm{S}}_f \preceq \left(\frac{\theta(t)}{2\rho(t)} + \gamma R\right) g,$$

where $\gamma < \frac{1}{2(n-1)}$, and the function $\rho(t)$ is strictly negative for all t in the interval.

(i) Suppose that the solution $\phi(y,t)$ vanishes identically at a fixed time $t_1 \in (0,T)$. Then $\phi(y,t)$ must be identically zero for all earlier times $t \in [t_0, t_1] \subset (0,T)$.

(ii) If the quantity $Q(t)$ associated with the parabolic frequency remains invariant with respect to time, then $\phi(y,t)$ satisfies the eigenvalue problem for the weighted Laplacian operator Δ_f.

Proof. (i) By the definitions of $J(t)$ and $Q(t)$, we have

$$\begin{aligned}\frac{d}{dt}\log J(t) &= \frac{J'(t)}{J(t)}\\ &= -2(1-2(n-1)\gamma)\frac{D(t)}{\rho(t)J(t)} + 2\vartheta(t)\\ &= -2(1-2(n-1)\gamma)\frac{1}{\rho(t)}\exp\left\{\int_{t_0}^{t}\frac{\theta(s)+\rho'(s)}{\rho(s)}ds\right\}Q(t) + 2\vartheta(t). \end{aligned} \tag{3.17}$$

Integrating the differential relation (3.17) across the subinterval $[t_2, t_1] \subset [t_0, t_1]$ and utilizing Theorem 3.1, we arrive at the following consequence:

$$\begin{aligned}&\log J(t_1) - \log J(t_2)\\ &\geq -2(1-2(n-1)\gamma)\int_{t_2}^{t_1}\exp\left(\int_{t_0}^{t}\frac{\theta(s)+\rho'(s)}{\rho(s)}ds\right)\frac{Q(t)}{\rho(t)}dt + 2\int_{t_2}^{t_1}\vartheta(t)dt\end{aligned}$$

$$\geq -2Q(t_0)(1-2(n-1)\gamma)\int_{t_2}^{t_1}\exp\left(\int_{t_0}^{t}\frac{\rho'(s)+\theta(s)}{\rho(s)}ds\right)\frac{1}{\rho(t)}dt+2\int_{t_2}^{t_1}\vartheta(t)dt.$$

Rearranging,

$$\frac{J(t_1)}{J(t_2)}\geq \exp\left\{2\int_{t_2}^{t_1}\vartheta(t)dt-2Q(t_0)(1-2(n-1)\gamma)\int_{t_2}^{t_1}\exp\Big(\int_{t_0}^{t}\frac{\theta(s)+\rho'(s)}{\rho(s)}ds\Big)\frac{dt}{\rho(t)}\right\}. \quad (3.18)$$

Consequently, if $\phi(\cdot,t_1)=0$, it follows that $J(t_1)=0$. Invoking (3.18), we deduce $J(t_2)=0$ for any $t_2\in[t_0,t_1]$, which in turn yields $J(t)\equiv 0$ throughout the interval $[t_0,t_1]$. This implies that $\phi(\cdot,t)$ vanishes identically on $[t_0,t_1]$.

(ii) In the case where the frequency functional $Q(t)$ remains constant over time, one has $Q'(t)=0$. This condition forces equality in the Cauchy–Schwarz inequality (3.16), from which we conclude that

$$\Delta_f\phi=c(t)\phi,$$

for some function $c(t)$ depending on t. Accordingly, ϕ is an eigenfunction of the weighted Laplacian Δ_f.

