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Vanishing quasi-conformal curvature tensor of a certain class of almost contact metric manifolds

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Abstract. This paper focuses on the essential role of the quasi-conformal curvature tensor in developing the theoretical foundation and applications of quasi-contact metric geometry. The quasi-conformal curvature tensor of $\mathtt{NC}_{10}$-manifolds using the space associated with G structures (AGS-space) has been investigated. By deriving explicit expressions for the tensor components, the necessary and sufficient conditions for considering the quasi-conformal $\mathtt{NC}_{10}$-manifold as flat quasi-conformal have been determined. Noteworthy, it indicated that any flat quasi-conformal $\mathtt{NC}_{10}$-manifold is locally isometric to the product of the Kähler manifold and the real line, thus connecting abstract tensor properties to classical geometric structures. Moreover, the conditions for the $\mathtt{NC}_{10}$-manifold that yield the η-Einstein structure are established. New analogues of Gray's identities for the quasi-conformal curvature tensor have been introduced for $\mathtt{NC}_{10}$-manifold.

Keywords. Quasi-conformal tensor, $\mathtt{NC}_{10}$-manifold, η-Einstein manifold

1 Introduction

This study arises from the necessity to enhance our comprehension of the curvature characteristics of generalized metric structures exhibiting near-perfect connectedness, particularly in $\mathtt{NC}_{10}$-manifolds. Although classical literature has examined standard curvature tensors, including Riemannian, Nijenhuis, and Ricci tensors, the quasi-conformal curvature tensor has received less focus in this area of study. This tensor's examination yields novel insights into the fundamental geometry of $\mathtt{NC}_{10}$-manifolds and establishes significant connections with established geometric structures, including cosymplectic and Kähler manifolds. Specifically, by determining the conditions under which the $\mathtt{NC}_{10}$-manifold is identical flat and η-Einstein. This study opens the path for more accurate classifications and prospective applications in both pure mathematics and theoretical physics.

The initial study by Chinea and Gonzalez on $\mathtt{C}_{10}$-manifolds [9], together with further expansions by Rustanov [15], established an adequate basis through the examination of different curvature tensors in quasi-contact metric structures. Our current study uniquely concentrates

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on the quasi-conformal curvature tensor, an unexplored area based on these findings. Utilizing AGS-space enhances tensor analysis and establishes a connection between previous studies on Riemannian and Nijenhuis tensors and modern geometric classifications. This targeted methodology elucidates the distinctive curvature characteristics of $\mathtt{NC}_{10}$-manifolds and enhances our comprehension of their connection to symplectic and Kähler geometry. Motivated by a desire to further investigate these generalizations, Rustanov [15] presented the $\mathtt{NC}_{10}$-manifold, a construction that incorporates the almost contact metric manifold related to $\mathtt{C}_{10}$-manifolds, while providing a more flexible framework for investigating geometric properties. Rustanov's work [16] is distinguished by the integration of more complex structural features, particularly in formulating the essential structural equations for the $\mathtt{NC}_{10}$-manifold and in precisely computing the principal curvature components, such as the Riemannian curvature tensor, the Nijenhuis tensor, the Ricci tensor, and the scalar curvature, utilizing innovative applications of AGS-space. These results emphasize the significant interaction between generalized contact geometry and traditional symplectic theory. Moreover, they illustrate that the completely integrable $\mathtt{NC}_{10}$-manifold possesses characteristics that closely resemble those of symplectic manifolds. Moreover, Rustanov established that if the $\mathtt{NC}_{10}$-manifold is normal and integrable, it qualifies as a co-symplectic manifold. if the $\mathtt{NC}_{10}$-manifold possesses a closed contact form, its structure is enhanced, exceeding that of a co-symplectic manifold. Expanding upon these foundational findings, Rustanov et al. [16] advanced the theory by establishing new identities for the Riemannian curvature tensor and its subordinate curvature components.

It is worth mentioning that previous works have focused on particular forms of curvature tensors in which the Riemannian curvature tensor is an essential component, regarded as one of the fundamental components of the quasi-conformal curvature tensor under consideration. For more information, see citations [1], [2], [3], [4] and [5].

