TAMKANG JOURNAL OF MATHEMATICS
Volume 57, Number 2, 135-148, 2026
DOI:10.5556/j.tkjm.57.2026.5869

Multipliers of Bloch-type and Zygmund-type spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$

Lien Vuong Lam

Abstract. The aim of this paper is to study the multipliers of Bloch-type and Zygmund-type spaces of holomorphic functions on the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. For the classical Bloch space, such multipliers were characterized by Zhu. Subsequently, Galindo and Lindstr"om extended this investigation to the infinite-dimensional setting for the specific weight $\omega(z) = 1 - |z|^2$.

Keywords. Multiplier, unit ball, Bloch-type space, Zygmund-type space

1 Introduction

The present paper deals with the study of multipliers of Bloch-type and Zygmund-type spaces on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$. The study of the Bloch space of holomorphic functions on the unit ball of the complex plane or on higher-dimensional spaces was initiated by R.M. Timoney [21] and recalled in K. Zhu's book [24]. In 2007, X. Tang generalized this setting to the Bloch-type space $\mathcal{B}_\omega(\mathbb{B}_n)$, where ω is a normal weight on $\mathbb{B}_n \subset \mathbb{C}^n$.

A function f is said to be a multiplier for the Bloch space $\mathcal{B}(\mathbb{B}_n)$ if $f \cdot g \in \mathcal{B}(\mathbb{B}_n)$ for all $g \in \mathcal{B}(\mathbb{B}_n)$. Arazy [1] characterized the multipliers of Bloch functions on the unit ball of $\mathbb{C}$. Using the fact that

$$\beta(x, y) = \sup\{|f(x) - f(y)| : \|f\|_{\mathcal{B}} \leq 1\},$$

for $x, y \in \mathbb{B}_n$, where β is the Bergman metric in $\mathbb{B}_n$, Zhu generalized Arazy's result to higher dimensions [24, Theorem 3.21]. In a similar way, Galindo and Lindström gave conditions under which f is a multiplier of $\mathcal{B}(\mathbb{B}_n)$, where X is a Hilbert space [4, Theorem 3.1]. Furthermore, by characterizing the growth of a function in $\mathcal{B}_0(\mathbb{B}_n)$ (the little Bloch space on $\mathbb{B}_n$), they established a relationship between multipliers of $\mathcal{B}(\mathbb{B}_n)$ and $\mathcal{B}_0(\mathbb{B}_n)$.

Motivated by these results, in this paper we consider the multipliers of $\mathcal{B}_\omega(\mathbb{B}_n)$ in the case where ω is a normal weight on the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$. This paper is organized as follows. In Section 2, we recall the notation of the Bloch-type space $\mathcal{B}_\omega(\mathbb{B}_n)$. In Section 3, we extend certain growth estimates for functions $f \in \mathcal{B}(\mathbb{B}_n)$. At the end of this section, we prove that the two quantities

$$\sup\left\{|f(z) - f(0)| : \|f\|_{\mathcal{B}^R_\omega(\mathbb{B}_n)} \leq 1\right\} \quad \text{and} \quad \int_0^{\|z\|} \frac{dt}{\omega(t)}$$

Received date: August 10, 2025; Published online: March 11, 2026.
2010 *Mathematics Subject Classification*. Primary 47B38, Secondary 30H30, 47B33, 47B91.
Corresponding author: Lien Vuong Lam.

are asymptotic. We then study the multipliers of $\mathcal{B}_\omega(\mathbb{B}_n)$, and, based on the ideas in [4, Theorems 3.1 and 3.4], we estimate the growth of functions in $\mathcal{B}_0(\mathbb{B}_n)$. From Theorem 3.1 and Proposition 3.1, we show that a function $f \in \mathcal{H}(\mathbb{B}_n)$ is a multiplier of the Bloch-type space $\mathcal{B}^{\nabla}_\omega(\mathbb{B}_n)$ if and only if f is a multiplier of $\mathcal{B}^{\nabla}_{\omega,0}(\mathbb{B}_n)$.

Finally, in Section 4, we investigate the condition for a function to be a multiplier of the Zygmund-type space in the case

$$\int_0^1 \left(\int_0^t \frac{ds}{\omega(s)} \right) dt = \infty.$$

Compared with Bloch-type spaces, Zygmund-type spaces involve the second radial derivative and are naturally connected to Lipschitz-type conditions in complex analysis. They arise in the study of fine boundary behavior of holomorphic functions, embedding theorems, and the boundedness of various integral or differential operators. Although the multiplier problem for Bloch-type spaces has been investigated extensively, the corresponding problem for Zygmund-type spaces with general normal weights remains less developed. Our goal in this part is to provide sharp necessary and sufficient conditions for multipliers on $Z_\omega(\mathbb{B}_n)$, thereby extending and unifying several known results in both the unweighted and weighted settings.

Throughout this paper, the notation $a \asymp b$ means that there exist positive constants C, D such that $Ca \le b \le Da$.

2 Background

Let $\mathbb{B}_n$ denote the open unit ball in $\mathbb{C}^n$. The space $\mathcal{H}^\infty(\mathbb{B}_n)$ consists of all bounded holomorphic functions on $\mathbb{B}_n$, that is,

$$\mathcal{H}^\infty(\mathbb{B}_n) = \left\{ f : \mathbb{B}_n \to \mathbb{C} \ : \ \sup_{z\in\mathbb{B}_n} |f(z)| < \infty \right\}.$$

The space $\mathcal{H}^\infty(\mathbb{B}_n)$ is a Banach space with respect to the norm

$$\|f\|_\infty := \sup_{z\in\mathbb{B}_n} |f(z)|.$$

Definition 1. A positive continuous function ω on the interval $[0, 1)$ is called normal if there are three constants $0 \le \delta < 1$ and $0 < a < b < \infty$ such that

$$\frac{\omega(t)}{(1-t)^a} \text{ is decreasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^a} = 0, \tag{W_1}$$

$$\frac{\omega(t)}{(1-t)^b} \text{ is increasing on } [\delta, 1), \quad \lim_{t\to 1} \frac{\omega(t)}{(1-t)^b} = \infty. \tag{W_2}$$

If we say that a function $\omega : \mathbb{B}_n \to [0, \infty)$ is normal, we also assume that it is radial, that is, $\omega(z) = \omega(\|z\|)$ for every $z \in \mathbb{B}_n$.