□

4 Monotonicity of functional on YF

4.1 YF

It is a classical fact that on two-dimensional manifolds, the Yamabe and Ricci flows coincide. However, in dimensions $n\geq 3$, these flows display distinct behaviors. By choosing $\bar{\alpha}=1$ in equation (1.4) and omitting the parameter γ within this context, we examine the YF coupled to a linear heat-type evolution:

$$\begin{cases}\partial_t g(y,t)=-R(y,t)g(y,t),\\ \partial_t\phi(y,t)-\Delta_{g(t)}\phi(y,t)=\vartheta(t)\phi(y,t),\\ (g(y,0),\phi(y,0))=(g_0,\phi_0).\end{cases} \quad (4.1)$$

This system describes a geometric flow of metrics $(M,g(t))$ evolving over a time interval $t\in[0,T)$ on a compact Riemannian manifold (M,g) of dimension n. The evolution of the metric alone corresponds to the YF, governed by the equation

$$\frac{\partial}{\partial t}g(t)=-R(t)g(t),\quad g(0)=g_0. \quad (4.2)$$

First proposed by R. Hamilton in an unpublished manuscript from the 1980s [8], the YF emerged as a promising strategy to tackle the classical Yamabe problem [16]. This foundational question in differential geometry asks whether one can deform a given Riemannian metric g_0 on a closed n-manifold ($n\geq 3$) into a conformally equivalent metric whose scalar curvature is constant. For an in-depth account of the problem and its historical development, readers are referred to the comprehensive monograph by Aubin [2] and the survey by Lee and Parker [10].

4.2 Parabolic frequency functional on YF

Consider the linear heat equation

$$w_t = \Delta w + \vartheta(t)w, \quad w(0) = w_0, \tag{4.3}$$

on the evolving manifold $(M, g(t))$ under YF. For a smooth function $w : M \times [t_0, t_1] \to \mathbb{R}$ with $w(\cdot, t), w_t(\cdot, t) \in W_0^{2,2}(dV_{g(t)})$ and for any $t \in [t_0, t_1] \subset [0, T)$, define

$$J_2(t) = \int_M w^2 \, dV, \tag{4.4}$$

$$D_2(t) = \rho(t) \int_M |\nabla w|^2 \, dV, \tag{4.5}$$

Here, $\rho(t)$ denotes a time-dependent smooth function. Subsequently, we introduce the parabolic frequency functional $Q_2(t)$ as follows:

$$Q_2(t) = \exp\left\{ -\int_{t_0}^{t} \frac{\theta(s) + \rho'(s)}{\rho(s)} ds \right\} \frac{D_2(t)}{J_2(t)}, \tag{4.6}$$

where $\rho(t)$ and $\theta(t)$ are smooth functions.

4.3 Monotonicity of $Q_2(t)$ on $(M, g(t))$ evolving by (4.1)

Consider the backward time parameter $\sigma(t) = T - t$ under the YF (4.1). For a smooth, time-dependent function $f(t) \in C^\infty(M)$, define the function

$$H(t) := (4\pi\sigma(t))^{-\frac{n}{2}} e^{-f(t)},$$

which serves as a strictly positive solution to the conjugate heat equation

$$\partial_t H = -\Delta H + \frac{n}{2} R(g(t))H. \tag{4.7}$$

By utilizing the above definition and equation (4.7), it follows that $f(t)$ satisfies the conjugate heat equation along the flow:

$$\partial_t f = -\Delta f + |\nabla f|^2 - \frac{n}{2}\left(R - \frac{1}{\sigma(t)}\right). \tag{4.8}$$

Define the weighted measure under YF by

$$dV = H(t)d\mu = (4\pi\sigma(t))^{-\frac{n}{2}} e^{-f(t)} d\mu, \qquad \int_M dV = 1,$$

where $d\mu$ is the volume form of $g(t)$. The volume form evolves by $\partial_t d\mu = -\frac{n}{2} R d\mu$, and hence the weighted volume form satisfies

$$\partial_t dV = (H_t - \tfrac{n}{2} RH)d\mu = -\frac{\Delta H}{H} dV. \tag{4.9}$$

We now state the monotonicity theorem for the YF.

Theorem 4.1. *Let $(M^n, g(t))$, $t \in [0,T)$ with $n \geq 3$, be a solution to the YF (4.1) with*

$$S_f \leq \frac{\theta(t)}{2\rho(t)} g \quad \text{if } (M^n, g(t)) \text{ satisfies } S - \frac{1}{2}Rg = 0,$$

otherwise

$$\frac{1}{2}Rg + \operatorname{Hess} f \leq \frac{\theta(t)}{2\rho(t)} g.$$

(i) Suppose $\rho(t)$ remains positive for all times; under this condition, the parabolic frequency $Q_2(t)$ exhibits a monotonically decreasing behavior along the YF.

