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The Legendrian self-expander in the standard contact Euclidean five-space

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Abstract. Based on the geometric correspondence between Lagrangian and Legendrian submanifolds, we construct Legendrian 2-submanifolds in the standard contact Euclidean Five-space $\mathbb{R}^5$ satisfying the self-similarity equation $H+\theta\xi = \alpha F^{\perp}(\alpha > 0)$, with particular focus on their self-expander solutions under Legendrian mean curvature flow. This paper mainly generalizes Theorem C of the work by Joyce-Lee-Tsui [10].

Keywords. Sasaki-Einstein metric, Legendrian mean curvature flow, blow-up, Legendrian self-expander

1 Introduction

The mean curvature flow of Legendrian submanifolds was introduced by K. Smoczyk in 2003 [13]. He proved that such flow will preserve Legendrian condition given the ambient space is Sasakian pseudo-Einstein manifold. He also studied the 1 dimensional case and proved that closed Legendrian curves in Sasakian spaces converge to closed Legendrian geodesics under this flow. In 2024, Chang-Han-Wu [6] made further progress by proving the existence of long-time solutions and asymptotic convergence properties along the Legendrian mean curvature flow in $(2n+1)-$dimensional η-Einstein Sasakian manifold under small energy condition. Recently, Chang-Wu-Zhang classified Type-I Singularities of the Legendrian mean curvature flow by applying blow-up analysis [7].

It is well known that Type-I Singularities of the mean curvature flow are locally modeled by self-similar solutions. For the Lagrangian case, Castro-Lerma [4] and Joyce-Lee-Tsui [10] have constructed numerous examples of self-similar solutions and translating solitons for the Lagrangian mean curvature flow.

In this paper, we particularly focus on self-expanders of the Legendrian mean curvature flow and generalize Theorem C of Joyce-Lee-Tsui's paper [10] to Legendrian case in the standard contact Euclidean Five-space $\mathbb{R}^5$. From Theorems A and B in [10], we have already proven the following conclusions in $\mathbb{R}^5$ [8].

Let $\lambda_j, w_j, r_j, C, I, \phi, \phi_j$ as in Theorems A and B of [10] for $j = 1, 2$, and $\tilde{\theta} : I \to \mathbb{R}$ or $\tilde{\theta} : I \to \mathbb{R}/2\pi\mathbb{Z}$.

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Proposition 1.1. *Fix $s_0 \in I$. Define $u : I \to \mathbb{R}$ by*

$$u(s) = 2\int_{s_0}^{s} r_1(t)r_2(t)\cos(\phi(t) - \tilde{\theta}(t))dt. \tag{1.1}$$

For the above u, we have $r_j^2(s) = \alpha_j + \lambda_j u(s), j = 1,2, s \in I, \alpha_j = r_j^2(s_0)$. Define a degree 2 polynomial $Q(u)$ by $Q(u) = \prod_{j=1}^{2}(\alpha_j + \lambda_j u)$. Suppose that

$$\begin{cases} \dfrac{\mathrm{d}w_1}{\mathrm{d}s} = \lambda_1 e^{i\tilde{\theta}(s)}\bar{w}_2, \quad \dfrac{\mathrm{d}w_2}{\mathrm{d}s} = \lambda_2 e^{i\tilde{\theta}(s)}\bar{w}_1 \\ \dfrac{d\phi_j}{ds} = -\dfrac{\lambda_j Q(u)^{1/2}\sin(\phi - \tilde{\theta})}{\alpha_j + \lambda_j u}, j = 1,2, \\ \dfrac{\mathrm{d}z}{\mathrm{d}s} = \dfrac{C}{2}Q(u)^{1/2}\sin(\phi - \tilde{\theta}) \end{cases} \tag{1.2}$$

hold in I. Then the submanifold $\tilde{L}$ in $\mathbb{C}^2\times\mathbb{R}$ given by

$$\tilde{L} = \left\{ (x_1 w_1(s), x_2 w_2(s), z(s)) : s \in I, x_j \in \mathbb{R}, \sum_{j=1}^{2}\lambda_j x_j^2 = C \right\}, \tag{1.3}$$

is Legendrian.

Proposition 1.2. *In the situation of Proposition 1.1, let $\tilde{\alpha} \in \mathbb{R}$ be constant, and $\tilde{\theta}$ is a linear function of s. Suppose that*

$$\frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}s} = -\frac{\tilde{\alpha}Cr_1r_2\sin(\tilde{\theta} - \phi)}{4} \tag{1.4}$$

hold in I, then $\tilde{L}$ with Legendrian angle $\tilde{\theta}(s)$ at $(x_1w_1(s), x_2w_2(s), z(s))$. Its position vector $\tilde{F}$ and mean curvature vector $\tilde{H}$ satisty the relation $\tilde{H} + \tilde{\theta}\xi = \tilde{\alpha}F^{\perp}$. This implies that L is a self-expander when $\tilde{\alpha} > 0$ and a self-shrinker when $\tilde{\alpha} < 0$.