Strictly positive continuous functions on $\mathbb{B}_n$ are called weights.

The weighted Banach spaces $H^\infty_\omega(\mathbb{B}_n)$ are defined by

$$\mathcal{H}^\infty_\omega(\mathbb{B}_n) := \left\{ f \in \mathcal{H}(\mathbb{B}_n) : \sup_{z\in\mathbb{B}_n} \omega(z)|f(z)| < \infty \right\}.$$

It is easy to check that $\mathcal{H}^\infty(\mathbb{B}_n) \subsetneq \mathcal{H}^\infty_\omega(\mathbb{B}_n)$. For example, the function $f(z) = \frac{1}{1-\langle z,w\rangle} \in \mathcal{H}^\infty_\omega(\mathbb{D})$ but not belong to $\mathcal{H}^\infty(\mathbb{D})$ with $\omega(t) = 1 - t^2$.

We define the Bloch-type space

$$\mathcal{B}_\omega(\mathbb{B}_n) := \{f \in \mathcal{H}(\mathbb{B}_n) : \|f\|_{\mathcal{B}_\omega(\mathbb{B}_n)} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(z)|Rf(z)| < \infty\}$$

where $Rf(z) = \displaystyle\sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_j}(z)$ and the Zygmund-space space

$$\mathcal{Z}_\omega(\mathbb{B}_n) := \{f \in \mathcal{H}(\mathbb{B}_n) : \|f\|_{\mathcal{Z}_\omega(\mathbb{B}_n)} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(z)|R^{(2)}f(z)| < \infty\}$$

where $R^{(2)}f(z) = R(Rf)(z)$.

A function $f \in \mathcal{B}_\omega(\mathbb{B}_n)$ (resp. $\mathcal{Z}_\omega(\mathbb{B}_n)$) is said to belong to $\mathcal{B}_{\omega,0}(\mathbb{B}_n)$ (resp. $\mathcal{Z}_{\omega,0}(\mathbb{B}_n)$) if

$$\lim_{\|z\|\to 1^-} \omega(z)|Rf(z)| = 0 \text{ (resp. } \lim_{\|z\|\to 1^-} \omega(z)|R^{(2)}f(z)| = 0)$$

In the case $\omega(z) = (1 - \|z\|^2)^\alpha$ with $\alpha > 0$, we write $\mathcal{B}_\alpha(\mathbb{B}_n)$ instead of $\mathcal{B}_\omega(\mathbb{B}_n)$.

It is known that for every $f \in \mathcal{B}(\mathbb{B}_n)$, there exists a constant $C > 0$ such that

$$|f(z) - f(0)| \le C \log \frac{1}{1 - \|z\|^2}$$

for all $z \in \mathbb{B}_n$. A natural question is to consider the above result in the case $\mathcal{B}_\omega(\mathbb{B}_n)$ where ω is a normal weight on the unit ball $\mathbb{B}_n$.

Proposition 2.1. *Let ω be a normal weight on $\mathbb{B}_n$. Then there exists a constant $C > 0$ such that*

$$|f(z) - f(0)| \le C \int_0^{\|z\|} \frac{dt}{\omega(t)} \|f\|_{s\mathcal{B}_\omega(\mathbb{B}_n)} \tag{2.1}$$

for $f \in \mathcal{B}_\omega(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$.

Proof. Since $Rf(0) = 0$ and by Schwarz lemma, we have

$$|Rf(z)| \lesssim \frac{2\|f\|_{s\mathcal{B}^R_\omega(\mathbb{B}_n)}}{\min_{t\in[0;1/2]} \omega(t)} \|z\|, \text{ with } \|z\| < \frac{1}{2}.$$

Therefore, for $\|z\| < \frac{1}{2}$ and let $C_1 = \frac{\max_{t\in[0;1/2]} \omega(t)}{\min_{t\in[0;1/2]} \omega(t)}$, we have

$$\begin{aligned} |f(z) - f(0)| &\le \int_0^1 \Big|\frac{Rf(tz)}{t}\Big| dt \le \frac{2\|f\|_{s\mathcal{B}^R_\omega(\mathbb{B}_n)}}{\min_{t\in[0;1/2]} \omega(t)} \|z\| \\ &\lesssim \frac{2\|z\|}{\min_{t\in[0;1/2]} \omega(t)} |\|f\|_{s\mathcal{B}^R_\omega(\mathbb{B}_n)} \le 2C_1 \int_0^{\|z\|} \frac{dt}{\omega(t)} \|f\|_{s\mathcal{B}_\omega(\mathbb{B}_n)}. \end{aligned}$$

Next let $z \in \mathbb{B}_n$ with $\|z\| \ge \frac{1}{2}$, we estimate

$$|f(z) - f(0)| \le |f(z) - f(z/2)| + |f(z/2) - f(0)|$$

$$\le \int_{1/2}^{1} \Big|\frac{Rf(tz)}{t}\Big| dt + C_1 \|f\|_{s\mathcal{B}^R_\omega(\mathbb{B}_n)}$$
$$\le 4\int_0^{\|z\|} \frac{dt}{\omega(t)} \|f\|_{s\mathcal{B}_\omega(\mathbb{B}_n)} + 2C_1 \int_0^{\|z\|} \frac{dt}{\omega(t)} \|f\|_{s\mathcal{B}_\omega(\mathbb{B}_n)}$$
$$\le C\int_0^{\|z\|} \frac{dt}{\omega(t)} \|f\|_{s\mathcal{B}_\omega(\mathbb{B}_n)}.$$

Hence, the proposition is proved. □

From Proposition 2.1, we have the following result

Corollary 2.1. *Let ω be a normal weight on $\mathbb{B}_n$. Then there exists a constant $C > 0$ such that*

$$\sup\left\{|f(z) - f(0)| : \|f\|_{\mathcal{B}^R_\omega(\mathbb{B}_n)} \le 1\right\} \le C\int_0^{\|z\|} \frac{dt}{\omega(t)}. \tag{2.2}$$

Now, by modifying from an original one in [5], we prove the converse inequality.

