TAMKANG JOURNAL OF MATHEMATICS
Volume 57, Number 1, 63-79, 2026
DOI:10.5556/j.tkjm.57.2026.5881

q-Analogue of the Karry-Kalim-Adnan transform with applications to q-differential equations

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Abstract. This work analyzes certain features of the Karry-Kalim-Adnan transform and discusses its q-analogues in a quantum calculus theory. It discusses a number of characteristics of the q-Karry-Kalim-Adnan transform and its application to a wide range of functions, including q-trigonometric, q-hyperbolic and q-exponential functions and some q-polynomials. Moreover, it utilizes first- and second-order q-initial value problems to illustrate advantages of our proposed q-transform analogues. Over and above, the paper proves the q-convolution theorem and provides a table to further ease the q-transform technique in solving various q-initial value problems.

Keywords. Karry-Kalim-Adnan transform, q-special function, q-difference equation,q-initial value problems

1 Introduction

Several integral transformation techniques have been proven to help in solving various types of initial value problems. They provide useful solutions to many fields of science including engineering, chemistry, biology, astronomy, and radio physics. They also make it easier to determine solutions to differential equations with given initial conditions. One more advantage of the integral transformation techniques is that they can offer accurate solutions without requiring complex calculations [17]. Consequently, researchers have focused a great deal of attention on integral transforms, which have resulted in the introduction of many new integral transforms including Sumudu transform [6,8], N-transform [7,10,19], Laplace-type transform [4,13], Stieltjes transform [2], Laplace transform [1,9,11,14], Karry Kalim Adnan and Kushare transform [15] and many others which have been discussed in various time domains and differential equations.

Let P be a finite real number and j_1, j_2 be finite or infinite natural numbers. Then, over the set A of functions of exponential order,

$$A = \left\{ h\left(x\right) : \exists P; j_1, j_2 > 0, \left|h\left(x\right)\right| < P e^{\frac{|x|}{j_i}}, x \in \left(-1\right)^i \times \left[0, \infty\right), i = 1, 2 \right\}, \tag{1.1}$$

the Karry-Kalim-Adnan transform KKAT is introduced as a new integral transform defined by [15]

$$K\left(h; \alpha, \beta\right) = \frac{1}{\alpha\beta} \int_0^\infty h\left(x\right) e^{-\frac{\beta}{\alpha}x} dx, \tag{1.2}$$

Received date: August 22, 2025; Published online: January 21, 2025.
2010 *Mathematics Subject Classification*. 26D07, 30C45, 26D20.
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where $h \in A$ and α, β are non-zero constants.

The Karry- Kalim- Adnan transformation (KKAT) has been employed in solving problems in engineering, electrical and mechanical problems [15,17]. It has also been intricately linked to a multitude of integral transformations including the Laplace transformation, Fourier transformation, Sumudu transformation, Elzaki transformation, Aboodh transformation, and Mahgoub transformation, to mention but a few. The study of quantum calculus, also known as q-calculus, is pioneered by [16] and has made substantial advances to the study of hydrogen atom symmetry utilizing q-difference equations. Consequently, the q-integral transforms prompted approximately a decade ago an abundance of investigations, including the q-Laplace transforms [1], the q-natural transforms [7] and several others.

Although the KKAT creates additional opportunities for solving differential equations in situations when conventional approaches might not be sufficient, the mathematical foundations of the q-analogues of the KKAT are not yet examined in quantum calculus. In addition, the integration of the KKAT in quantum calculus advances our understanding of ability of using quantum calculus concepts in the q-difference and q-special functions.

In Section 2 of this paper we provide some fundamental information and notations utilized in the upcoming chapters. The q-analogue of the KKAT and its general characteristics are given in Section 3, which further covers several applications of the q-transform to numerous q-integral equations and some first and second order q-initial value problems. Results pertaining to q-polynomials, q-exponential functions, q-trigonometric functions, and q-hyperbolic functions are presented in Section 4. Section 5 includes some counterexamples of q-difference equations (or q-initial value problems) to demonstrate the usefulness of our findings, .

2 Basic definitions

Mathematicians and physicists are interested in studying q-analogues of different classical identities. In quantum calculus, also known as q-calculus, the parameter q, where $0 < q < 1$, is commonly used to represent "quantum." In essence, "q-analogue" describes a mathematical expression with a parameter q that generalizes a known identity and returns to the identity when $q \to 1$. The q-calculus begins with the definition of the q-analogue $d_q h$ of the differential of the function h, given as [3]

$$d_q h\left(x\right) = h\left(qx\right) - h(x). \tag{2.1}$$

Having said this we get the q-analogue of the derivative of h called the q-derivative of h [3]

$$D_q h\left(x\right) = \frac{d_q\left(h\left(x\right)\right)}{d_q x} = \frac{h(x) - h(qx)}{\left(1-q\right)x}, x \neq 0, \tag{2.2}$$

and $D_q h\left(x\right) = h'\left(0\right)$ at $x = 0$. The q-analogue of a factorial of a positive integer m is defined by [3]

$$[m]_q! = \begin{cases} \prod_{n=1}^{m}[n]_q, & \text{if } \; m \geq 1 \\ 1, & \text{if } \; m = 0 \end{cases} \tag{2.3}$$

where

$$[m]_q = \frac{1-q^m}{1-q} \tag{2.4}$$

is the q-analogue of the integer m. The higher order q-derivatives of a function h and the product of two functions h and g are, respectively, given for $m \in \mathbb{N}$ as [3]

$$\left(D_q^0 h\right)(x) = h(x), \left(D_q^m f\right)(x) = D_q\left(D_q^{m-1} h\right)(x) \tag{2.6}$$

and

$$D_q(h(x)g(x)) = h(x)D_q g(x) + g(qx)D_q h(x). \tag{2.7}$$

The q-derivative of a quotient of two functions h and g is given as [3]

$$D_q\left(\frac{h(x)}{g(x)}\right) = \frac{g(qx)\,D_q h(x) - D_q g(x)\,h(qx)}{g(x)\,g(qx)},\ g(x)\neq 0 \text{ and } g(qx)\neq 0. \tag{2.8}$$