The present study enhances our comprehension of the geometric and curvature characteristics of $\mathtt{NC}_{10}$-manifolds and establishes the foundation for prospective applications in mathematical physics and dynamical systems.

2 Preliminaries

This section is devoted to clarifying the principal topics pertaining to our work. We meticulously described the complete structural equations and the distinct components of the Riemannian curvature tensor of the $\mathtt{NC}_{10}$-manifold.

An almost contact metric structure (ALM_C-structure) is a quaternary $\Gamma = (\Phi, \varsigma, g, \psi)$ of tensors, where Φ is a $(1,1)$-tensor called a *structure endomorphism of a* $C^\infty(\mathsf{M})$*-module* $\mathfrak{X}(\mathsf{M})$; ς is a one-form called a *differential contact* 1*-form*; $g = \langle ., . \rangle$ is a Riemannian metric and, ψ is a vector field called *a characteristic vector*, besides the above attached terms, the following hold:

1. $\Phi(\psi) = 0$;
2. $\varsigma \circ \Phi = 0$;
3. $\varsigma(\psi) = 1$;
4. $\Phi^2 = -id + \varsigma \otimes \psi$;
5. $\langle \Phi T, \Phi H \rangle = \langle T, H \rangle - \varsigma(T)\varsigma(H),\ T, H \in \mathfrak{X}(\mathsf{M})$.

In such a status, the $2n+1$-dimensional manifold M escorted by the aforesaid structure Γ is called an ALM_C-manifold [6]. In this paper, we refer to the $2n+1$-dimensional manifold M^{2n+1} as M.

In the module $\mathfrak{X}(\mathsf{M})$ on the ring of smooth functions $C^\infty(\mathsf{M})$, assign two alternately complementary projections υ and ϱ where $\upsilon = \varsigma \otimes \psi$ and $\varrho = -\Phi^2$; consequently $\mathfrak{X}(\mathsf{M}) = \mathfrak{V} \oplus \mathfrak{D}$, where $\mathfrak{V}$ =Im$\Phi = ker\varsigma$ and $\mathfrak{D}$ =Im$\upsilon = ker\Phi$ [11].

Lemma 2.1. [12] In ALM_C-manifold, the tensors Φ_m and g_m have the following formulas:

$$(\Phi^t_k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \ (g_{tk}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},$$

where I_n is the n-identity matrix.

For more details about the basics of the associated G-structure space (AGS-space), we refer the reader to review the citations [12] and [13].

Definition 2.1. [6] In ALM_C-structure Γ, the skew-symmetric tensor of kind, $2\ \Omega(D,E) = g(D, \Phi E)$ is called a fundamental form, where $D,\ E \in \mathfrak{X}(\mathsf{M})$.

Definition 2.2. [15] An ALM_C-structure Γ is called an $\mathtt{NC}_{10}$-structure if the condition below hold:

$$\nabla_D(\Phi)E + \nabla_E(\Phi)D = \psi\nabla_D(\varsigma)\Phi E + \psi\nabla_E(\varsigma)\Phi D + \varsigma(D)\nabla_{\Phi E}\psi + \varsigma(E)\nabla_{\Phi D}\psi; \quad D, E \in \mathfrak{X}(\mathsf{M})$$

A manifold M escorted by the $\mathtt{NC}_{10}$-structure is called an $\mathtt{NC}_{10}$-manifold.

Lemma 2.2. [15] The complete structure equations of $\mathtt{NC}_{10}$-manifold, derived from considering the AGS-space, are presented in the following forms:

1. $d\omega - F^{ab}\omega_a \wedge \omega_b = F_{ab}\omega^a \wedge \omega^b;$
2. $d\omega_a - C_{abc}\omega^b \wedge \omega^c = -\omega^b_a \wedge \omega_b + F_{ab}\omega^b \wedge \omega;$
3. $d\omega^a - C^{abc}\omega_b \wedge \omega_c = -\omega^a_b \wedge \omega^b + F^{ab}\omega_b \wedge \omega;$
4. $d\omega^a_b + \omega^a_c \wedge \omega^c_b = (A^{ad}_{bc} - C^{adh}C_{hbc} - F^{ad}F_{bc})\omega^c \wedge \omega_d;$
5. $dF_{ab} - F_{cb}\omega^c_a - F_{ac}\omega^c_b = 0;$
6. $dF^{ab} + F^{cb}\omega^a_c + F^{ac}\omega^b_c = 0;$
7. $dC^{abc} + C^{dbc}\omega^a_d + C^{adc}\omega^b_d + C^{abd}\omega^d_c = C^{abcd}\omega_d;$
8. $dC_{abc} - C_{dbc}\omega^d_a - C_{adc}\omega^d_b - C_{abd}\omega^d_c = C_{abcd}\omega^d,$
9. $dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\omega^h - A^{adh}_{bc}\omega_h.$