In this paper, our primary work generalizes Theorem C in [10] by extending the Lagrangian framework in flat space $\mathbb{R}^4$ to Legendrian submanifolds in contact space $(\mathbb{R}^5, \phi, \xi, \eta, g)$.

For surfaces taking the form of (1.3), put $\lambda_1 = \lambda_2 = C = 1$ and let $w_j(y) = e^{i\phi_j(y)}r(y)$ for $j = 1,2$, where

$$r_1(y) = r_2(y) = r(y) = \sqrt{\frac{1}{a} + y^2}, \quad \phi_j = \psi_j(y) + \int_0^y \frac{dt}{(\frac{1}{a} + t^2)\sqrt{P(t)}}, \tag{1.5}$$

with $P(t) = \dfrac{1}{t^2}((1 + at^2)^2 e^{\alpha t^2} - 1), a > 0, \alpha > 0$, then we have the main theorem:

Theorem 1.1. *Suppose that*

$$\begin{cases} \dfrac{\mathrm{d}\tilde{\theta}}{\mathrm{d}y} = -\dfrac{\alpha}{\sqrt{P(y)}} \\ \dfrac{\mathrm{d}z}{\mathrm{d}y} = -\dfrac{1}{2\sqrt{p(y)}} \end{cases} \tag{1.6}$$

holds in I*. Then the submanifolds* $\tilde{L}$ *in* $\mathbb{C}^2 \times \mathbb{R}$ *given by*

$$\tilde{L} = \left\{ (x_1 w_1(y), x_2 w_2(y), z(y)) : x_j \in \mathbb{R}, \sum_{j=1}^{2} x_j^2 = 1 \right\} \tag{1.7}$$

is an embedded Legendrian diffeomorphic to $\mathbb{S} \times \mathbb{R}$*, with a Legendrian angle* $\tilde{\theta}(y)$ *at*

$$(x_1 w_1(y), x_2 w_2(y), z(y)).$$

Its position vector $\tilde{F}^{\perp}$ *and mean curvature vector* $\tilde{H}$ *satisfy* $\tilde{H} + \tilde{\theta}\xi = 4\alpha \tilde{F}^{\perp}$ *with* $\alpha > 0$*. This implies that* $\tilde{L}$ *is a self-expander.*

2 Preliminaries

In this section, we have provided the necessary background knowledge, laying the theoretical foundation for the subsequent discussion. For the problem of the contact Euclidean five space, we have constructed an adapted frame field $\{E_i\}_{i=1}^5$. This constitutes the essential foundation for all computations in our work, without which subsequent procedures cannot proceed.

We consider the Sasakian space form $(\mathbb{R}^5, \phi, \xi, \eta, g)$ with coordinates (x_1, y_1, x_2, y_2, z), with the standard contact form $\eta = \dfrac{1}{2}dz - \dfrac{1}{4}\sum_{i=1}^2 (y_i dx_i - x_i dy_i)$, the associated mertic $g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^2((dx_i)^2 + (dy_i)^2)$, the Reeb vector fold $\xi = 2\frac{\partial}{\partial z}$ and the $(1,1)-$tensor ϕ which can be respresented as

$$\phi \sim \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \frac{x_1}{2} & \frac{x_2}{2} & \frac{y_1}{2} & \frac{y_2}{2} & 0 \end{pmatrix}. \tag{2.1}$$

Definition 1. For any vector X orthogonal to the Reeb vector field ξ, the sectional curvature $K(X, \phi X)$ denoted by $\mathcal{H}(x)$ is called $\phi-$sectional curvature.

Remark 1. $\phi-$sectional curvature plays the role in Sasakian geometry that holomorphic sectional curvtaure plays in Kähler geometry, and it determines the curvature tensor completely.

Remark 2. In other literatures, η form is chosn to be $\tilde{\eta} = \dfrac{1}{2}dz - \dfrac{1}{2}\sum_{i=1}^2 y_i dx_i$, with compatible $\tilde{\phi}$ tensor and associated metirc $\tilde{g} = \tilde{\eta} \otimes \tilde{\eta} + \frac{1}{4}\sum_{i=1}^2((dx_i)^2 + (dy_i)^2)$, then $(\mathbb{R}^5, \tilde{\phi}, \xi, \tilde{\eta}, \tilde{g})$ is called the standard Sasakian space form with constant $\phi-$sectional curvature -3.

The choice of ours is to use the symmetry of η for the convenience of computation. The next proposition states that it is contactmorphism and isometric to the standard Sasakian space form $(\mathbb{R}^5, \tilde{\phi}, \xi, \tilde{\eta}, \tilde{g})$.

Proposition 2.1. $(\mathbb{R}^5, \phi, \xi, \eta, g)$ *is a Sasaki manifold with* $\phi-$*sectional curvature* -3*.*

Proof. For details, we refer to [8]. ∎

In our paper, on a Legendrian submanifold $\tilde{L}$, the mean curvature vector $\tilde{H}$ and the Legendre pseudo angle $\tilde{\theta}$ satisfy the relationship

$$\tilde{H} = -\phi \nabla \tilde{\theta}, \tag{2.2}$$

where ∇ is the gradient operator on $\tilde{L}$.