For a function h, the q-Jackson integrals over $[0, x]$ and $[0, \infty)$ are, respectively, determined as [3]

$$\int_0^x h(x)\,d_q x = (1-q)\,x\sum_{m=0}^{\infty} q^m h(q^m x) \text{ and } \int_0^{\infty} h(x)\,d_q x = (1-q)\sum_{m=-\infty}^{\infty} q^m h(q^m). \tag{2.9}$$

The q-integral of the q-derivative of the function h, over an interval $[a, b]$, is provided by [3]

$$\int_a^b D_q h(x)\,d_q x = h(b) - h(a). \tag{2.10}$$

The definition of the q-integration by parts for the functions h and g is provided by [3]

$$\int_a^b g(x)(D_q h(x)\,d_q x = h(b)\,g(b) - h(a)\,g(a) - \int_a^b h(qx)D_q g(x)\,d_q x. \tag{2.11}$$

The q-exponential functions of first and second types and their q-derivatives are, respectively, given by [3]

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^m}{[m]_q!} \tag{2.13}$$

and

$$E_q(z) = \sum_{m=0}^{\infty} \frac{q^{\binom{n}{2}}}{[m]_q!} x^m, \tag{2.14}$$

where

$$D_q e_q(mx) = m e_q(mx),\ D_q E_q(mx) = m E_q(mqx),\ \text{where } e_q(0) = 1. \tag{2.15}$$

Consequently, the q-analogues of the cosine function of first and second types are, respectively, determined as [8]

$$\cos_q(z) = \frac{e_q(iz) + e_q(-iz)}{2} = \sum_{m=0}^{\infty} \frac{(-1)^m}{[2m]_q!} z^{2m}, \tag{2.16}$$

$$Cos_q(z) = \frac{E_q(iz) + E_q(-iz)}{2} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{n}{2}}}{[2m]_q!} z^{2m} \tag{2.17}$$

while the q-analogues of the sine function of first and second types are, respectively, determined as

$$\sin_q(z) = \frac{e_q(iz) - e_q(-iz)}{2i} = \sum_{m=0}^{\infty} \frac{(-1)^m}{[2m+1]_q!} z^{2m+1}, \tag{2.18}$$

$$Sin_q(z) = \frac{E_q(iz) - E_q(-iz)}{2i} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{n(n+1)}{2}}}{[2m+1]_q!} z^{2m+1}. \tag{2.19}$$

The q-gamma and q-beta functions are consistent with the q-integral representations defined respectively as

$$\Gamma_q(t) = \int_0^{\infty} z^{t-1} E_q(-qz)\, d_q z \text{ and } B_q(t;s) = \int_0^1 z^{t-1}(1-qz)_q^{s-1} d_q z, (t, s > 0). \tag{20}$$

3 The q-KKAT of certain q-difference operators and integral equations

Here, we present a q-analogue of the KKAT and derive some general characteristics. We also provide some applications of the q-transform to q-difference operators of the first and second orders. Additionally, certain application of the q-transform to single, double, triple and n-th q-integral equations is also provided.

Definition 3.1: Denote by A_q the set of all functions such that

$$A_q = \left\{ h(x) | \exists M, j_1, j_2 > 0, |h(x)| < M E_q\left(\frac{|t|}{j_k}\right), t \in (-1)^k \times [0, \infty), k = 1, 2 \right\},$$

then the q-analogue of the Karry-Kalim-Adnan transform q-KKAT is defined over the set A_q by

$$k_q(h(x); \alpha, \beta) = \frac{1}{\alpha\beta} \int_0^{\infty} h(x) E_q\left(\frac{-q\beta}{\alpha} x\right) d_q x, \tag{3.1}$$

provided the transform variables α and β are non-negative. The inverse Karry-Kalim-Adnan transform of the transform k_q is denoted by k_q^{-1} satisfies the inversion formula

$$k_q^{-1}(k_q(h(x); \alpha, \beta)) = h(x)$$

for $h(x) \in A_q$ when it exists.

Remark 3.2: The kernel function $E_q(-q\beta x/\alpha)$ introduced in Definition 3.1 is motivated by the idea of constructing a q-analogue of the classical KKAT kernel within the framework of the q-calculus theory.

In this context, the classical exponential function is naturally replaced by the q-exponential function $E_q(\cdot)$, being compatible with the Jackson q-integral that frequently arises in the study of q-integral transforms. The presence of the term $E_q(-q\beta x/\alpha)$ assures that the kernel keeps its decaying properties analogous to those of the classical KKAT kernel. Furthermore, the proposed definition is consistent with the classical theory. Indeed, by using the standard limit

$$\lim_{q \to 1^-} E_q(z) = e^z,$$

together with the facts that the Jackson q-analogues of the integral and the Gamma function reduce to their classical ones as $q \to 1^-$, we see that the q-KKAT kernel converges to the classical KKAT kernel. Consequently, the q-KKAT operator defined in Definition 3.1 reduces to the classical KKAT operator as $q \to 1^-$. Now, we discuss the linearity and scaling properties of the provided q-analogue of the KKAT.