The tensors C^{abc} and F_{ab} are referred to as the first structure tensor and the second structure tensor, respectively, ω^i and ω^i_j are the components of the displacement form and the connection form of the Riemannian connection $\bigtriangledown$, respectively. At the same time, the following equalities are hold:

1. $C^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}}; C_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c}; C_{[abc]} = C_{abc}; C^{[abc]} = C^{abc}; \overline{C^{abc}} = C_{abc};$

2. $F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}}; F_{ab} = -\sqrt{-1}\Phi^0_{a,b}; F_{ab} + F_{ba} = 0; F^{ab} + F^{ba} = 0; \overline{F^{ab}} = F_{ab}; C_{a[bcd]} = F_{a[b}F_{cd]}; C^{a[bcd]} = F^{a[b}F^{cd]}$;

3. $A^{ad}_{[bc]} = A^{[ad]}_{bc} = 0; F_{ad}C^{dbc} = F^{ad}C_{dbc} = 0; A^{ad}_{[bch]} = A^{a[dh]}_{bc} = 0.$

Here$\{\Phi^i_{j,k}\}$ are the components of the covariant differentials of the tensor with respect to the Riemannian connection.

Lemma 2.3. [13] An ALM_C-structure Γ is normal iff, $\Phi^{\hat{a}}_{b,c}; \Phi^0_{a,0}; \Phi^{\hat{a}}_{b,0}, \Phi^0_{a,b}; \Phi^a_{\hat{b},0}; \Phi^0_{\hat{a},\hat{b}}; \Phi^a_{\hat{b},\hat{c}}$ and $\Phi^0_{\hat{a},0}$ are identical to zero.

Definition 2.3. [10] An ALM_C-structure Γ is called an almost co-symplectic structure (ALM_S-structure) if $d\Omega = 0$ and $d\varsigma = 0$.

Definition 2.4. [6] A normal ALM_S-structure is said to be co-symplectic.

The structural equations of the aforementioned structure by taking advantage of the AGS-space, are listed below [12]:

1. $d\omega = 0$;

2. $d\omega^a = -\omega^b \wedge \omega^a_b$; (2.1)

3. $d\omega_a = \omega_b \wedge \omega^b_a$.

The next lemma submits the non vanishing components for one of the important tensors of the $\mathtt{NC}_{10}$-structure by utilizing the AGS-space.

Lemma 2.4. [16] The non-zero essential components of the Riemannian tensor of the $\mathtt{NC}_{10}$-manifold are expressed as follows:

1. $\mathsf{R}^0_{a0\hat{b}} = -F_{ac}F^{cb}$;

2. $\mathsf{R}^a_{bc\hat{d}} = A^{ad}_{bc} - C^{adh}C_{hbc}$;

3. $\mathsf{R}^a_{\hat{b}cd} = 2C^{abh}C_{hcd}$;

4. $\mathsf{R}^a_{\hat{b}\hat{c}\hat{d}} = C_{acdb} - F_{ab}F_{cd}$.

It's worth noting that the other components can be obtain regarding the properties of R.

A tensor r of kind $(2,0)$ which is defined by $\mathsf{r}_{kl} = -\mathsf{R}^t_{klt}$ is called a Ricci tensor [8], and its components of the $\mathtt{NC}_{10}$-manifold with the benefit of the AGS-space are given below [16]:

1. $\mathsf{r}_{a\hat{b}} = r_{\hat{b}a} = A^{bc}_{ac} - 3C^{bcd}C_{dca} - F_{ac}F^{cb}$;

2. $\mathsf{r}_{oo} = -2F_{ab}F^{ba}$; (2.2)

3. $\mathsf{r}_{ab} = \mathsf{r}_{a0} = 0$.