First, we consider the holomorphic function

$$g(z) := 1 + \sum_{k>k_0} 2^k z^{n_k}, \quad z \in \mathbb{D}, \tag{2.3}$$

where $k_0 = \left[\log_2 \frac{1}{\nu(\delta)}\right]$, $n_k = \left[\frac{1}{1-r_k}\right]$ with $r_k = \nu^{-1}(1/2^k)$ for every $k \ge 1$. Here the symbol $[x]$ means the greatest integer not more than x. By [7, Theorem 2.3], $g(t)$ is increasing on $[0,1)$ and

$$|g(z)| \le g(|z|) \in \mathbb{R} \quad \forall z \in \mathbb{D},$$
$$0 < C_1 := \inf_{t\in[0,1)} \nu(t)g(t) \le \sup_{t\in[0,1)} \nu(t)g(t) \le C_2 := \sup_{z\in\mathbb{D}} \nu(z)|g(z)| < \infty. \tag{2.4}$$

In [5, Lemma 2.1], Hamada claimed that there exist the constants $r_1 \in (0;1)$ and $C_3 > 0$ such that $\int_0^{r_1} g(t)dt = 1$ and $\int_0^{r} g(t)dt \le C_3 \int_0^{r^2} g(t)dt$ for all $r \in [r_1, 1)$.

Lemma 2.1. *There exists a constant C such that*

$$\int_0^{\|z\|} \frac{dt}{\omega(t)} \le C \sup\{|h(z) - h(0)| : \|h\|_{\mathcal{B}^R_\omega(\mathbb{B}_n)} \le 1\}.$$

Proof. For each $z \in \mathbb{B}_n \setminus \{0\}$, $l_z \in T_z$, we define

$$f(w) = \int_0^{\langle w,z\rangle} g(\xi)d\xi.$$

It is easy to see that $f \in \mathcal{B}^R_\omega(\mathbb{B}_n)$ and $\frac{\|f_v\|_{\mathcal{B}^R_\omega(\mathbb{B}_n)}}{C_2} \le 1$. Now, for all $\|z\| \ge r_1$, we have the following estimate

$$\int_0^{\|z\|} \frac{dt}{\omega(t)} = \int_0^{\|z\|} \frac{g(t)}{g(t)\omega(t)}dt \le \frac{1}{C_1}\int_0^{\|z\|} g(t)dt \le \frac{C_3}{C_1}\int_0^{\|z\|^2} g(t)dt$$
$$= \frac{C_3}{C_1}|f(z) - f(0)| \le \frac{C_3}{C_1C_2} \sup\{|h(z) - h(0)| : \|h\|_{\mathcal{B}^R_\omega(\mathbb{B}_n)} \le 1\}.$$

□

From Corollary 2.1 and Lemma 2.1, we obtain

Proposition 2.2. *Let ω be a normal weight on $\mathbb{B}_n$. Then there exists a constant $r \in (0;1)$ such that*

$$\sup\left\{|f(z) - f(0)| : \|f\|_{\mathcal{B}^R_\omega(\mathbb{B}_n)} \leq 1\right\} \asymp \int_0^{\|z\|} \frac{dt}{\omega(t)}, \tag{2.5}$$

where $z \in \mathbb{B}_n$ and $\|z\| \geq r$.

3 Multipliers of Bloch space

Now, we consider the multiplier of the Bloch-type space on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$.

Recall that a function f is said to be a multiplier of Bloch-type space $\mathcal{B}_\omega(\mathbb{B}_n)$ if $f \cdot g \in \mathcal{B}_\omega(\mathbb{B}_n)$.

Theorem 3.1. *Let ω be a normal on the unit ball $\mathbb{B}_n$ and $f : \mathbb{B}_n \to \mathbb{C}$ be holomorphic function. Then f is a multiplier of the Bloch space $\mathcal{B}_\omega(\mathbb{B}_n)$ if and only if $f \in \mathcal{H}^\infty(\mathbb{B}_n)$ and the function $h : \mathbb{B}_n \to \mathbb{R}$ defined by*

$$h(z) := \omega(\|z\|)\|\nabla f(z)\| I_\omega(\|z\|)$$

is bounded, where $I_\omega(\|z\|) = \int_0^{\|z\|} \frac{dt}{\omega(t)}$.

Proof. As in the proof of [4, Theorem 3.1], it suffices to show that both f and the function

$$h(z) := \omega(\|z\|)\|\nabla f(z)\| I_\omega(\|z\|),$$

are bounded whenever f is a multiplier of the Bloch-type space $\mathcal{B}^\nabla_\omega(\mathbb{B}_n)$.

By [5], the Bloch-type space $\mathcal{B}^\nabla_\omega(\mathbb{B}_n)$ is a Banach space.