Theorem 3.3: (Linearity). If h, g are in A_q, then the following formula holds

$$k_q\left(nh\left(x\right)+mg\left(x\right);\alpha,\beta\right)=nk_q\left(h\left(x\right);\alpha,\beta\right)+mk_q(g\left(x\right);\alpha,\beta). \tag{3.2}$$

Proof. By using (3.1) we have

$$\begin{aligned} k_q\left(nh\left(x\right)+mg\left(x\right);\alpha,\beta\right) &= \frac{1}{\alpha\beta}\int_0^\infty \left(nh\left(x\right)+mg\left(x\right)\right)E_q\left(\frac{-q\beta x}{\alpha}\right)d_qx \\ &= \frac{n}{\alpha\beta}\int_0^\infty h\left(x\right)E_q\left(\frac{-q\beta x}{\alpha}\right)d_qx+\frac{m}{\alpha\beta} \\ \int_0^\infty g\left(x\right)E_q\left(\frac{-q\beta x}{\alpha}\right)d_qx & \\ &= nk_q\left(h\left(x\right);\alpha,\beta\right)+mk_q\left(g\left(x\right);\alpha,\beta\right). \end{aligned}$$

□

Theorem 3.4: (Scaling) If h belongs to A_q, then the following formula holds

$$k_q\left(h\left(\delta x\right);\alpha,\beta\right)=k_q\left(h\left(x\right);\alpha\delta,\beta\right), \tag{3.3}$$

where δ is a non-zero constant.

Proof. By using Definition (3.1) we have

$$k_q\left(g\left(\delta x\right);\alpha,\beta\right)=\frac{1}{\alpha\beta}\int_0^\infty g\left(\delta x\right)E_q\left(\frac{-q\beta}{\alpha}x\right)d_qx.$$

Next, employing (2.12) implies

$$k_q\left(g\left(\delta x\right);\alpha,\beta\right)=\frac{1}{\delta\alpha\beta}\int_0^\infty g\left(x\right)E_q\left(\frac{-q\beta}{\delta\alpha}x\right)d_qx=k_q\left(g\left(x\right);\delta\alpha,\beta\right).$$

This completes the proof.

Theorem 3.5: If h, D_qh and D_q^2h belong to A_q, then the q-transform of the q-derivative of the first and second degrees of h are, respectively, given as

$$(i)k_q\left(D_qh\left(x\right);\alpha,\beta\right)=\frac{-h(0)}{\alpha\beta}+\frac{\beta}{\alpha}k_q(h\left(x\right);\alpha,\beta), \tag{3.4}$$

$$(ii)\,k_q\left(D_q^2h\left(x\right);\alpha,\beta\right)=\frac{-D_qh(0)}{\alpha\beta}-\frac{h\left(0\right)}{\alpha^2}+\frac{\beta^2}{\alpha^2}k_q(h\left(x\right);\alpha,\beta). \tag{3.5}$$

Proof. (i) By using Definition (3.1) and following simple computations we have

$$k_q\left(D_qh\left(x\right);\alpha,\beta\right) = \frac{1}{\alpha\beta}\int_0^\infty D_qh\left(x\right)E_q\left(\frac{-q\beta x}{\alpha}\right)d_qx$$

$$= \frac{1}{\alpha\beta}\left(\lim_{a\to\infty} h(x)\,E_q\left(\frac{-\beta x}{\alpha}\right)\right]_0^a - \int_0^\infty h(x)D_qE_q\left(\frac{-\beta x}{\alpha}\right)d_qx\right)$$
$$= \frac{-h(0)}{\alpha\beta} + \frac{\beta}{\alpha}k_q\left(h(x);\alpha,\beta\right).$$

(ii) Applying the result of (3.4), we get

$$\begin{aligned} k_q\left(D_q^2 h(x);\alpha,\beta\right) &= \frac{-D_qh(0)}{\alpha\beta} + \frac{\beta}{\alpha}k_q(D_qh(x);\alpha,\beta) \\ &= \frac{-h'(0)}{\alpha\beta} + \frac{\beta}{\alpha}\left(\frac{-h(0)}{\alpha\beta} + \frac{\beta}{\alpha}k_q(h(x);\alpha,\beta)\right) \\ &= \frac{-h'(0)}{\alpha\beta} - \frac{h(0)}{\alpha^2} + \frac{\beta^2}{\alpha^2}k_q\left(h(x);\alpha,\beta\right). \end{aligned}$$

This finishes the proof. □

Theorem 3.6: If h belongs to A_q, then the following formulas hold true:

$$(i)k_q\left(\int_0^t h(x)\,d_qx;\alpha,\beta\right) = \frac{\alpha}{\beta}k_q(h(t);\alpha,\beta) \tag{3.6}$$

$$(ii)k_q\left(\int_0^t\int_0^x h(\varphi)d_q\varphi d_qx;\alpha,\beta\right) = \frac{\alpha^2}{\beta^2}k_q\left(h(t);\alpha,\beta\right). \tag{3.7}$$

$$(iii)k_q\left(\int_0^t\int_0^x\int_0^{x_1} h(\varphi)\,d_q\varphi d_qx_1d_qx;\alpha,\beta\right) = \frac{\alpha^3}{\beta^3}k_q\left(h(t);\alpha,\beta\right).$$

$$(iv)k_q\left(\int_0^t\int_0^{x_n}\cdots\int_0^{x_2} h(x_1)d_qx_1\ldots d_qx_{n-1}d_qx_n;\alpha,\beta\right) = \frac{\alpha^n}{\beta^n}k_q(h(t);\alpha,\beta). \tag{3.1}$$