Additionally, the scalar curvature is known as $\kappa = g^{kl}r_{kl}$; thus, by using the AGS-space, the scalar curvature κ takes the form

$$\kappa = 2A^{ab}_{ab} - 6C^{abc}C_{cba} - 2F_{ab}F^{ba}.$$

Definition 2.5. [17] In an $2n+1$-ALM_C-manifold M, the tensor $\check{C}$ of the type $(4;0)$ which is defined by the formula

$$\begin{aligned}\check{C}(X,Y)Z =& \mathfrak{A}\mathsf{R}(X,Y)Z + \mathfrak{B}\left[\mathsf{r}(Y,Z)X - \mathsf{r}(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\right] \\ & - \frac{\kappa}{2n+1}\left(\frac{\mathfrak{A}}{2n} + 2\mathfrak{B}\right)\left[g(Y,Z)X - g(X,Z)Y\right].\end{aligned}$$

is called a quasi-conformal curvature tensor (QC-tensor), where $\mathfrak{A}$ and $\mathfrak{B}$ are the constants that are not synchronously zero and $\check{C}_{ijkl} = -\check{C}_{jikl} = -\check{C}_{ijlk} = \check{C}_{klij}$.

Definition 2.6. [7] An ALM_C-manifold $\mathit{\Gamma} = (\mathsf{M}, \Phi, \varsigma, g, \psi)$ is said to be an η-Einstein manifold if the Ricci tensor fulfills the equalization

$$\mathsf{r}(T,H) = \alpha\langle T,H\rangle + \beta\varsigma(T)\varsigma(H) \qquad \forall T,H \in \mathfrak{X}(\mathsf{M}), \tag{2.3}$$

Here α and β belong to the set of smooth functions $C^\infty(\mathsf{M})$. Concerning β, if it is identical to zero, then M will be an Einstein manifold.

The last equation (2.3) is equivalent to the expressions presented below in the AGS-space [14]:

$$\mathsf{r}_{a\hat{b}} = \mathsf{r}_{\hat{b}a} = \alpha\delta_b^a, \quad \mathsf{r}_{00} = \alpha + \beta, \tag{2.4}$$

3 Quasi-conformal curvature tensor of flat $\mathtt{NC}_{10}$-manifold

Chapter three clarifies the intricacies of the quasi-conformal curvature tensor of $\mathtt{NC}_{10}$-manifolds. This chapter is significant since it derives explicit formulations for the tensor components and establishes the specific conditions under which an $\mathtt{NC}_{10}$-manifold transforms into a quasi-conformal space. Our study indicates that this flatness signifies a local equivalency between the product of the Kähler manifold and the real line, a finding that clarifies the fundamental geometric structure and connects contemporary abstract theory with classical constructs. These findings establish a foundation for further study and classification of these intricate manifolds.

Theorem 3.1. The non-zero essential components of the quasi-conformal curvature tensor of the $\mathtt{NC}_{10}$-manifold, on the AGS-space are expressed as follows:

1. $\check{C}_{abcd} = \mathfrak{A}(C_{acdb} - F_{ab}F_{cd})$;
2. $\check{C}_{\hat{a}\hat{b}cd} = 2\mathfrak{A}C^{abh}C_{hcd} + \mathfrak{B}[(A^{bh}_{dh} - 3C^{bhg}C_{ghd} - F_{dh}F^{hb})\delta^a_c + (A^{ah}_{ch} - 3C^{ahg}C_{ghc} - F_{ch}F^{ha})\delta^b_d - (A^{ah}_{dh} - 3C^{ahg}C_{ghd} - F_{dh}F^{ha})\delta^b_c - (A^{bh}_{ch} - 3C^{bhg}C_{ghc} - F_{ch}F^{hb})\delta^a_d] - \dfrac{\mathfrak{A}+4\mathfrak{B}}{n(2n+1)}(A^{hg}_{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{ga})\delta^{ab}_{cd}$;
3. $\check{C}_{\hat{a}bc\hat{d}} = \mathfrak{A}(A^{ad}_{bc} - C^{adh}C_{hbc}) + \mathfrak{B}[(A^{bh}_{dh} - 3C^{bhg}C_{ghd} - F_{dh}F^{hb})\delta^a_c + (A^{ah}_{ch} - 3C^{ahg}C_{ghc} - F_{ch}F^{ha})\delta^b_d] - \dfrac{\mathfrak{A}+4\mathfrak{B}}{n(2n+1)}(A^{hg}_{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{ga})\delta^a_c\delta^d_b$;
4. $\check{C}_{0a0\hat{b}} = -\mathfrak{A}F_{ac}F^{cb} + \mathfrak{B}((A^{bc}_{ac} - 3C^{bcd}C_{dca} - F_{ac}F^{cb} - 2F_{cd}F^{dc}\delta^b_a) - \dfrac{\mathfrak{A}+4\mathfrak{B}}{n(2n+1)}(A^{hg}_{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{ga})\delta^b_a$.