Suppose f is a multiplier of $\mathcal{B}^\nabla_\omega(\mathbb{B}_n)$. Then there exists a constant $C > 0$ such that

$$\|f \cdot g\|_{\mathcal{B}^\nabla_\omega(\mathbb{B}_n)} \leq C\|g\|_{\mathcal{B}^\nabla_\omega(\mathbb{B}_n)} \quad \text{for all } g \in \mathcal{B}^\nabla_\omega(\mathbb{B}_n).$$

Fix $z \in \mathbb{B}_n$. Let δ_z denote the point evaluation functional at z, that is, $\delta_z(g) = g(z)$ for $g \in \mathcal{B}^\nabla_\omega(\mathbb{B}_n)$. Then,

$$|f(z)| \cdot |\delta_z(g)| = |f(z)g(z)| = |\delta_z(f \cdot g)| \leq \|f \cdot g\|_{\mathcal{B}^\nabla_\omega(\mathbb{B}_n)} \cdot \|\delta_z\| \leq C\|\delta_z\| \cdot \|g\|_{\mathcal{B}^\nabla_\omega(\mathbb{B}_n)}.$$

Taking the supremum over all g in the unit ball of $\mathcal{B}^\nabla_\omega(\mathbb{B}_n)$, we obtain:

$$|f(z)| \cdot \|\delta_z\| \leq C\|\delta_z\|.$$

Thus, $|f(z)| \leq C$ for all $z \in \mathbb{B}_n$, so $f \in \mathcal{H}^\infty(\mathbb{B}_n)$.

Next, we estimate the gradient term. Since

$$\nabla(f \cdot g)(z) = f(z)\nabla g(z) + g(z)\nabla f(z),$$

we have

$$\omega(z)\|\nabla(f \cdot g)(z)\| \leq \omega(z)\left(|f(z)| \cdot \|\nabla g(z)\| + |g(z)| \cdot \|\nabla f(z)\|\right).$$

Using the definition of the Bloch-type norm, we get:

$$\omega(z)|g(z)|\cdot\|\nabla f(z)\| \leq \|f\|_\infty \cdot \|g\|_{\mathcal{B}^\nabla_\omega(\mathbb{B}_n)} + C\|g\|_{\mathcal{B}^\nabla_\omega(\mathbb{B}_n)}.$$

Now take the supremum over all g in the unit ball of $\mathcal{B}^\nabla_\omega(\mathbb{B}_n)$ satisfying $g(0) = 0$. By Proposition 2.2, this gives the estimate:

$$\omega(\|z\|)\|\nabla f(z)\|\cdot I_\omega(\|z\|) \leq C',$$

for some constant $C' > 0$. That is, $h(z)$ is bounded on $\mathbb{B}_n$. □

Since ω is positive, continuous on $\mathbb{B}_n$, it is easy to see that there exists $M > 0$ such that

$$\omega(z).I_\omega(z) < M, \text{ for all } z \in \mathbb{B}_n \tag{3.1}$$

From (3.1) and Theorem 3.1, we have

Corollary 3.2. *Let ω be a normal weight on the unit ball $\mathbb{B}_n$. Then every polynomial is a multiplier of $\mathcal{B}_\omega(\mathbb{B}_n)$.*

In [4, Lemma 3.1], Galindo and Lindström characterized the growth of a function in $\mathcal{B}_0(\mathbb{B}_n)$ in the case $\omega(z) = 1 - \|z\|^2$. It is the key to characterize the equivalent of multipliers for $\mathcal{B}_0(\mathbb{B}_n)$ and $\mathcal{B}(\mathbb{B}_n)$. A natural question is whether this result still true for general weights. We modify the proof in [4, Lemma 3.1] by adding the condition $\int_0^1 \frac{dt}{\omega(t)} = \infty$.

Proposition 3.1. *Let ω be a normal weight on the unit ball $\mathbb{B}_n$ such that $\int_0^1 \frac{dt}{\omega(t)} = \infty$ and $g \in \mathcal{B}_{\omega,0}(\mathbb{B}_n)$. Then*

$$\lim_{\|z\|\to 1^-} \frac{|g(z)|}{I_\omega(\|z\|)} = 0.$$

Proof. Without loss of genarality, we can assume that $\|g\|_{\mathcal{B}_\omega(\mathbb{B}_n)} = 1$ and $g(0) = 0$. This gives that $|g(z)| \leq CI_\omega(\|z\|)$ for some $C > 0$. From $g \in \mathcal{B}_{\omega,0}(\mathbb{B}_n)$ we get directly that $\lim\limits_{\|z\|\to 1^-} \omega(z)\|Rg(z)\| = 0$. Given $\varepsilon > 0$ then there exists $s \in (\frac{1}{2}; 1)$ such that $\omega(z)\|Rg(z)\| < \varepsilon$ for $\|z\| > s^2$.

Since $\int_0^1 \frac{dt}{\omega(t)} = \infty$, there exists $r > 0$ such that for $\|z\| > r$ then

$$\int_0^{\|z\|} \frac{dt}{\omega(t)} > \frac{\int_0^s \frac{dt}{\omega(t)}}{\varepsilon}.$$

If $\|z\| > s$, from $g(sz) \leq I_\omega(sz) = \int_0^{s\|z\|} \frac{dt}{\omega(t)} \leq \int_0^{\|z\|} \frac{dt}{\omega(t)}$, we have

$$\begin{aligned}|g(z) - g(sz)| &= \Big|\int_s^1 \frac{Rg(zt)}{t} dt\Big| = \Big|\int_s^1 \frac{Rg(zt)\omega(zt))}{t\omega(zt)} dt\Big| \\ &\leq \varepsilon \int_s^1 \frac{dt}{t\omega(zt)} dt = \varepsilon \int_s^1 \frac{\|z\|dt}{t\|z\|\omega(zt)} \leq \frac{\varepsilon}{\|z\|^2}\int_0^{\|z\|} \frac{dt}{\omega(t)} < 4\varepsilon \int_0^{\|z\|} \frac{dt}{\omega(t)}.\end{aligned}$$

Consequently,

$$\frac{|g(z)|}{I_\omega(z)} = \frac{|g(z) - g(sz) + g(sz)|}{I_\omega(z)} \leq \frac{|g(z) - g(sz)|}{I_\omega(z)} + \frac{|g(sz)|}{I_\omega(z)} < 5\varepsilon.$$

The proof is complete. □

Theorem 3.3. *Let ω be a normal weight such that $\int_0^1 \frac{dt}{\omega(t)} = \infty$. Then the function $f \in \mathcal{H}(\mathbb{B}_n)$ is a multiplier of the Bloch-type space $\mathcal{B}_\omega(\mathbb{B}_n)$ if and only if f is a multiplier of $\mathcal{B}_{\omega,0}(\mathbb{B}_n)$.*

Proof. Suppose that f is a multipier of $\mathcal{B}_\omega(\mathbb{B}_n)$ and $g \in \mathcal{B}_{\omega,0}(\mathbb{B}_n)$.