Remark 3. Formula (2.2) is equivalent to $d\eta \lrcorner H = d\tilde{\theta}$.

Definition 2. Let $F : L^2 \to M^5$ be a submanifold of a Sasakian manifold $(M^5, \phi, \xi, \eta, g)$. It is called isotropic if it is normal to the contact structure ξ, that is $F^*\eta = 0$. Consequently, $F^*d\eta = 0$ also holds. A Legendrian submanifold L is an isotropic submanifold of maximal dimension 2.

Definition 3. Let $F : L^2 \to M^5$ be an $n-$dimensional Legendrian submanifold in Sasakian manififold. We call the Legendrian immersed manifold L^2 a self-similar solution if it satisfies the quasilinear elliptic system

$$H + \theta\xi = \alpha F^{\perp}, \tag{2.3}$$

for some constant α in $\mathbb{R}$, where $F^{\perp}$ is the projection of the position vector F in M^5 to the normal bundle of L^2, and H is the Legendrian mean curvature vector. It is called a self-shrinker if $\alpha < 0$ and self-expander if $\alpha > 0$.

3 Proofs of Theorem

This section is the core part of the article, providing detailed proof of Theorem 1.1. Moreover, the importance of the selected frame $\{E_i\}_{i=1}^5$ can be seen in the proof process.

Proof of Theorem 1.1. Firstly, we prove that $\tilde{L}$ is a Legendrian submanifold in $\mathbb{R}^5$. To facilitate matters, we can express equation (1.7) in the following form with $w_j = r(y)e^{i\phi_j(y)}$:

$$\tilde{L} = (x_1 r(y)\cos\phi_1(y), x_1 r(y)\sin\phi_1(y), x_2 r(y)\cos\phi_2(y), x_2 r(y)\sin\phi_2(y), z(t,y)). \tag{3.1}$$

Consider the standard contact form in $\mathbb{R}^5$ is $\eta = \frac{1}{2}d\bar{z} - \frac{1}{4}\sum_1^2(\bar{y}_i d\bar{x}_i - \bar{x}_i d\bar{y}_i)$ with coordinates $(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{z})$. From the second equation of (1.5) we have

$$\begin{aligned}
2dz &= r\cos t\sin\phi_1(-r\sin t\cos\phi_1 dt + \cos t\cos\phi_1 dr - r\cos t\sin\phi_1 d\phi_1)\\
&\quad - r\cos t\cos\phi_1(-r\sin t\sin\phi_1 dt + \cos t\sin\phi_1 dr + r\cos t\cos\phi_1 d\phi_1)\\
&\quad + r\sin t\sin\phi_2(r\cos t\cos\phi_2 dt + \sin t\cos\phi_2 dr - r\sin t\sin\phi_2 d\phi_2)\\
&\quad - r\sin t\cos\phi_2(r\cos t\sin\phi_2 dt + \sin t\sin\phi_2 dr + r\sin t\cos\phi_2 d\phi_2)\\
&= -r^2\cos^2 t d\phi_1 - r^2\sin^2 t d\phi_2\\
&= -r^2\cos^2 t\frac{1}{r^2\sqrt{P(y)}}dy - r^2\sin^2 t\frac{1}{r^2\sqrt{P(y)}}dy\\
&= -\frac{1}{\sqrt{P(y)}}dy,
\end{aligned}$$

here $d\phi_j = \frac{1}{r^2P(y)}$. On the other hand, $2dz = 2z_y \mathrm{d}y + 2z_t \mathrm{d}t = -\frac{1}{\sqrt{P(y)}}dy$ follows from the second equation of (1.6). Thus $\tilde{L}^*\eta = 0$, and it can be deduced that $z_t = 0$. Hence $\tilde{L}$ is a Legendrian submanifold in $\mathbb{C}^2\times\mathbb{R}$.

We shall prove that $\tilde{L}$ is a self-expander and satisfies $\tilde{H}+\tilde{\theta}\xi = 4\alpha\tilde{F}^{\perp}$ as follows. Choosing the orthonormal frame $\{E_i\}_{i=1}^4, \xi$ and parameterizing the equation $x_1^2+x_2^2=1$. Thus (1.7) can be rewritten as

$$\begin{aligned}\tilde{L} &= (r\cos t\cos\phi_1, r\cos t\sin\phi_1, r\sin t\cos\phi_2, r\sin t\sin\phi_2)\\ &= r\cos t\cos\phi_1(\frac{1}{2}E_1 - \frac{r\cos t\sin\phi_1}{4}\xi) + r\cos t\sin\phi_1(-\frac{1}{2}E_3 + \frac{r\cos t\cos\phi_1}{4}\xi)\\ &+ r\sin t\cos\phi_2(\frac{1}{2}E_2 - \frac{r\sin t\sin\phi_2}{4}\xi) + r\sin t\sin\phi_2(-\frac{1}{2}E_4 + \frac{r\sin t\cos\phi_2}{4}\xi) + \frac{z}{2}\xi\\ &= \frac{r}{2}\cos t\cos\phi_1 E_1 + \frac{r}{2}\sin t\cos\phi_2 E_2 - \frac{r}{2}\cos t\sin\phi_1 E_3 - \frac{r}{2}\sin t\sin\phi_2 E_4 + \frac{z}{2}\xi.\end{aligned} \tag{3.2}$$