The remaining components can be derived from the symmetry property of $\check{C}$, where $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^a_d\delta^b_c$.

Proof. By employing Definition 2.5, Lemmas 2.4 and 2.2, we calculate the components $\check{C}_{ijkl}$ for all likely indecies as:

1. Setting a, b, c and d instead of i, j, k and l respectively, we get

$$\begin{aligned}\check{C}_{abcd} &= \mathfrak{A}\mathsf{R}_{abcd} + \mathfrak{B}(\mathsf{r}_{bd}g_{ac} + \mathsf{r}_{ac}g_{bd} - \mathsf{r}_{ad}g_{bc} - r_{bc}g_{ad}) - \frac{\kappa}{2n+1}(\frac{\mathfrak{A}}{2n} + 2\mathfrak{B})[g_{ac}g_{bd} - g_{ad}g_{bc}]\\ &= \mathfrak{A}(C_{acdb} - F_{ab}F_{cd}).\end{aligned}$$

2. Substitute $\hat{a}, \hat{b}, c$ and d instead of i, j, k and l, to derive the equation

$$\begin{aligned}\check{C}_{\hat{a}\hat{b}cd} &= \mathfrak{A}\mathsf{R}_{\hat{a}\hat{b}cd} + \mathfrak{B}(\mathsf{r}_{\hat{b}d}g_{\hat{a}c} + \mathsf{r}_{\hat{a}c}g_{\hat{b}d} - \mathsf{r}_{\hat{a}d}g_{\hat{b}c} - \mathsf{r}_{\hat{b}c}g_{\hat{a}d}) - \frac{\kappa}{2n+1}(\frac{\mathfrak{A}}{2n} + 2\mathfrak{B})[g_{\hat{a}c}g_{\hat{b}d} - g_{\hat{a}d}g_{\hat{b}c}]\\ &= 2\mathfrak{A}C^{abh}C_{hcd} + \mathfrak{B}[(A^{bh}_{dh} - 3C^{bhg}C_{ghd} - F_{dh}F^{hb})\delta^a_c + (A^{ah}_{ch} - 3C^{ahg}C_{ghc} - F_{ch}F^{ha})\\ &\quad \delta^b_d - (A^{ah}_{dh} - 3C^{ahg}C_{ghd} - F_{dh}F^{ha})\delta^b_c - (A^{bh}_{ch} - 3C^{bhg}C_{ghc} - F_{ch}F^{hb})\delta^a_d] - \frac{\mathfrak{A} + 4\mathfrak{B}}{n(2n+1)}\\ &\quad (A^{hg}_{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{ga})\delta^{ab}_{cd}.\end{aligned}$$

By applying the previously mentioned technique, we finalize the remaining components. □

Definition 3.1. An $\mathtt{NC}_{10}$-manifold M is called a quasi-conformally flat(QC-flat), if the QC-tensor vanishes.

Theorem 3.2. An $\mathtt{NC}_{10}$-manifold it is of QC-flat iff is a cosymplectic manifold.