(ii) Conversely, if $\rho(t)$ is strictly negative throughout the evolution, then $Q_2(t)$ increases monotonically as the flow progresses.

Lemma 4.1. *Let $(g, w) = (g(y,t), w(y,t))$ solve (4.1). Then the following identities hold:*

$$\partial_t |\nabla w|^2 = Rg(\nabla w, \nabla w) + 2\langle \nabla w, \nabla \Delta w\rangle + 2\vartheta(t)|\nabla w|^2, \tag{4.10}$$

$$(\partial_t - \Delta)|\nabla w|^2 = 2\left(\tfrac{1}{2}Rg - S\right)(\nabla w, \nabla w) + 2\vartheta(t)|\nabla w|^2 - 2|\operatorname{Hess} w|^2. \tag{4.11}$$

Proof. Using the standard formula for the evolution of the gradient norm under a time-dependent metric,

$$\partial_t |\nabla w|^2 = -[\partial_t g](\nabla w, \nabla w) + 2\langle \nabla w, \nabla \partial_t w\rangle.$$

From (4.1), $-\partial_t g = Rg$ and $\partial_t w = \Delta w + \vartheta(t) w$, establishing (4.10). Combining with the Bochner formula yields (4.11). □

Proof of Theorem 4.1

By integration by parts and using (4.9), we calculate the derivative of $J_2(t)$:

$$\begin{aligned} J_2'(t) &= \int_M \left(2ww_t - w^2\frac{\Delta H}{H}\right) dV \\ &= \int_M \left(2ww_t - \Delta w^2\right) dV \\ &= 2\int_M w(w_t - \Delta w) dV - 2\int_M |\nabla w|^2 dV. \end{aligned}$$

Using

$$\int_M (w_t - \Delta w) w dV = \vartheta(t) J_2(t), \qquad \int_M |\nabla w|^2 dV = \frac{D_2(t)}{\rho(t)},$$

we obtain

$$J_2'(t) = 2\vartheta(t) J_2(t) - 2\frac{D_2(t)}{\rho(t)}. \tag{4.12}$$

Define

$$\hat{J}_2(t) = \exp\left\{-\int_{t_0}^t 2\vartheta(s) ds\right\} J_2(t).$$

Then,

$$\hat{J}_2'(t) = -\frac{2}{\rho(t)} \exp\left\{-\int_{t_0}^t 2\vartheta(s) ds\right\} D_2(t).$$

Next, we compute $D_2'(t)$ using the definition, (4.9), and integration by parts:

$$\begin{aligned} D_2'(t) &= \rho(t)\int_M \left(\partial_t|\nabla w|^2 - |\nabla w|^2\frac{\Delta H}{H}\right)dV + \rho'(t)\int_M |\nabla w|^2 dV \\ &= \rho(t)\int_M (\partial_t - \Delta)|\nabla w|^2 dV + \rho'(t)\int_M |\nabla w|^2 dV. \end{aligned} \tag{4.13}$$

Substituting (4.11) into (4.13) gives

$$\begin{aligned} D_2'(t) &= (2\vartheta(t)\rho(t) + \rho'(t))\int_M |\nabla w|^2 dV + 2\rho(t)\int_M \left(\frac{1}{2}Rg - S\right)(\nabla w, \nabla w)dV \\ &\quad - 2\rho(t)\int_M |\mathrm{Hess}\, w|^2 dV. \end{aligned}$$

Using the weighted Bochner formula and integration by parts leads to

$$\begin{aligned} D_2'(t) &= (2\vartheta(t)\rho(t) + \rho'(t))\int_M |\nabla w|^2 dV + 2\rho(t)\int_M \left(\frac{1}{2}Rg - S\right)(\nabla w, \nabla w)dV \\ &\quad - 2\rho(t)\int_M \left[-S_f(\nabla w, \nabla w) + (\Delta_f w)^2\right] dV. \end{aligned} \tag{4.14}$$