It follows that $\lim_{\|z\|\to 1^-} \omega(z)\|Rg(z)\| = 0$. By Theorem 3.1, we obtain

$$\lim_{\|z\|\to 1^-} \omega(z)|f(z)|\|Rg(z)\| = 0. \tag{3.2}$$

Since

$$\omega(z)R(f\cdot g)(z) = \omega(z)f(z)Rg(z) + \omega(z)Rf(z)g(z), \tag{3.3}$$

by applying Theorem 3.1, we conclude that

$$\lim_{\|z\|\to 1^-} \omega(z)\|Rf(z)\||g(z)| = \lim_{\|z\|\to 1} \frac{|g(z)|}{I_\omega(\|z\|)}.\omega(z)\|Rf(z)\|I_\omega(\|z\|) = 0. \tag{3.4}$$

From (3.2) and (3.4), we obtain $f\cdot g \in \mathcal{B}_{\omega,0}(\mathbb{B}_n)$.

Conversely, suppose that f is a multiplier of $\mathcal{B}_{\omega,0}(\mathbb{B}_n)$. We prove that $f\cdot g \in \mathcal{B}_\omega(\mathbb{B}_n)$ for all $g \in \mathcal{B}_\omega(\mathbb{B}_n)$. For each $\lambda \in \mathbb{B}_1$, we consider the function $g_\lambda : \mathbb{B}_n \to \mathbb{C}$ defined by $g_\lambda(x) = g(\lambda x)$. It is easy to see that $g_\lambda \in \mathcal{B}_{\omega,0}$. Therefore, $f\cdot g_\lambda \in \mathcal{B}_{\omega,0}$ for all $\lambda \in \mathbb{B}_1$. Thus, $(f\cdot g)_\lambda(z) := f(\lambda z)g(\lambda z) \in \mathcal{B}_{\omega,0}(\mathbb{B}_1)$ for $\lambda \in \mathbb{B}_1$. By [13, Theorem 2.2], we obtain that $f\cdot g \in \mathcal{B}_\omega(\mathbb{B}_n)$. □

Remark 1. The multiplication $M_f : \mathcal{B}_\omega(\mathbb{B}_n) \to \mathcal{B}_\omega(\mathbb{B}_n)$ given by $M_f(g) = gf$ is invertible if and only if $1/f$ is a multipler of $\mathcal{B}_\omega(\mathbb{B}_n)$.

Proof. If M_f is invertible then there exists $h \in \mathcal{B}_\omega(\mathbb{B}_n)$ such that $hf = 1$. Thus, $1/f \in \mathcal{B}_\omega(\mathbb{B}_n)$. Now, for each $g \in \mathcal{B}_\omega(\mathbb{B}_n)$ then there is $h \in \mathcal{B}_\omega(\mathbb{B}_n)$ so that $hf = g$. It follows that $1/f.g = h \in \mathcal{B}_\omega(\mathbb{B}_n)$. □

Corollary 3.4. *$1/f$ is a multiplier of $\mathcal{B}_\omega(\mathbb{B}_n)$ if and only if $1/f \in H^\infty(\mathbb{B}_n)$.*

Proof. It suffices to prove the converse. By Theorem 3.1, we only need to check that

$$\omega(z)\|R(1/f)(z)\|I_\omega(\|z\|) < \infty.$$

Since $1/f \in H^\infty(\mathbb{B}_n)$, there is a constan $a > 0$ such that $a \le |f(z)|$ for all $z \in \mathbb{B}_n$. This imples that f is a multiplier of $\mathcal{B}_\omega(\mathbb{B}_n)$. By Theorem 3.1, we have

$$\sup_{z\in\mathbb{B}_n} \omega(z)\|Rf(z)\|I_\omega(\|z\|) < \infty.$$

Moreover,

$$|R(1/f)(z)| = \left|\frac{-Rf(z)}{f^2(z)}\right| \le \frac{1}{a^2}.|Rf(z)|.$$

It follows that

$$\sup_{z\in\mathbb{B}_n} \omega(z)|R(1/f)(z)|I_\omega(\|z\|) < \infty.$$

□

4 Multipliers of Zygmund-type spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$

In this section, we study pointwise multipliers on the weighted Zygmund-type space of holomorphic functions defined on the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$.

We show that this condition is precisely characterized by the integral function

$$I_\omega^{(2)}(\|z\|) := \int_0^{\|z\|} \left(\int_0^t \frac{ds}{\omega(s)} \right) dt.$$

Recall that a function $f \in \mathcal{H}(\mathbb{B}_n)$ is said to belong to the Zygmund-type space $\mathcal{Z}_\omega(\mathbb{B}_n)$ if

$$\sup_{z\in\mathbb{B}_n} \omega(z) \left| R^{(2)} f(z) \right| < \infty,$$

where the second radial derivative is defined by $R^{(2)}f(z) := R(Rf)(z)$ and

$$Rf(z) := \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

We denote by $\mathcal{Z}_{\omega,0}(\mathbb{B}_n)$ the closed subspace of $\mathcal{Z}_\omega(\mathbb{B}_n)$ consisting of functions vanishing near the boundary in the weighted second radial derivative sense:

$$\mathcal{Z}_{\omega,0}(\mathbb{B}_n) := \left\{ f \in \mathcal{Z}_\omega(\mathbb{B}_n) : \lim_{\|z\|\to 1^-} \omega(z)|R^{(2)}f(z)| = 0 \right\}.$$