From (1.5) we have $\frac{\mathrm{d}r}{\mathrm{d}y} = \frac{y}{r}, \frac{\mathrm{d}\phi_j}{\mathrm{d}y} = \frac{1}{r^2\sqrt{P(y)}}$. So the tangent vectors to be

$$\frac{\partial\tilde{L}}{\partial t} = -\frac{r}{2}\sin t\cos\phi_1 E_1 + \frac{r}{2}\cos t\cos\phi_2 E_2 + \frac{r}{2}\sin t\sin\phi_1 E_3 - \frac{r}{2}\cos t\sin\phi_2 E_4, \tag{3.3}$$

and

$$\begin{aligned}\frac{\partial\tilde{L}}{\partial y} =&(\frac{y}{r}\cos t\cos\phi_1 - \frac{\cos t\sin\phi_1}{r\sqrt{P(y)}}, \frac{y}{r}\cos t\sin\phi_1 + \frac{\cos t\cos\phi_1}{r\sqrt{P(y)}},\\ &\frac{y}{r}\sin t\cos\phi_2 - \frac{\sin t\sin\phi_2}{r\sqrt{P(y)}}, \frac{y}{r}\sin t\sin\phi_2 + \frac{\sin t\cos\phi_2}{r\sqrt{P(y)}}, -\frac{1}{2\sqrt{P(y)}})\\ =&\frac{\cos t}{2r}(y\cos\phi_1 - \frac{\sin\phi_1}{\sqrt{P(y)}})E_1 + \frac{\sin t}{2r}(y\cos\phi_2 - \frac{\sin\phi_2}{\sqrt{P(y)}})E_2\\ &-\frac{\cos t}{2r}(y\sin\phi_1 + \frac{\cos\phi_1}{\sqrt{P(y)}})E_3 - \frac{\sin t}{2r}(y\sin\phi_2 + \frac{\cos\phi_2}{\sqrt{P(y)}})E_4.\end{aligned} \tag{3.4}$$

We write $\psi^2(y) = \frac{1}{4}r^2$, and $h^2(y) = \frac{1}{P(y)} + y^2$. Then the associated metric is

$$||\frac{\partial\tilde{L}}{\partial t}||^2 = \psi^2, ||\frac{\partial\tilde{L}}{\partial y}||^2 = \frac{\psi^2 h^2}{r^4}.$$

Hence, we choose the orthonormal basis to be

$$e_1 = -\frac{r}{2\psi}\sin t\cos\phi_1 E_1 + \frac{r}{2\psi}\cos t\cos\phi_2 E_2 + \frac{r}{2\psi}\sin t\sin\phi_1 E_3 - \frac{r}{2\psi}\cos t\sin\phi_2 E_4, \tag{3.5}$$

and

$$\begin{aligned}e_2 =&\frac{r\cos t}{2\psi h}(y\cos\phi_1 - \frac{\sin\phi_1}{\sqrt{P(y)}})E_1 + \frac{r\sin t}{2\psi h}(y\cos\phi_2 - \frac{\sin\phi_2}{\sqrt{P(y)}})E_2\\ &-\frac{r\cos t}{2\psi h}(y\sin\phi_1 + \frac{\cos\phi_1}{\sqrt{P(y)}})E_3 - \frac{r\sin t}{2\psi h}(y\sin\phi_2 + \frac{\cos\phi_2}{\sqrt{P(y)}})E_4.\end{aligned} \tag{3.6}$$

Thus

$$\Phi e_1 = -\frac{r}{2\psi}\sin t\sin\phi_1 E_1 + \frac{r}{2\psi}\cos t\sin\phi_2 E_2 - \frac{r}{2\psi}\sin t\cos\phi_1 E_3 + \frac{r}{2\psi}\cos t\cos\phi_2 E_4, \tag{3.7}$$

and

$$\begin{aligned}\Phi e_2 =&\frac{r\cos t}{2\psi h}(y\sin\phi_1+\frac{\cos\phi_1}{\sqrt{P(y)}})E_1+\frac{r\sin t}{2\psi h}(y\sin\phi_2+\frac{\cos\phi_2}{\sqrt{P(y)}})E_2\\&+\frac{r\cos t}{2\psi h}(y\cos\phi_1-\frac{\sin\phi_1}{\sqrt{P(y)}})E_3+\frac{r\sin t}{2\psi h}(y\cos\phi_2-\frac{\sin\phi_2}{\sqrt{P(y)}})E_4.\end{aligned}\tag{3.8}$$