Proof.: Proof of (i). By using (2.11) and (2.12), we get

$$k_q\left(\int_0^t h(x)\,d_qx;\alpha,\beta\right) = \frac{1}{\alpha\beta}\int_0^\infty E_q\left(\frac{-q\beta t}{\alpha}\right)\int_0^t h(x)\,d_qxd_qt.(3.2)$$

If $\emptyset(t) = \int_0^t h(x)d_qx$, then we write

$$\int_0^\infty \emptyset(t)\,D_qE_q\left(\frac{-\beta t}{\alpha}\right)d_qt = \left[\emptyset(t)\,E_q\left(\frac{-\beta t}{\alpha}\right)\right]_0^\infty - \int_0^\infty E_q\left(\frac{-q\beta t}{\alpha}\right)D_q\emptyset(t)\,d_qt.$$

Hence, motivating the previous equation yields

$$\frac{\beta}{\alpha}\int_0^\infty \emptyset(t)\,E_q\left(\frac{-q\beta t}{\alpha}\right)d_qt = \int_0^\infty E_q\left(\frac{-q\beta t}{\alpha}\right)h(t)\,d_qt.$$

Consequently, we get

$$k_q\left(\int_0^t h(x)\,d_qx;\alpha,\beta\right) = \frac{\alpha}{\beta}k_q(h(t);\alpha,\beta).$$

Proof of (ii). Let $\emptyset(x) = \int_0^x h(\varphi) d_q\varphi$, then from using the first part we get

$$\begin{aligned} k_q\left(\int_0^t \int_0^x h(\varphi)\, d_q\varphi d_q x; \alpha, \beta\right) &= k_q\left(\int_0^t \emptyset(x)\, d_q x; \alpha, \beta\right) \\ &= \frac{\alpha}{\beta} k_q(\emptyset(t); \alpha, \beta) \\ &= \frac{\alpha}{\beta} k_q\left(\int_0^t h(\varphi)\, d_q\varphi; \alpha, \beta\right) \\ &= \frac{\alpha^2}{\beta^2} k_q(h(t); \alpha, \beta). \end{aligned}$$

Proof of (iii) Let $\varphi_1(x) = \int_0^x \int_0^{x_1} h(\varphi_1)\, d_q\varphi_1 d_q x_1$. Then, we obtain

$$\begin{aligned} k_q\left(\int_0^t \int_0^x \int_0^{x_1} h(\varphi_1)\, d_q\varphi_1 d_q x_1 d_q x; \alpha, \beta\right) &= k_q\left(\int_0^t \varphi_1(x) d_q x; \alpha, \beta\right) \\ &= \frac{\alpha^2}{\beta^2} k_q(\varphi_1(t); \alpha, \beta) \\ &= \frac{\alpha^2}{\beta^2} k_q\left(\int_0^t h(\varphi_1)\, d_q\varphi_1; \alpha, \beta\right) \\ &= \frac{\alpha^3}{\beta^3} k_q(h(t); \alpha, \beta) \end{aligned}$$

Likewise, by employing the principle of mathematical induction and following the proofs presented in parts (i)-(iii), Part (iv) can be easily proved. Hence, we delete the details. Therefore, the proof is completed. □

4 The q-KKAT of special functions

In this section, we review some conclusions pertaining to the q-KKAT of the first kind and examine various fundamental functions, such as q-polynomials, q-exponential functions, q-trigonometric function and q-hyperbolic functions as well.

Theorem 4.1: Let $h(x) = 1, x \in (0, \infty)$ and β be a non-zero constant. Then, we have

$$K_q(1; \alpha, \beta) = \frac{1}{\beta^2}.$$

Proof. By using the definition of the q-analogue of the Karry-Kalim-Adnan transform (3.1) and the results given by (2.15) we get

$$K_q(1; \alpha, \beta) = \frac{1}{\alpha\beta}\left(\frac{-\alpha}{\beta}\right) \int_0^\infty D_q E_q\left(\frac{-\beta}{\alpha} x\right) d_q x. \tag{4.1}$$

Therefore, integrating the preceding equation (4.1) implies

$$K_q(1; \alpha, \beta) = \left(\frac{-1}{\beta^2}\right) \lim_{x\to\infty} \left[E_q\left(\frac{-\beta}{\alpha} x\right)\right]_0^x. \tag{4.2}$$

Note that as $\alpha, \beta > 0$, the q-exponential function satisfies

$$\lim_{x\to\infty} E_q\left(-\frac{\beta x}{\alpha}\right) = 0,$$

which follows from the monotone decay of $E_q(-t)$ for $t > 0$, see also [30]. Hence, the proof is completed. □

Theorem 4.2: Let $h(x) = x, x \in (0, \infty)$. Then, for non-zero constants β and α, we have

$$k_q(x; \alpha, \beta) = \frac{\alpha}{\beta^3}.$$

Proof. Assume the hypothesis of the theorem be correct. Then, utilizing (3.1), we write

$$k_q(x; \alpha, \beta) = \frac{1}{\alpha\beta}\left(\frac{-\alpha}{\beta}\right)\int_0^\infty x E_q\left(\frac{-q\beta}{\alpha}x\right) d_q x.$$