Let ψ be a holomorphic function on $\mathbb{B}_n$. We say that ψ is a pointwise multiplier of $\mathcal{Z}_\omega(\mathbb{B}_n)$ if

$$M_\varphi(f) := \psi f \in \mathcal{Z}_\omega(\mathbb{B}_n), \quad \text{for all } f \in \mathcal{Z}_\omega(\mathbb{B}_n).$$

The set of all such multipliers is denoted by $\mathrm{Mult}(\mathcal{Z}_\omega(\mathbb{B}_n))$. We are now interested in giving a necessary and sufficient condition for a function φ to define a bounded multiplier. For this purpose, we first recall the formula for the second radial derivative of a product:

$$R^{(2)}(\psi f)(z) = \psi(z)R^{(2)}f(z) + 2R\psi(z)Rf(z) + R^{(2)}\psi(z)f(z).$$

Lemma 4.1. *Let ω be a normal weight on $\mathbb{B}_n$. Then there exist constants $C > 0$ and $r \in (0,1)$ such that for all $z \in \mathbb{B}_n$ with $\|z\| \geq r$, we have*

$$\int_0^{\|z\|} \int_0^s \frac{dt\,ds}{\omega(t)} \leq C \sup\left\{ |h(z) - h(0)| : \|h\|_{\mathcal{Z}_\omega} \leq 1 \right\}.$$

Proof. Fix $z \in \mathbb{B}_n \setminus \{0\}$ and set $\zeta := z/\|z\|$. Consider the function

$$h(w) := \int_0^{\langle w,\zeta\rangle} \int_0^t \frac{d\xi\,dt}{\omega(\xi)}, \quad w \in \mathbb{B}_n.$$

A direct computation shows that

$$R^{(2)}h(w) = \frac{1}{\omega(\langle w,\zeta\rangle)}.$$

In particular, for $w = z$ we have $\langle z, \zeta\rangle = \|z\|$, hence

$$|R^{(2)}h(z)| = \frac{1}{\omega(\|z\|)}.$$

Therefore

$$\|h\|_{\mathcal{Z}_\omega} = \sup_{u\in\mathbb{B}_n} \omega(u)|R^{(2)}h(u)| = 1.$$

Moreover,

$$h(z) - h(0) = \int_0^{\|z\|}\int_0^t \frac{d\xi\, dt}{\omega(\xi)} = I_\omega^{(2)}(\|z\|).$$

This shows that

$$I_\omega^{(2)}(\|z\|) = |h(z) - h(0)| \le \sup\{|h(z) - h(0)| : \|h\|_{\mathcal{Z}_\omega} \le 1\}.$$

Finally, since ω is normal, the above estimate holds uniformly for $\|z\| \ge r$ with an absolute constant $C > 0$, giving the desired inequality. □

Proposition 4.1. *Let ω be a normal weight on $\mathbb{B}_n$. Then*

$$|f(z) - f(0)| \le \int_0^{\|z\|}\int_0^t \frac{ds\, dt}{\omega(s)} = I_\omega^{(2)}(\|z\|).$$

for all $f \in \mathcal{Z}_\omega^R(\mathbb{B}_n)$ such that $\|f\|_{\mathcal{Z}_\omega} \le 1$.

Proof. Let $f \in \mathcal{Z}_\omega(\mathbb{B}_n)$ with $\|f\|_{\mathcal{Z}_\omega} \le 1$. Fix $z \in \mathbb{B}_n \setminus \{0\}$ and define the radial path $\gamma(t) = t\zeta$, where $\zeta = z/\|z\| \in \partial\mathbb{B}_n$ and $t \in [0, \|z\|]$.

We can write the Taylor expansion of f along this radial line:

$$f(z) - f(0) = \int_0^{\|z\|} Rf(t\zeta)\, dt = \int_0^{\|z\|}\left(\int_0^t R^{(2)}f(s\zeta)\, ds\right) dt.$$

Taking absolute values and using Fubini's theorem:

$$|f(z) - f(0)| \le \int_0^{\|z\|}\left(\int_0^t |R^{(2)}f(s\zeta)|\, ds\right) dt = \int_0^{\|z\|}\int_0^t |R^{(2)}f(s\zeta)|\, ds\, dt.$$

Since $\|f\|_{\mathcal{Z}_\omega} \le 1$, we have:

$$|R^{(2)}f(s\zeta)| \le \frac{1}{\omega(s)}.$$

Hence,

$$|f(z) - f(0)| \le \int_0^{\|z\|}\int_0^t \frac{ds\, dt}{\omega(s)} = I_\omega^{(2)}(\|z\|).$$

□

Proposition 4.2. *Let ω be a normal weight on $\mathbb{B}_n$. Then there exists a constant $r \in (0;1)$ such that*

$$\sup\left\{|f(z) - f(0)| : \|f\|_{\mathcal{Z}_\omega(\mathbb{B}_n)} \le 1\right\} \asymp \int_0^{\|z\|}\int_0^t \frac{ds\, dt}{\omega(s)}, \tag{4.1}$$

where $z \in \mathbb{B}_n$ and $\|z\| \ge r$.

This motivates the following characterization:

Theorem 4.1. *Let ω be a normal weight on the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$, and let $\psi \in \mathcal{H}(\mathbb{B}_n)$. Then ψ is a multiplier of the weighted Zygmund-type space $\mathcal{Z}_\omega(\mathbb{B}_n)$ if and only if $\psi \in \mathcal{H}^\infty(\mathbb{B}_n)$ and the function*

$$H(z) := \omega(\|z\|)\left(|R^{(2)}\psi(z)| \cdot I_\omega^{(2)}(\|z\|) + |R\psi(z)| + |\psi(z)|\right)$$

is bounded on $\mathbb{B}_n$, where

$$I_\omega^{(2)}(\|z\|) := \int_0^{\|z\|}\int_0^t \frac{ds\,dt}{\omega(s)}.$$

Proof. Assume first that ψ is a multiplier of $\mathcal{Z}_\omega(\mathbb{B}_n)$. Then there exists a constant $C > 0$ such that

$$\|\psi f\|_{\mathcal{Z}_\omega} \leq C\|f\|_{\mathcal{Z}_\omega}, \quad \forall f \in \mathcal{Z}_\omega(\mathbb{B}_n).$$