Thus we have

$$\begin{aligned}\overline{\nabla}_{e_1}e_1 =& -\frac{r\cos t\cos\phi_1}{2\psi^2}E_1+\frac{r^2\sin^2 t\sin\phi_1\cos\phi_1}{4\psi^2}\xi-\frac{r\sin t\cos\phi_2}{2\psi^2}E_2+\frac{r^2\cos^2 t\sin\phi_2\cos\phi_2}{4\psi^2}\xi\\&+\frac{r\cos t\sin\phi_1}{2\psi^2}E_3-\frac{r^2\sin^2 t\sin\phi_1\cos\phi_1}{4\psi^2}\xi+\frac{r\sin t\sin\phi_2}{2\psi^2}E_4-\frac{r^2\cos^2 t\sin\phi_2\cos\phi_2}{4\psi^2}\xi\\=&-\frac{r\cos t\cos\phi_1}{2\psi^2}E_1-\frac{r\sin t\cos\phi_2}{2\psi^2}E_2+\frac{r\cos t\sin\phi_1}{2\psi^2}E_3+\frac{r\sin t\sin\phi_2}{2\psi^2}E_4,\end{aligned}$$

Denote

$$\nu_1(y,t)=\frac{r\cos t}{2\psi h}(y\cos\phi_1-\frac{\sin\phi_1}{\sqrt{P(y)}}),\quad \nu_2(y,t)=\frac{r\sin t}{2\psi h}(y\cos\phi_2-\frac{\sin\phi_2}{\sqrt{P(y)}}),$$

and

$$\nu_3(y,t)=\frac{r\cos t}{2\psi h}(y\sin\phi_1+\frac{\cos\phi_1}{\sqrt{P(y)}}),\quad \nu_4(y,t)=\frac{r\sin t}{2\psi h}(y\sin\phi_2+\frac{\cos\phi_2}{\sqrt{P(y)}}),$$

then

$$\overline{\nabla}_{e_2}e_2=\frac{r^2}{\psi h}\frac{\partial\nu_1}{\partial y}E_1+\frac{r^2}{\psi h}\frac{\partial\nu_2}{\partial y}E_2-\frac{r^2}{\psi h}\frac{\partial\nu_3}{\partial y}E_3-\frac{r^2}{\psi h}\frac{\partial\nu_4}{\partial y}E_4,$$

where

$$\begin{aligned}\frac{\partial\nu_1}{\partial y} =& \frac{\cos t}{2}\frac{\partial}{\partial y}(\frac{ry\cos\phi_1}{\psi h}-\frac{r\sin\phi_1}{\psi h\sqrt{P(y)}})\\=&\frac{y^2\cos t\cos\phi_1}{2\psi hr}+\frac{r\cos t\cos\phi_1}{2\psi h}-\frac{y\cos t\sin\phi_1}{2\psi hr\sqrt{P(y)}}-\frac{ry\cos t\cos\phi_1\psi_y}{2\psi^2 h}-\frac{ry\cos t\cos\phi_1 h'}{2\psi h^2}\\&-\frac{y\cos t\sin\phi_1}{2\psi hr\sqrt{P(y)}}-\frac{\cos t\cos\phi_1}{2\psi hrP(y)}+\frac{r\cos t\sin\phi_1\psi_y}{2\psi^2 h\sqrt{P(y)}}+\frac{r\cos t\sin\phi_1 h'}{2\psi h^2\sqrt{P(y)}}+\frac{r\cos t\sin\phi_1 P'(y)}{4\psi hP(y)\sqrt{P(y)}},\\\frac{\partial\nu_2}{\partial y} =& \frac{sint}{2}\frac{\partial}{\partial y}(\frac{ry\cos\phi_2}{\psi h}-\frac{r\sin\phi_2}{\psi h\sqrt{P(y)}})\\=&\frac{y^2\sin t\cos\phi_2}{2\psi hr}+\frac{r\sin t\cos\phi_2}{2\psi h}-\frac{y\sin t\sin\phi_2}{2\psi hr\sqrt{P(y)}}-\frac{ry\sin t\cos\phi_2\psi_y}{2\psi^2 h}-\frac{ry\sin t\cos\phi_2 h'}{2\psi h^2}\\&-\frac{y\sin t\sin\phi_2}{2\psi hr\sqrt{P(y)}}-\frac{\sin t\cos\phi_2}{2\psi hrP(y)}+\frac{r\sin t\sin\phi_2\psi_y}{2\psi^2 h\sqrt{P(y)}}+\frac{r\sin t\sin\phi_2 h'}{2\psi h^2\sqrt{P(y)}}+\frac{r\sin t\sin\phi_2 P'(y)}{4\psi hP(y)\sqrt{P(y)}},\\\frac{\partial\nu_3}{\partial y} =& \frac{\cos t}{2}\frac{\partial}{\partial y}(\frac{ry\sin\phi_1}{\psi h}+\frac{rcos\phi_1}{\psi h\sqrt{P(y)}})\\=&\frac{y^2\cos t\sin\phi_1}{2\psi hr}+\frac{r\cos t\sin\phi_1}{2\psi h}+\frac{y\cos t\cos\phi_1}{2\psi hr\sqrt{P(y)}}-\frac{ry\cos t\sin\phi_1\psi_y}{2\psi^2 h}-\frac{ry\cos t\sin\phi_1 h'}{2\psi h^2}\\&+\frac{y\cos t\cos\phi_1}{2\psi hr\sqrt{P(y)}}-\frac{\cos t\sin\phi_1}{2\psi hrP(y)}-\frac{r\cos t\cos\phi_1\psi_y}{2\psi^2 h\sqrt{P(y)}}-\frac{r\cos t\cos\phi_1 h'}{2\psi h^2\sqrt{P(y)}}-\frac{r\cos t\cos\phi_1 P'(y)}{4\psi hP(y)\sqrt{P(y)}},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \nu_4}{\partial y} &= \frac{\sin t}{2}\frac{\partial}{\partial y}(\frac{ry\sin\phi_2}{\psi h} + \frac{r\cos\phi_2}{\psi h\sqrt{P(y)}}) \\
&= \frac{y^2\sin t\sin\phi_2}{2\psi hr} + \frac{r\sin t\sin\phi_2}{2\psi h} + \frac{y\sin t\cos\phi_2}{2\psi hr\sqrt{P(y)}} - \frac{ry\sin t\sin\phi_2\psi_y}{2\psi^2 h} - \frac{ry\sin t\sin\phi_2 h'}{2\psi h^2} \\
&+ \frac{y\sin t\cos\phi_2}{2\psi hr\sqrt{P(y)}} - \frac{\sin t\sin\phi_2}{2\psi hrP(y)} - \frac{r\sin t\cos\phi_2\psi_y}{2\psi^2 h\sqrt{P(y)}} - \frac{r\sin t\cos\phi_2 h'}{2\psi h^2\sqrt{P(y)}} - \frac{r\sin t\cos\phi_2 P'(y)}{4\psi hP(y)\sqrt{P(y)}},
\end{aligned}
$$