Proof. Let M be a QC-flat $\mathtt{NC}_{10}$-manifold. Considering Theorem 3.1, item 1, we conclude

$$\mathfrak{A}(C_{acdb} - F_{ab}F_{cd}) = 0.$$

Then it means that $\mathfrak{A} = 0$ or $C_{acdb} - F_{ab}F_{cd} = 0$.
Let's consider the first case, i.e. $\mathfrak{A} = 0$. Then the identities of Theorem 3.1 will take the form:

1. $\check{C}_{abcd} = 0;$
2. $$\begin{aligned}\check{C}_{\hat{a}\hat{b}cd} &= \mathfrak{B}(\mathsf{r}_{\hat{b}d}\delta^a_c + \mathsf{r}_{\hat{a}c}\delta^b_d - \mathsf{r}_{\hat{a}d}\delta^b_c - \mathsf{r}_{\hat{b}c}\delta^a_d) - \frac{2\mathfrak{B}\kappa}{n(2n+1)}\delta^{ab}_{cd} = \mathfrak{B}[(A^{bh}_{dh} - 3C^{bhg}C_{ghd} - F_{dh}F^{hb})\\ &\quad \delta^a_c + (A^{ah}_{ch} - 3C^{ahg}C_{ghc} - F_{ch}F^{ha})\delta^b_d - (A^{ah}_{dh} - 3C^{ahg}C_{ghd} - F_{dh}F^{ha})\delta^b_c - (A^{bh}_{ch} -\\ &\quad 3C^{bhg}C_{ghc} - F_{ch}F^{hb})\delta^a_d] - \frac{4\mathfrak{B}}{n(2n+1)}(A^{hg}_{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh})\delta^{ab}_{cd}; \qquad (3.1)\end{aligned}$$
3. $$\begin{aligned}\check{C}_{\hat{a}bc\hat{d}} &= \mathfrak{B}(\mathsf{r}_{b\hat{d}}\delta^a_c + \mathsf{r}_{\hat{a}c}\delta^d_b) - \frac{\mathfrak{B}b\kappa}{n(2n+1)}\delta^a_c\delta^d_b = \mathfrak{B}[(A^{bh}_{dh} - 3C^{bhg}C_{ghd} - F_{dh}F^{hb})\delta^a_c + (A^{ah}_{ch} -\\ &\quad 3C^{ahg}C_{ghc} - F_{ch}F^{ha})\delta^b_d] - \frac{\mathfrak{B}b}{n(2n+1)}(A^{hg}_{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh})\delta^a_c\delta^d_b;\end{aligned}$$

4. $\check{C}_{0a0\hat{b}} = \mathfrak{B}(\mathrm{r}_{a\hat{b}} + \mathrm{r}_{00}\delta_a^b) - \frac{2\mathfrak{B}\kappa}{n(2n+1)}\delta_a^b = \mathfrak{B}(A_{ac}^{bc} - 3C^{bcd}C_{dca}) - \mathfrak{B}F_{ac}F^{cb} - \frac{\mathfrak{B}b}{n(2n+1)}(A_{hg}^{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh})\delta_a^b.$

Let's consider the relation $\check{C}_{0a0\hat{b}} = 0$, i.e.,

$$\mathfrak{B}\left(A_{ac}^{bc} - 3C^{bcd}C_{dca}\right) - \mathfrak{B}F_{ac}F^{cb} - \frac{\mathfrak{B}b}{n\left(2n+1\right)}\left(A_{hg}^{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh}\right)\delta_a^b = 0. \tag{3.2}$$

Let's reduce this equality by indices a and b, then we get

$$\frac{\mathfrak{B}\left(2n-3\right)}{2n+1}\left(A_{hg}^{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh}\right) = 0.$$

From the last equality, it follows that: 1. $\mathfrak{B} = 0$; 2. $A_{hg}^{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh} = 0$. The case when $\mathfrak{A} = \mathfrak{B} = 0$ is not interesting, then $A_{hg}^{hg} - 3C^{hgf}C_{fgh} - F_{hg}F^{gh} = 0$, i.e., the manifold has zero scalar curvature. Formula (3.2) will take the form