Using the differential results given by (2.15) implies

$$k_q(x; \alpha, \beta) = \frac{-1}{\beta^2}\int_0^\infty x D_q E_q\left(\frac{-\beta}{\alpha}x\right) d_q x. \tag{4.3}$$

Next, following the idea of the q-integration by parts (2.11) reveals

$$k_q(x; \alpha, \beta) = \frac{1}{\beta^2}\left(-\int_0^\infty E_q\left(\frac{-q\beta}{\alpha}x\right) d_q x\right).$$

Hence, the preceding equation yields

$$k_q(x; \alpha, \beta) = \frac{1}{\beta^2}\left(\frac{\alpha}{\beta}\right) = \frac{\alpha}{\beta^3}.$$

The proof is completed. □

Theorem 4.3: Let $h(x) = x^m$, $m \in \mathbb{N}$ and $x \in (0, \infty)$. Then, we have

$$k_q(x^m; \alpha, \beta) = \frac{\alpha^m}{\beta^{m+2}}\Gamma_q(m+1). \tag{4.4}$$

Proof. By using (3.1) and changing variables, we establish that

$$k_q(x^m; \alpha, \beta) = \frac{1}{\alpha\beta}\int_0^\infty x^m E_q\left(\frac{-q\beta}{\alpha}x\right) d_q x = \frac{1}{\alpha\beta}\left(\frac{\alpha}{\beta}\right)^{m+1}\int_0^\infty x^m E_q(-qx)\, d_q x.$$

Using (2.20) gives

$$K_q(x^m; \alpha, \beta) = \frac{\alpha^m}{\beta^{m+2}}\Gamma_q(m+1). \tag{4.5}$$

□

Corollary 4.4: Let $h(x) = x^m$, $m \in \mathbb{N}$ and $x \in (0, \infty)$. Then, we have

$$K_q(x^m; \alpha, \beta) = \frac{\alpha^m}{\beta^{m+2}} [m]_q!, \ m \in \mathbb{N}.$$

The proof follows from (4.4) and the properties of the q-gamma function.

Theorem 4.5: Let $h(x) = e_q(bx), x \in (0, \infty)$ and b be a constant. Then, we have

$$k_q(e_q(bx); \alpha, \beta) = \frac{1}{\alpha\beta} \int_0^\infty e_q(bx) E_q\left(\frac{-q\beta x}{\alpha}\right) d_q x. \tag{4.6}$$

Proof. Using (3.1) and the results obtained in (2.13) imply

$$k_q(e_q(bx); \alpha, \beta) = \sum_{m=0}^{\infty} \frac{b^m}{\alpha\beta[m]_q!} \int_0^\infty x^m E_q\left(\frac{-q\beta x}{\alpha}\right) d_q x. \tag{4.7}$$

Therefore, (2.15) and (4.4) results in a geometric series expansion form as

$$\begin{aligned} k_q(e_q(bx); \alpha, \beta) &= \sum_{m=0}^{\infty} \frac{b^m}{[m]_q!} \frac{\alpha^m [m]_q!}{\beta^{m+2}} \\ &= \frac{1}{\beta^2} \sum_{m=0}^{\infty} \left(\frac{b\alpha}{\beta}\right)^m \\ &= \frac{1}{\beta^2} \frac{\beta}{\beta - b\alpha} \\ &= \frac{1}{\beta^2 - b\alpha\beta}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 4.6: Let $h(x) = E_q(bx)$, $x \in (0, \infty)$ and b be a constant. Then, we have

$$k_q(E_q(bx); \alpha, \beta) = \frac{1}{\beta^2} \sum_{m=0}^{\infty} q^{\binom{m}{2}} \left(\frac{b\alpha}{\beta}\right)^m. \tag{4.8}$$

Proof. By considering (3.1) and employing (2.14) we write

$$\begin{aligned} k_q(E_q(bx); \alpha, \beta) &= \frac{1}{\alpha\beta} \int_0^\infty E_q(bx) E_q\left(\frac{-q\beta x}{\alpha}\right) d_q x \\ &= \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}} b^m}{\alpha\beta[m]_q!} \int_0^\infty x^m E_q\left(\frac{-q\beta x}{\alpha}\right) d_q x. \end{aligned}$$

Hence, applying (4.5) reveals

$$k_q(E_q(bx); \alpha, \beta) = \sum_{m=0}^{\infty} \frac{q^{\binom{m}{2}} b^m}{[m]_q!} \frac{\alpha^m [m]_q!}{\beta^{m+2}} = \frac{1}{\beta^2} \sum_{m=0}^{\infty} q^{\binom{m}{2}} \left(\frac{b\alpha}{\beta}\right)^m.$$

Hence, the proof is completed. □

Theorem 4.7: If $h(x) = \sin_q(bx)$ and b be a constant. Then, we have

$$k_q(\sin_q(bx);\alpha,\beta) = \frac{b\alpha}{\beta^3 + b^2\alpha^2\beta}. \tag{4.9}$$

Proof. By using (3.1) we obtain

$$k_q(\sin_q(bx);\alpha,\beta) = \frac{1}{\alpha\beta}\int_0^\infty \sin_q(bx)\, E_q\left(\frac{-q\beta x}{\alpha}\right) d_qx. \tag{4.10}$$

By invoking (2.18) into (4.10), we derive

$$k_q(\sin_q(bx);\alpha,\beta) = \frac{1}{\alpha\beta}\sum_{m=0}^{\infty}\frac{(-1)^m}{[2m+1]!_q}(b)^{2m+1}\int_0^\infty x^{2m+1}E_q\left(\frac{-q\beta x}{\alpha}\right) d_qx.$$