If $(M, g(t))$ adimts in $S - \frac{1}{2}Rg = 0$, then (4.14) reduces to

$$D_2'(t) = (2\vartheta\rho + \rho')\int_M |\nabla w|^2 dV - 2\rho\int_M \left[(\Delta_f w)^2 - S_f(\nabla w, \nabla w)\right] dV. \tag{4.15}$$

Otherwise, if $S - \frac{1}{2}Rg \neq 0$, from (4.14) we have

$$D_2'(t) = (2\vartheta\rho + \rho')\int_M |\nabla w|^2 dV - 2\rho\int_M \left[(\Delta_f w)^2 - \left(\frac{R}{2}g + \mathrm{Hess} f\right)(\nabla w, \nabla w)\right] dV. \tag{4.16}$$

Invoking the curvature bound

$$S_f \leq \frac{\theta(t)}{2\rho(t)}g \quad \text{if } S - \frac{1}{2}Rg = 0, \quad \text{or} \quad \frac{1}{2}Rg + \mathrm{Hess} f \leq \frac{\theta(t)}{2\rho(t)}g \quad \text{otherwise},$$

we conclude in either case that

$$D_2'(t) \leq \left(2\vartheta + \frac{\theta(t) + \rho'}{\rho}\right) D_2(t) - 2\rho\int_M (\Delta_f w)^2 dV. \tag{4.17}$$

Define

$$\hat{D}_2(t) = \exp\left\{-\int_{t_0}^t \left(2\vartheta(s) + \frac{\theta(s) + \rho'(s)}{\rho(s)}\right) ds\right\} D_2(t).$$

Then,

$$\hat{D}_2'(t) \leq -2\rho(t)\exp\left\{-\int_{t_0}^t \left(2\vartheta(s) + \frac{\rho'(s) + \theta(s)}{\rho(s)}\right) ds\right\}\int_M (\Delta_f w)^2 dV. \tag{4.18}$$

Recalling the definitions of $\hat{J}_2$ and $\hat{D}_2$, an equivalent formulation for the parabolic frequency takes the form:

$$Q_2(t) = \frac{\hat{D}_2(t)}{\hat{J}_2(t)}.$$

Using the derivatives of $\hat{J}_2$ and $\hat{D}_2$, it follows that

$$\begin{aligned}\hat{J}_2^2(t)Q_2'(t) &= \hat{D}_2'(t)\hat{J}_2(t) - \hat{J}_2'(t)\hat{D}_2(t)\\ &\leq -2\rho(t)e^{-\Gamma(t)}\left[\int_M (\Delta_f w)^2 dV \int_M w^2 dV - \left(\int_M |\nabla w|^2 dV\right)^2\right],\end{aligned}$$

where

$$\Gamma(t) := \int_{t_0}^{t}\left(4\vartheta(s) + \frac{\theta(s)+\rho'(s)}{\rho(s)}\right) ds.$$

By the Cauchy–Schwarz inequality,

$$\left(\int_M |\nabla w|^2 dV\right)^2 = \left(\int_M w\Delta_f w\, dV\right)^2 \leq \int_M (\Delta_f w)^2 dV \int_M w^2 dV.$$

As a consequence, we deduce that $Q_2'(t) \leq 0$, implying that $Q_2(t)$ exhibits a non-increasing trend throughout the evolution governed by the YF.

□

Remark 1. The identity $S - \frac{1}{2}Rg = 0$ guarantees that the curvature of the scalar R is zero, and thus describes the Riemannian manifold (M, g) as Ricci-flat. In the framework of general relativity, a metric satisfying $S = \lambda g$ for some $\lambda \in \mathbb{R}$ corresponds to a vacuum solution of Einstein's field equations in the presence of a cosmological constant.

Corollary 4.2. *Consider a solution $(M^n, g(t))$ of the YF (4.1), defined over the interval $t \in [0, T)$, subject to the curvature bounds stated in Theorem 4.1. Assume additionally that the function $\rho(t)$ is negative-valued over time.*

(i) If the function $w(\cdot, t_1)$ vanishes identically at time t_1, then it remains identically zero for all times $t \in [t_0, t_1] \subset (0, T)$.