and $\psi_y = \frac{y}{4\psi}, \psi_t = \frac{\sin t\cos t}{4\psi}(r^2 - r^2) = 0, h' = \frac{1}{2h}(2y - \frac{P'(y)}{P^2(y)})$.

From above equations we find that

$$
\begin{aligned}
\langle\overline{\nabla}_{e_1}e_1, \Phi e_1\rangle =& \frac{r^2\sin t\cos t\sin\phi_1\cos\phi_1}{4\psi^3} - \frac{r^2\sin t\cos t\sin\phi_2\cos\phi_2}{4\psi^3} \\
&- \frac{r^2\sin t\cos t\sin\phi_1\cos\phi_1}{4\psi^3} + \frac{r^2\sin t\cos t\sin\phi_2\cos\phi_2}{4\psi^3} \\
=&0,
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\langle\overline{\nabla}_{e_2}e_2, \Phi e_1\rangle &= \frac{r^2y\sin t\cos t}{4\psi^3h^2\sqrt{P(y)}} + \frac{r^2y\sin t\cos t}{4\psi^3h^2\sqrt{P(y)}} - \frac{r^4y\sin t\cos t}{16\psi^5h^2\sqrt{P(y)}} \\
&- \frac{r^4\sin t\cos t}{8\psi^3h^4\sqrt{P(y)}}(2y - \frac{P'(y)}{P^2(y)}) - \frac{r^4\sin t\cos tP'(y)}{8\psi^3h^2P(y)\sqrt{P(y)}} \\
&- \frac{r^2y\sin t\cos t}{4\psi^3h^2\sqrt{P(y)}} - \frac{r^2y\sin t\cos t}{4\psi^3h^2\sqrt{P(y)}} + \frac{r^4y\sin t\cos t}{16\psi^5h^2\sqrt{P(y)}} \\
&+ \frac{r^4\sin t\cos t}{8\psi^3h^4\sqrt{P(y)}}(2y - \frac{P'(y)}{P^2(y)}) + \frac{r^4\sin t\cos tP'(y)}{8\psi^3h^2P(y)\sqrt{P(y)}} \\
&= 0.
\end{aligned}
\tag{3.10}
$$