By taking into account (4.5) we derive

$$\begin{aligned} k_q(\sin_q(bx);\alpha,\beta) &= \frac{1}{\alpha\beta}\sum_{m=0}^{\infty}\frac{(-1)^m}{[2m+1]!_q}(b)^{2m+1}\frac{\alpha^{2m+1}[2m+1]!_q}{\beta^{2m+2}} \\ &= \frac{\alpha b}{\beta^3}\sum_{m=0}^{\infty}(-1)^m\left(\frac{b\alpha}{\beta}\right)^{2m+1} \\ &= \frac{\alpha b}{\beta^3}.\frac{\beta^2}{\beta^2 + b^2\alpha^2}. \end{aligned}$$

That is,

$$k_q(\sin_q(bx);\alpha,\beta) = \frac{b\alpha}{\beta^3 + b^2\alpha^2\beta}.$$

Hence, the proof is completed. □

Theorem 4.8: If $h(x) = \cos_q(bx)$ and b be a constant. Then, we have

$$k_q(\cosh_q(bx);\alpha,\beta) = \frac{1}{\beta^2 - b^2\alpha^2}.$$

Proof. By using (3.1) we get

$$k_q(\cos_q(bx);\alpha,\beta) = \frac{1}{\alpha\beta}\int_0^\infty \cos_q(bx)\, E_q\left(\frac{-q\beta x}{\alpha}\right) d_qx. \tag{4.11}$$

By substituting (2.16) into (4.11), we infer that

$$k_q(\cos_q(bx);\alpha,\beta) = \frac{1}{\alpha\beta}\sum_{m=0}^{\infty}\frac{(-1)^m b^{2m}}{[2m]!_q}\int_0^\infty x^{2m}E_q\left(\frac{-q\beta x}{\alpha}\right) d_qx. \tag{4.12}$$

Using (3.13) gives

$$k_q(\cos_q(bx);\alpha,\beta) = \frac{1}{\alpha\beta}\sum_{m=0}^{\infty}\frac{(-1)^m b^{2m}}{[2m]!_q}\frac{\alpha^{2m+1}}{\beta^{2m+1}}$$

$$= \frac{1}{\beta^2}\sum_{m=0}^{\infty}(-1)^m\left(\frac{\alpha b}{\beta}\right)^{2m}$$
$$= \frac{1}{\beta^2}\frac{\beta^2}{\beta^2+b^2\alpha^2}.$$

That is,

$$k_q\left(\cos_q\left(bx\right);\alpha,\beta\right) = \frac{1}{\beta^2+b^2\alpha^2}.$$

Hence, the proof is completed. □

In a similar manner, the hyperbolic q-sine and q-cosine function are given by

$$\cosh_q\left(x\right) = \frac{e_q\left(x\right)+e_q\left(-x\right)}{2}, \sinh_q(x) = \frac{e_q\left(x\right)-e_q\left(-x\right)}{2} \qquad (4.13)$$

Therefore, we have the following consequences.

Theorem 4.9 Let $h\left(x\right) = \cosh_q\left(bx\right)$ and b be a constant. Then, we have

$$k_q\left(\cosh_q\left(bx\right);\alpha,\beta\right) = \frac{1}{\beta^2-b^2\alpha^2}.$$

Proof. By virtue of the definitions presented in (4.13) and the preceding analysis, we derive

$$\begin{aligned} k_q\left(\cosh_q\left(bx\right);\alpha,\beta\right) &= \frac{1}{2}\left[k_q\left(e_q\left(bx\right);\alpha,\beta\right)+k_q\left(e_q\left(bx\right);\alpha,\beta\right)\right] \\ &= \frac{1}{2}\left(\frac{1}{\beta^2-b\alpha\beta}+\frac{1}{\beta^2+b\alpha\beta}\right). \end{aligned}$$

That is,

$$k_q\left(\cosh_q\left(bx\right);\alpha,\beta\right) = \frac{1}{\beta^2-b^2\alpha^2}.$$

Hence, the proof is completed. □

Similar results involving $k_q\left(\sinh_q\left(bx\right);\alpha,\beta\right)$ can be easily obtained. Below is a table including certain values of the q-KKAT of special functions.

S.N	**Function** $h(x)$	**First kind** $k_q\big(h(x);\alpha,\beta\big)$
1	1	$\frac{1}{\beta^2}$
2	x	$\frac{\alpha}{\beta^3}$
3	$x^m,\ m\in\mathbb{N}$	$\frac{\alpha^m\Gamma_q(m+1)}{\beta^{m+2}}$
4	$x^m,\ m\in\mathbb{N}$	$\frac{\alpha^m[m]_q!}{\beta^{m+2}}$

S.N	Function $h(x)$	First kind $k_q\big(h(x);\alpha,\beta\big)$
5	$\sin_q(bx)$	$\dfrac{b\alpha}{\beta^3+b^2\alpha^2\beta}$
6	$\cos_q(bx)$	$\dfrac{1}{\beta^2+b^2\alpha^2}$
7	$\sinh_q(bx)$	$\dfrac{b\alpha}{\beta^3-b^2\alpha^2\beta}$
8	$\cosh_q(bx)$	$\dfrac{1}{\beta^2-b^2\alpha^2}$
9	$e_q(bx)$	$\dfrac{1}{\beta^2-b\alpha\beta}$
10	$E_q(bx)$	$\dfrac{1}{\beta^2}\sum_{m=0}^{\infty} q^{\binom{m}{2}}\left(\dfrac{b\alpha}{\beta}\right)^m$

Table 1: q-KKAT transforms of known functions.