(ii) If the parabolic frequency $Q_2(t)$ remains constant throughout the flow, then $w(y, t)$ must satisfy the eigenvalue equation associated with the weighted Laplacian Δ_f.

Proof. The argument mirrors that employed in the demonstration of Corollary 3.2, and hence is omitted. □

Conclusion

This study has focused on the time evolution of a parabolic-type energy quantity, denoted by $Q_2(t)$, in the context of compact Riemannian manifolds evolving under the YF. By introducing a frequency-like functional tailored to the heat-type setting, we established its monotonic behavior under sign conditions on a time-dependent weight function. These results reveal a fundamental interaction between conformal geometric deformations and analytic quantities such as weighted Laplacians and conjugate heat solutions. The monotonicity properties obtained herein may serve as a foundation for further investigations into the spectral and geometric implications of the YF.

Declarations

Funding

This work did not receive any specific funding.

Conflict of interest

We declare that we have no commercial or financial relationships that could be construed as a potential conflict of interest in connection with this work.

Availability of data and materials

All data generated or analyzed during this study are included in this published article.

Authors' contributions

All authors contributed equally to the preparation of this manuscript.

References

[1] F. J. Almgren, Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents. Minimal submanifolds and geodesics, 1-6, North Holland, Amsterdam, 1976.

[2] T. Aubin, Some nonlinear problems in Riemannian geometry, Springer Monographs in Mathematics, Springer, 1998.

[3] J. P. Bourguignon, Ricci curvature and Einstein metrics, Global differential geometry and global analysis (Berlin,1979) Lecture Notes in Math. vol. 838, Springer, Berlin, 1981, 42-63.

[4] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza and L. Mazzieri, The Ricci-Bourguignon flow, Pacific J. Math. **287** (2)(2017), 337–370.

[5] T. H. Colding and W. P. Minicozzi II, Parabolic frequency on manifolds, International Mathematics Research Notice, 2022 (15) (2022), 11878-11890.

[6] N. Garofalo and F. H. Lin, Monotonicity properties of variational integrals, A_p weights and unique continuation, Indiana Univ. Math. J., **35** (2) (1986), 245-268.

[7] N. Garofalo and F. H. Lin, Unique continuation for elliptic operators: a geometric-variational approach, Comm. Pure Appl. Math., **40**(3) (1987), 347-366.

[8] R. S. Hamilton, Lectures on geometric flows (1989), unpublished.

[9] J. Baldauf and D. Kim, Parabolic frequency on Ricci flows, International Mathematics Research Notice, 2022; rnac128,https://doi.org/10.1093/imrn/rnac128.

[10] J. M. Lee and T. H. Parker, The Yamabe problem, Bull. Amer. Math. Soc., 17 (1987), 37–91.

[11] C. Li, Y. Li and K. Xu, Parabolic frequency monotonicity on Ricci-flow and Ricci-harmonic flow with bounded curvature. arxiv.org/abs/2205.07702v1

[12] X. Li and K. Wang, Parabolic frequency monotonicity on compact manifolds, Calc. Var. Partial Differential Equations **58** (2019), no. 6, Paper No. 189, 18 pp.

[13] F. H. Lin, Nodal sets of solutions of elliptic and parabolic equations, Comm. Pure Appl. Math., **44**(3) (1991), 287-308.

[14] L. Ni, Parabolic frequency monotonicity and a theorem of Hardy-Pólya-Szegö. Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong, 203-210, Contemp. Math., 644, Amer. Math. Soc., Providence, RI, 2015.

[15] C. C. Poon, Unique continuation for parabolic equations, Comm. Partial Differential Equations, **21** (1996), no. 3-4, 521-539.

[16] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. **12** (1960), 21–37.

Shahroud Azami Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

E-mail: azami@sci.ikiu.ac.ir

Abimbola Abolarinwa Department of Mathematics, University of Lagos, Akoka, Lagos State, Nigeria

E-mail: a.abolarinwa1@gmail.com