Hence,

$$
\begin{aligned}
\langle\overline{\nabla}_{e_1}e_1, \Phi e_2\rangle &= -\frac{r^2\cos^2 t\cos\phi_1}{4\psi^3h}(y\sin\phi_1 + \frac{\cos\phi_1}{\sqrt{P(y)}}) - \frac{r^2\sin^2 t\cos\phi_2}{4\psi^3h}(y\sin\phi_2 + \frac{\cos\phi_2}{\sqrt{P(y)}}) \\
&+ \frac{r^2\cos^2 t\sin\phi_1}{4\psi^3h}(y\cos\phi_1 - \frac{\sin\phi_1}{\sqrt{P(y)}}) + \frac{r^2\sin^2 t\sin\phi_2}{4\psi^3h}(y\cos\phi_2 - \frac{\sin\phi_2}{\sqrt{P(y)}}) \\
&= -\frac{r^2\cos^2 t}{4\psi^3h\sqrt{P(y)}} - \frac{r^2\sin^2 t}{4\psi^3h\sqrt{P(y)}} \\
&= -\frac{r^2}{4\psi^3h\sqrt{P(y)}},
\end{aligned}
\tag{3.11}
$$

$$\begin{aligned}
\langle\overline{\nabla}_{e_2}e_2,\Phi e_2\rangle &= -\frac{r^2y^2\cos^2 t}{4\psi^3h^3\sqrt{P(y)}} - \frac{r^2y^2\cos^2 t}{4\psi^3h^3\sqrt{P(y)}} + \frac{r^4y\cos^2 t\psi_y}{4\psi^4h^3\sqrt{P(y)}} + \frac{r^4y\cos^2 th'}{4\psi^3h^4\sqrt{P(y)}}\\
&+ \frac{r^4y\cos^2 tP'(y)}{8\psi^3h^3P(y)\sqrt{P(y)}} + \frac{r^2y^2\cos^2 t}{4\psi^3h^3\sqrt{P(y)}} + \frac{r^4\cos^2 t}{4\psi^3h^3\sqrt{P(y)}} - \frac{r^4y\cos^2 t\psi_y}{4\psi^4h^3\sqrt{P(y)}}\\
&- \frac{r^4y\cos^2 th'}{4\psi^3h^4\sqrt{P(y)}} - \frac{r^2\cos^2 t}{4\psi^3h^3P(y)\sqrt{Py)}} - \frac{r^2y^2\sin^2 t}{4\psi^3h^3\sqrt{P(y)}} - \frac{r^2y^2\sin^2 t}{4\psi^3h^3\sqrt{P(y)}}\\
&+ \frac{r^4y\sin^2 t\psi_y}{4\psi^4h^3\sqrt{P(y)}} + \frac{r^4y\sin^2 th'}{4\psi^3h^4\sqrt{P(y)}} + \frac{r^4y\sin^2 tP'(y)}{8\psi^3h^3P(y)\sqrt{P(y)}} + \frac{r^2y^2\sin^2 t}{4\psi^3h^3\sqrt{P(y)}}\\
&+ \frac{r^4\sin^2 t}{4\psi^3h^3\sqrt{P(y)}} - \frac{r^4y\sin^2 t\psi_y}{4\psi^4h^3\sqrt{P(y)}} - \frac{r^4y\sin^2 th'}{4\psi^3h^4\sqrt{P(y)}} - \frac{r^2\sin^2 t}{4\psi^3h^3P(y)\sqrt{P(y)}}\\
&= \frac{r^2yP'(y)}{2\psi^3h^3P(y)\sqrt{P(y)}}(\frac{r^2\sin^2 t}{4} + \frac{r^2\cos^2 t}{4}) + \frac{r^2}{\psi^3h^3\sqrt{P(y)}}(\frac{r^2\sin^2 t}{4} + \frac{r^2\cos^2 t}{4})\\
&- \frac{r^2}{4\psi^3h^3\sqrt{P(y)}}(y^2 + \frac{1}{P(y)})(\sin^2 t + \cos^2 t)\\
&= \frac{r^2yP'(y)}{2\psi h^3P(y)\sqrt{P(y)}} + \frac{r^2}{\psi h^3\sqrt{P(y)}} - \frac{r^2}{4\psi^3h\sqrt{P(y)}}.
\end{aligned} \tag{3.12}$$

From equations (3.9)-(3.12) the mean curvature vector $\tilde{H}$ is determined to be

$$\begin{aligned}
\tilde{H} &= (\overline{\nabla}_{e_i}e_i)^{\perp}\\
&= \langle\overline{\nabla}_{e_1}e_1,\Phi e_1\rangle\Phi e_1 + \langle\overline{\nabla}_{e_1}e_1,\Phi e_2\rangle\Phi e_2 + \langle\overline{\nabla}_{e_2}e_2,\Phi e_1\rangle\Phi e_1 + \langle\overline{\nabla}_{e_2}e_2,\Phi e_2\rangle\Phi e_2\\
&= [\frac{r^2yP'(y)}{2\psi h^3P(y)\sqrt{P(y)}} + \frac{r^2}{\psi h^3\sqrt{P(y)}} - \frac{r^2}{2\psi^3h\sqrt{P(y)}}]\Phi e_2.
\end{aligned} \tag{3.13}$$