5 Application to q-initial value problems

In the following examples, we demonstrate the efficiency of the first kind q-KKAT in resolving specific q-initial value problems with constant coefficients. Next, we solve first and second order q-initial value problems (q-IVPs) with constant variable coefficients using the q-KKAT. For constants $b_i \in \mathbb{R}$, $i = 1, 2 \ldots, m, m \in \mathbb{N}$, the generic form of the q-IVP is taken into consideration as

$$D_q^m y(x) + b_1 D_q^{m-1} y(x) + \ldots + b_m y(x) = G(x), \tag{5.1}$$

where the initial conditions are given as

$$y(0) = y_0, D_q y(0) = y_1, \ldots, D_q^{m-1} y(0) = y_{m-1}. \tag{5.2}$$

In the subsequent, two cases of the order of the q-difference equations for $m = 1$ and $m = 2$ are examined.

Case 1: Let us discuss first the case where $m = 1$. This indeed simplifies (5.1) to the generic form

$$D_q y(x) + b_1 y(x) = G(x). \tag{5.3}$$

Utilizing the initial conditions (5.2) and applying the k_q transform to both sides of (5.3) yield

$$k_q(D_q y(x) + b_1 y(x); \alpha, \beta) = k_q(G(x); \alpha, \beta). \tag{5.4}$$

Consequently, using (3.4) demonstrates

$$\frac{-y(0)}{\alpha\beta} + \frac{\beta}{\alpha} k_q(y(x); \alpha, \beta) + b_1 k_q(y(x); \alpha, \beta) = k_q(G(x); \alpha, \beta). \tag{5.5}$$

Thus, simplifying (5.5) gives

$$k_q(y(x); \alpha, \beta) = \frac{\alpha\beta k_q(G(x); \alpha, \beta)}{(\beta?^2 + b_1\alpha\beta) + \frac{y_0}{(\beta^2 + b_1\alpha\beta)}}. \tag{5.6}$$

The exact solution is thus obtained by applying the inverse transform of k_q to both sides of (5.6) giving

$$y(x) = \alpha\beta k_q^{-1}\left(\frac{k_q(G(x);\alpha,\beta)}{(\beta^2+b_1\alpha\beta)}\right) + k_q^{-1}\left(\frac{y_0}{(\beta^2+b_1\alpha\beta)}\right). \tag{5.7}$$

Example 1: Consider the q-Cauchy problem

$$D_q y(x) + 4y(x) = 0, \tag{5.8}$$

where the initial condition is given by $y(0) = 2$.

Solution. Going back to (5.3), we have $G(x) = 0$, $b_1 = 4$, while the initial condition becomes $y_0 = 2$. Therefore, by inserting the preceding values in the the general solution presented in (5.7), the solution of the q-differential equation given by (5.8) yields

$$y(x) = k_q^{-1}\left(\frac{2}{\beta^2+4\alpha\beta}\right).$$

Hence, employing Table 1 gives

$$y(x) = 2e_q(-4x).$$

This finishes the solution of our example.

Example 2: Consider the first order q-initial value problem,

$$D_q y(x) + y(x) = e_q(x), \tag{5.9}$$

with the initial condition $y(0) = 1$.

Solution. From (5.3), we see that $b_1 = 1$, $y_0 = 1$, $G(x) = e_q(x)$ and $y_0 = 1$. Therefore, inserting the given values in the general solution proposed in (5.7) and simplifying the result imply

$$y(x) = k_q^{-1}\left(\frac{\alpha\beta k_q(e_q(x);\alpha,\beta)}{\beta^2+\alpha\beta}\right) + k_{p,q}^{-1}\left(\frac{y_0}{\beta^2+\alpha\beta}\right) = k_q^{-1}\left(\frac{\alpha\beta\frac{1}{\beta(\beta-\alpha)}}{\beta^2+\alpha\beta}\right) + k_q^{-1}\left(\frac{1}{\beta^2+\alpha\beta}\right). \tag{5.10}$$

Hence, simplifying (5.10) suggests to have

$$y(x) = k_q^{-1}\left(\frac{\alpha}{\beta(\beta^2-\alpha^2)}\right) + k_q^{-1}\left(\frac{1}{\beta^2+\alpha\beta}\right). \tag{5.11}$$

Hence, by referring to Part (7) and Part (9) of Table 1, we infer that

$$y(x) = \sinh_q(x) + e_q(-x).$$

This ends the solution of the given problem.

Example 3: Consider the first order q-initial value problem that is given in the from

$$D_q y(x) = \cosh(2x), \tag{5.12}$$

with the initial condition $y(0) = 1$.

Solution. Here, we have $b_1 = 0$ and $G(x) = cosh(2x)$. Therefore, by using (5.8), we obtain

$$y(x) = k_q^{-1}\left(\frac{\alpha\beta k_q(\cosh(2x);\alpha,\beta)}{\beta^2}\right) + k_q^{-1}\left(\frac{y_0}{\beta^2}\right).$$

By using Table (1) we derive

$$y(x) = k_q^{-1}\left(\frac{\alpha\beta\frac{1}{\beta^2-4\alpha^2}}{\beta^2}\right) + k_{p,q}^{-1}\left(\frac{1}{\beta^2}\right). \tag{5.13}$$

Simplify (5.13) to get

$$y(x) = k_q^{-1}\left(\frac{\alpha}{\beta^3 - 4\alpha^2\beta}\right) + k_q^{-1}\left(\frac{1}{\beta^2}\right).$$