By the definition of the norm, we have

$$\|f\|_{\mathcal{Z}_\omega} := |f(0)| + \sup_{z\in\mathbb{B}_n} \omega(\|z\|)\,|R^{(2)}f(z)|.$$

Step 1: Boundedness of ψ. Choose $f \equiv 1 \in \mathcal{Z}_\omega(\mathbb{B}_n)$. Then

$$R^{(2)}f(z) = 0, \quad f(0) = 1,$$

so

$$\|\psi\|_{\mathcal{Z}_\omega} = |\psi(0)| < \infty.$$

Thus, $\psi \in \mathcal{H}^\infty(\mathbb{B}_n)$ since we can replace f with constants to estimate $|\psi(z)|$ at any point by evaluation as a multiplier.

Step 2: Boundedness of $H(z)$. Let $f \in \mathcal{Z}_\omega(\mathbb{B}_n)$ be arbitrary. By the product rule for holomorphic functions,

$$R^{(2)}(\psi f) = (R^{(2)}\psi)f + 2(R\psi)(Rf) + \psi(R^{(2)}f).$$

Hence,

$$|R^{(2)}(\psi f)(z)| \leq |R^{(2)}\psi(z)|\,|f(z)| + 2|R\psi(z)|\,|Rf(z)| + |\psi(z)|\,|R^{(2)}f(z)|.$$

Now take supremum and multiply by $\omega(\|z\|)$:

$$\omega(\|z\|)\,|R^{(2)}(\psi f)(z)| \leq \omega(\|z\|)\left(|R^{(2)}\psi(z)|\,|f(z)| + 2|R\psi(z)|\,|Rf(z)| + |\psi(z)|\,|R^{(2)}f(z)|\right).$$

Therefore,

$$\omega(\|z\|)\,|R^{(2)}(\psi f)(z)| \lesssim \left[\omega(\|z\|)\left(|R^{(2)}\psi(z)|\,I_\omega^{(2)}(\|z\|) + |R\psi(z)| + |\psi(z)|\right)\right] \cdot \|f\|_{\mathcal{Z}_\omega}.$$

Moreover, since

$$(\psi f)(0) = \psi(0)f(0),$$

we have

$$|(\psi f)(0)| \leq \|\psi\|_\infty\,|f(0)| \leq \|\psi\|_\infty \cdot \|f\|_{\mathcal{Z}_\omega}.$$

Combining both parts, we get

$$\|\psi f\|_{\mathcal{Z}_\omega} \lesssim \sup_{z\in\mathbb{B}_n} H(z)\cdot \|f\|_{\mathcal{Z}_\omega},$$

where

$$H(z) := \omega(\|z\|)\left(|R^{(2)}\psi(z)|\, I_\omega^{(2)}(\|z\|) + |R\psi(z)| + |\psi(z)|\right).$$

So H must be bounded.

Conversely, suppose that $\psi \in \mathcal{H}^\infty(\mathbb{B}_n)$ and $H(z)$ is bounded on $\mathbb{B}_n$. Then the above estimates work in reverse and yield

$$\|\psi f\|_{\mathcal{Z}_\omega} \le C\|f\|_{\mathcal{Z}_\omega}, \quad \forall f \in \mathcal{Z}_\omega(\mathbb{B}_n).$$

Hence, ψ is a multiplier of $\mathcal{Z}_\omega(\mathbb{B}_n)$. □

Proposition 4.3. *Let ω be a normal weight on the unit ball $\mathbb{B}_n$ such that*

$$\int_0^1\int_0^t \frac{ds}{\omega(s)}dt = \infty,$$

and let $f \in \mathcal{Z}_{\omega,0}(\mathbb{B}_n)$. Then

$$\lim_{\|z\|\to 1^-} \frac{|f(z)|}{I_\omega^{(2)}(\|z\|)} = 0,$$

where

$$I_\omega^{(2)}(r) := \int_0^r \left(\int_0^t \frac{ds}{\omega(s)}\right) dt.$$

Proof. Without loss of generality, assume that $\|f\|_{\mathcal{Z}_\omega} = 1$ and $f(0) = Rf(0) = 0$. Then we have:

$$|f(z)| \le C I_\omega^{(2)}(\|z\|)$$

for some constant $C > 0$.

Since $f \in \mathcal{Z}_{\omega,0}(\mathbb{B}_n)$, we have

$$\lim_{\|z\|\to 1^-} \omega(\|z\|)|R^{(2)} f(z)| = 0.$$

Fix $\varepsilon > 0$. Then there exists $s \in \left(\frac{1}{2}, 1\right)$ such that

$$\omega(\|z\|)|R^{(2)} f(z)| < \varepsilon \quad \text{whenever } \|z\| > s^2.$$

Since $\int_0^1\int_0^t \frac{ds}{\omega(s)}dt = \infty$,, it follows that $I_\omega^{(2)}(r) \to \infty$ as $r \to 1^-$. So we can choose $r_0 \in (0, 1)$ such that for $\|z\| > r_0$,

$$I_\omega^{(2)}(\|z\|) > \frac{1}{\varepsilon}\int_0^s \left(\int_0^t \frac{ds}{\omega(s)}\right) dt.$$

Now fix $z \in \mathbb{B}_n$ with $\|z\| > \max\{s, r_0\}$. Consider the Taylor expansion with radial derivatives:

$$f(z) = f(sz) + Rf(sz)(1-s)\|z\| + \int_s^1 R^{(2)} f(tz)(1-t)\|z\|^2 dt.$$

Then:

$$|f(z) - f(sz)| \le |Rf(sz)|(1-s)\|z\| + \int_s^1 |R^{(2)}f(tz)|(1-t)\|z\|^2 dt.$$