Using (3.2), (3.7) and (3.8), it follows that

$$\langle\tilde{F},\Phi e_1\rangle = 0, \tag{3.14}$$

and

$$\begin{aligned}
\langle\tilde{F},\Phi e_2\rangle &= \frac{r^2\cos^2 t\cos\phi_1}{4\psi h}(y\sin\phi_1 + \frac{\cos\phi_1}{\sqrt{P(y)}}) + \frac{r^2\sin^2 t\cos\phi_2}{4\psi h}(y\sin\phi_2 + \frac{\cos\phi_2}{\sqrt{P(y)}})\\
&- \frac{r^2\cos^2 t\sin\phi_1}{4\psi h}(y\cos\phi_1 - \frac{\sin\phi_1}{\sqrt{P(y)}}) - \frac{r^2\sin^2 t\sin\phi_2}{4\psi h}(y\cos\phi_2 - \frac{\sin\phi_2}{\sqrt{P(y)}})\\
&= \frac{r^2\cos^2 t\cos^2\phi_1}{4\psi h\sqrt{P(y)}} + \frac{r^2\sin^2 t\cos^2\phi_2}{4\psi h\sqrt{P(y)}} + \frac{r^2\cos^2 t\sin^2\phi_1}{4\psi h\sqrt{P(y)}} + \frac{r^2\sin^2 t\sin^2\phi_2}{4\psi h\sqrt{P(y)}}\\
&= \frac{r^2}{4\psi h\sqrt{P(y)}},
\end{aligned} \tag{3.15}$$

and $\langle \tilde{F}, \xi \rangle = \frac{z}{2}$. Thus the normal projection of the position vector is

$$\tilde{F}^{\perp} = \frac{r^2}{4\psi h\sqrt{P(y)}}\Phi e_2 + \frac{z}{2}\xi. \tag{3.16}$$

From $P(t) = \frac{1}{t^2}((1+at^2)^2 e^{\alpha t^2} - 1)$, we have

$$P'(y) = \frac{4a}{y}(1+ay^2)e^{\alpha y^2} + \frac{2\alpha}{y}(1+ay^2)^2 e^{\alpha y^2} - \frac{2}{y^3}(1+ay^2)^2 e^{\alpha y^2} + \frac{2}{y^3},$$

and

$$h^2 = \frac{1}{P(y)} + y^2 = \frac{y^2(1+ay^2)^2 e^{\alpha y^2}}{(1+ay^2)^2 e^{\alpha y^2} - 1}.$$

It was obtained

$$\frac{y}{h^2} = \frac{(1+ay^2)^2 e^{\alpha y^2} - 1}{y(1+ay^2)^2 e^{\alpha y^2}},$$

and

$$\frac{P'(y)}{P(y)} = \frac{4ay(1+ay^2)e^{\alpha y^2}}{(1+ay^2)^2 e^{\alpha y^2} - 1} + \frac{2\alpha y(1+ay^2)^2 e^{\alpha y^2}}{(1+ay^2)^2 e^{\alpha y^2} - 1} - \frac{2(1+ay^2)^2 e^{\alpha y^2}}{y[(1+ay^2)^2 e^{\alpha y^2} - 1]} + \frac{2}{y[(1+ay^2)^2 e^{\alpha y^2} - 1]}.$$

It follows that

$$\begin{aligned} &-\frac{1}{\psi^2} + \frac{2}{h^2} + \frac{yP'(y)}{h^2 P(y)} \\ &= -\frac{4a}{1+ay^2} + \frac{2(1+ay^2)^2 e^{\alpha y^2} - 2}{y^2(1+ay^2)^2 e^{\alpha y^2}} + \frac{4a}{1+ay^2} + 2\alpha - \frac{2}{y^2} + \frac{2}{y^2(1+ay^2)^2 e^{\alpha y^2}} \\ &= 2\alpha, \end{aligned} \tag{3.17}$$

Thus (1.6), (3.13), (3.16) and (3.17) give

$$\tilde{H} + \tilde{\theta}\xi = (-\frac{2}{\psi^2} + \frac{4}{h^2} + \frac{2yP'(y)}{h^2 P(y)})\tilde{F}^{\perp} = 4\alpha \tilde{F}^{\perp}.$$

On the other hand, since the frames e_1, e_2 are orthonormal, locally the metric tensor g is the identity matrix. Thus,

$$d\tilde{\theta} = (\nabla_{e_1}\tilde{\theta})e_1 + (\nabla_{e_2}\tilde{\theta})e_2 = -\frac{\alpha r^2}{\psi h\sqrt{P(y)}}e_2,$$

and $\Phi\nabla\tilde{\theta} = \Phi(g^{-1}d\tilde{\theta}) = -H$. In summary, when $\alpha > 0$, $\tilde{L}$ is a self-expander and satisfies $\tilde{H} + \tilde{\theta}\xi = 4\alpha\tilde{F}^{\perp}$. And it follows from equation (3.3) that $\tilde{L}$ is embedded Legendrian diffeomorphic to $\mathbb{S} \times \mathbb{R}$, which is similar to the proof in [10]. □

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