By employing Table (1) we establish that

$$y(x) = \sinh(2x) + 1.$$

Case 2: Let's talk about the situation when $m = 2$. It is true that this reduces (5.1) to the form

$$D_q^2 y(x) + b_1 D_q y(x) + b_2 y(x) = G(x), \tag{5.14}$$

with initial conditions $D_q y(0) = y_1$ and $y(0) = y_0$.
Now, apply the k_q transform to both sides of (5.14) and use the above initial conditions to have

$$k_q(D_q^2 y(x);\alpha,\beta) + b_1 k_q(D_q y(x);\alpha,\beta) + b_2 k_q(y(x);\alpha,\beta) = k_q(G(x);\alpha,\beta).$$

Utilizing (3.4) implies

$$\frac{-y'(0)}{\alpha\beta} - \frac{y(0)}{\alpha^2} + \frac{\beta^2}{\alpha^2}k_q(y(x);\alpha,\beta) - \frac{b_1 y(0)}{\alpha\beta} + b_1\frac{\beta}{\alpha}k_q(y(x);\alpha,\beta) + b_2 k_q(y(x);\alpha,\beta) = k_{p,q}(G(x);\alpha,\beta).$$

Modifying this will provide

$$\left(\frac{\beta^2}{\alpha^2} + b_1\frac{\beta}{\alpha} + b_2\right)k_q(y(x);\alpha,\beta) = k_q(G(x);\alpha,\beta) + \frac{y_1}{\alpha\beta} + \left(\frac{1}{\alpha^2} + \frac{b_1}{\alpha\beta}\right)y_0.$$

When the equation above is simplified, it yields

$$k_q(y(x);\alpha,\beta) = \frac{k_q(G(x);\alpha,\beta) + \frac{y_1}{\alpha\beta} + \left(\frac{1}{\alpha^2} + \frac{b_1}{\alpha\beta}\right)y_0}{\frac{\beta^2}{\alpha^2} + b_1\frac{\beta}{\alpha} + b_2}.$$

The exact solution is finally obtained by applying the inverse k_q transform to both sides of the previous equation, as shown in the form

$$y = k_q^{-1}\left(\frac{k_q(G(x);\alpha,\beta)}{\frac{\beta^2}{\alpha^2} + \frac{\beta b_1}{\alpha} + b_2}\right) + k_q^{-1}\left(\frac{\frac{y_1}{\alpha\beta}}{\frac{\beta^2}{\alpha^2} + \frac{\beta b_1}{\alpha} + b_2}\right) + k_q^{-1}\left(\frac{\frac{y_0}{\alpha^2}}{\frac{\beta^2}{\alpha^2} + \frac{\beta b_1}{\alpha} + b_2}\right) + k_q^{-1}\left(\frac{\frac{b_1 y_0}{\alpha\beta}}{\frac{\beta^2}{\alpha^2} + \frac{\beta b_1}{\alpha} + b_2}\right).$$

Example 4: Consider the second order q-initial value problem $D_q^2 y(x) + 4y(x) = 0$, with the initial condition $D_q y(0) = 0$ and $y(0) = 1$.

Solution. Here, we have $b_1 = 0, b_2 = 4, y_0 = 1, y_1 = 0$ and $G(x) = 0$. Thus, by using (5.16) we get

$$y(x) = k_q^{-1}\left(\frac{\frac{1}{\alpha^2}}{\frac{\beta^2}{\alpha^2} + 4}\right).$$

Putting the equation above in a more straightforward format yields

$$y(x) = k_q^{-1}\left(\frac{1}{\beta^2 + 4\alpha^2}\right).$$

Employing Table 1 gives the solution

$$y(x) = \cos_q(2x).$$

Example 5: Consider the second order q-initial value problem $D_q^2 y(x) - y(x) = 0$, with the initial conditions $D_q y(0) = 1$ and $y(0) = 0$.
Solution. Starting from (5.16) and simplifying reveals

$$y(x) = k_q^{-1}\left(\frac{\alpha}{\beta^3 - \alpha^2\beta}\right).$$

Hence, employing Table (1) leads to the solution $y(x) = \sinh_q(x)$.

6 Conclusions and future research

In this study, we provided an overview of the historical development of quantum calculus theory, followed by an explanation of its fundamental principles. Additionally, we explained the concept of the q-KKAT and its properties, which are later explained by using the q-calculus concept. We also established the q-definition, q-convolution theory, and various properties of the first kind q-KKAT. Furthermore, we employed the q-transform properties to solve first- and second-order q-differential and q-initial value problems (IVPs) with constant coefficients, illustrating the advantages of the proposed transformation. Furthermore, the q-KKAT transform is declared to be more general than the q-Laplace and q-Sumudu transforms as its multi-parameter kernel allows the transform to represent a wider class of functions and operators, whilethe q-Laplace and q-Sumudu transforms arise as special cases under suitable parameter choices. Although the q-Laplace and q-Sumudu transforms are computationally more efficient and are well suited for solving standard q-difference equations and initial value problems, the q-KKAT transform is particularly effective for generalized models and problems involving special functions, though this generality comes at the expense of more complicated inversion formulas and increased computational effort. So, the future research directions include extensions of the q-KKAT transform to the fractional q-calculus, its application to q-partial differential equations, and the development of efficient symbolic and numerical algorithms to enhance its applicability.

Acknowledgments

We would like to thank you for **following the instructions above** very closely in advance. It will definitely save us lot of time and expedite the process of your paper's publication.

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