Note that $|Rf(sz)| \le \|f\|_{\mathcal{Z}_\omega} = 1$, and for $\|tz\| > s^2$, we have $|R^{(2)}f(tz)| < \frac{\varepsilon}{\omega(t\|z\|)}$. So:

$$\begin{aligned}
|f(z) - f(sz)| &\le (1-s)\|z\| + \varepsilon\|z\|^2 \int_s^1 \frac{1-t}{\omega(t\|z\|)} dt \\
&\le (1-s)\|z\| + \varepsilon\|z\|^2 \int_0^{\|z\|} \left(\int_0^t \frac{ds}{\omega(s)} \right) dt \\
&= (1-s)\|z\| + \varepsilon\|z\|^2 I_\omega^{(2)}(\|z\|).
\end{aligned}$$

Also, since $f(sz)$ is bounded by $I_\omega^{(2)}(s\|z\|) \le I_\omega^{(2)}(\|z\|)$, we have:

$$\frac{|f(z)|}{I_\omega^{(2)}(\|z\|)} \le \frac{|f(z) - f(sz)|}{I_\omega^{(2)}(\|z\|)} + \frac{|f(sz)|}{I_\omega^{(2)}(\|z\|)} \le \frac{(1-s)\|z\|}{I_\omega^{(2)}(\|z\|)} + \varepsilon\|z\|^2 + 1.$$

When $\|z\|$ is close to 1, the first term tends to 0 (since denominator blows up), so:

$$\limsup_{\|z\| \to 1^-} \frac{|f(z)|}{I_\omega^{(2)}(\|z\|)} \le \varepsilon \cdot 1^2 + 1 \cdot 0 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. □

References

[1] J. Arazy, Multiplier of Bloch functions, University of Haifa Mathematics Publiactions, **54** (1982).

[2] O. Blasco, P. Galindo, A. Miralles The Bloch space on the unit ball of an infinite dimensional Hilbert space, J. Func. Anal, **267**(2014), 1188-1204.

[3] J. Chen, M. Wang, Integration operatpor between Bloch type spaces and growth spaces on the unit ball of a complex Banach space, Annales Polonici Mathematici, **127** (2021), 177-199.

[4] P. Galindo, M. Lindström, The Bloch space on the unit ball of a Hilbert space: Maximality and Multipliers, Acta Mathematica Scientia, 2021 **41B**(3), 899-906.

[5] H. Hamada, Bloch-type spaces and extended Cesàro operators in the unit ball of a complex Banach space, Sci. China Math., **62**(4) (2019), 617-628.

[6] H. Hardy, E. Littlewood, Some properties of fractional integrals, II, Math.Z. **34** (1932), 403-439.

[7] Z.J. Hu, S.S. Wang, Composition operators on Bloch-type spaces, Proc. Roy. Soc. Edinburgh Sect. A, **135** (2005), 1229-1239.

[8] Y. Liu, Y. Yu, Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball J. Math. Anal. Appl., **423** (2015), 76-93.

[9] Y. Liu, J. Zhou, On an operator $M_u\mathcal{R}$ from mixed norm spaces to Zygmund-type spaces on the unit ball, Complex Anal.Oper. Theory, **7** (2013), 593-606.

[10] S. Li, X. Zhang, Composition operators on the normal weight Zygmund spaces in high dimensions, J. Math. Anal. Appl. 487 (2020) 124000.

[11] W. Lusky Growth conditions for harmonic and holomorphic functions, In: Functional Analysis (Trier 1994), de Gruyter, Berlin, 1996, 281-291.

[12] K.M. Madigan, A. Matheson, Compact composition operators on Bloch spaces, Trans. Amer.Math. Soc., 347 (1995), 2679-2687.

[13] T. T. Quang, Banach-valued Bloch-type functions on the unit of Hilbert space and weak spaces of Bloch-type, Constructive Mathematical Analysis 6(2023), No. 1, pp. 6-21.

[14] S. Stević, On a new operator from the logarithmic Bloch space to the Bloch-type space on the unit ball, Appl. Math. Comput., 206 (2008), 313-320.

[15] S. Stević, On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces,Nonlinear Anal., 71 (2009) , 6323-6342.

[16] S. Stević, Weighted radial operator from the mixed-norm space to the nth weighted-type space on the unit ball,Appl. Math. Comput., 218 (2012), 9241-9247.

[17] S. Stević, Weighted iterated radial composition operators between some spaces of holomorphic functions on the unit ball,Abstr. Appl. Anal., 2010 (2010), Article ID 801264, 14 pp.

[18] S. Stević, On a product-type operator from Bloch spaces to weighted-type spaces on the unit ball,Appl. Math. Comput., 217 (12) (2011), 5930-5935.

[19] A. L. Shields, D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, Trans. Amer. Math. Soc., 162(1971), 287-302.

[20] X. Tang, Extended Cesàro operators between Bloch-type spaces in the unit ball of $\mathbb{C}^n$, J. Math. Anal. Appl., 326 (2007), 1199-1211.

[21] R. M. Timoney, Bloch functions in several complex variables, I, Bull. London. Math. Soc, 12 (1980), 241-267.

[22] J. Zhou, Y. Liu, Products of radial derivative and multiplication operators from $F(p,q,s)$ to weighted-type spaces on the unit ball,Taiwanese J. Math., 17 (2013), 161-178.

[23] J. Zhou, Y. Liu, Products of radial derivative and multiplication operator between mixed norm spaces and Zygmund-type spaces on the unit ball, Math. Inequal. Appl., 17 (2014), 349-366.

[24] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball,Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005.

[25] X. Zhu, Volterra composition operator on logarithmic Bloch space, Banach J. Math. Anal., 3 (2009), 122-130.

[26] Z. Xu, Bloch type spaces on the unit ball of a Hilbert space, Czechoslovak Math.J., 69 (2019), 695-711.

[27] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

Lien Vuong Lam Department of Mathematics, Pham Van Dong University, 986 Quang Trung, Quang Ngai, Vietnam.

E-mail: lvlam@pdu.edu.vn