$$\mathfrak{B}\left(A_{ac}^{bc} - 3C^{bcd}C_{dca} - F_{ac}F^{cb}\right) = 0,$$

That means,

$$A_{ac}^{bc} - 3C^{bcd}C_{dca} - F_{ac}F^{cb} = 0. \tag{3.3}$$

Now let's consider the case $C_{acdb} - F_{ab}F_{cd} = 0$. Let's collapse this equality with the object F^{bh}, since $C_{abch}F^{hd} = 0$, then $F_{ab}F_{cd}F^{bh} = 0$, i.e., $F_{ab}F^{bh}F_{cd} = 0$, which implies

$$F_a^h F_{cd} = 0, \tag{3.4}$$

where $F_a^h = F_{ab}F^{bh}$. The endomorphism $\mathcal{F}$ defined in the A-frame by the matrix F_b^a is Hermitian symmetric and, therefore, diagonalizable in a suitable A-frame, i.e., in this frame, we have

$$F_b^a = F_b\delta_b^a, \tag{3.5}$$

where $\{F_a\}$ are the eigenvalues of this endomorphism. Moreover, the Hermitian form $\mathcal{F}(X,Y) = F_b^a X^b Y_a$ corresponding to this endomorphism is positive semidefinite, since

$$\mathcal{F}\left(X,X\right) = F_b^a X^b X_a = F_{bc}F^{ca}X^b X_a = \sum_c \left|F_{ac}X^a\right|^2 \geq 0.$$

Therefore, $F_a \geq 0; a = 1,\ldots,n$. Substituting (3.5) into (3.4), we obtain $F_a\delta_a^h F_{cd} = 0$, which leads to

$$F_a F_{cd} = 0. \tag{3.6}$$

The last equality is possible if and only if $F_{cd} = 0$, which means M is a closely cosymplectic manifold. Then the equality $C_{acdb} - F_{ab}F_{cd} = 0$ takes the form $C_{acdb} = 0$, i.e., the first structure tensor is covariantly constant in the Riemannian connection.

Equality (3.3) takes the form

$$A_{ac}^{bc} - 3C^{bcd}C_{dca} = 0.$$

Therefore, a QC-flat $\mathtt{NC}_{10}$-manifold is a cosymplectic manifold. It is easy to see that a cosymplectic manifold is a QC-flat $\mathtt{NC}_{10}$-manifold. □

Taking into account the equalities $F_{cd}=0$ and $A^{bc}_{ac}-3C^{bcd}C_{dca}=0$, the relations of Theorem 3.1 will take the form:

$$
\begin{aligned}
&1.\quad \check{C}_{abcd}=0;\\
&2.\quad \check{C}_{\hat{a}\hat{b}cd}=2\mathfrak{A}C^{abh}C_{hcd};\\
&3.\quad \check{C}_{\hat{a}bc\hat{d}}=\mathfrak{A}\left(A^{ad}_{bc}-C^{adh}C_{hbc}\right);\\
&4.\quad \check{C}_{0a0\hat{b}}=0.
\end{aligned}
\tag{3.7}
$$

Since $\mathfrak{A}\neq 0$, it follows from (3.7:2) that $C^{abc}=0$. Then from (3.7:3) it follows that $A^{ad}_{bc}=0$, that means the $\mathtt{NC}_{10}$-manifold is a flat cosymplectic manifold. Since a flat cosymplectic manifold is locally equivalent to the product of a flat Kähler manifold and the real line. Then, according to the Hawley-Igus theorem, a flat Kähler manifold is holomorphically isometric to the complex Euclidean space $\mathbb{C}^n$, equipped with the standard Hermitian metric $\langle\langle .,.\rangle\rangle = ds^2$ in the canonical atlas given by the relation $ds^2=\sum_{a=1}^{n} dZ^a \otimes d\bar{Z}^a)$ by the real line. Thus, we get the following theorem.

Theorem 3.3. A QC-flat $\mathtt{NC}_{10}$-manifold M is locally equivalent to the product of the complex Euclidean space $\mathbb{C}^n$ (equipped with the standard Hermitian metric $\langle\langle .,.\rangle\rangle = ds^2$ in the canonical atlas given by the relation $ds^2=\sum_{a=1}^{n} dZ^a \otimes d\bar{Z}^a)$ by the real line.

The subsequent example illustrates that when the quasi-conformal curvature tensor vanishes on an $\mathtt{NC}_{10}$-manifold, the manifold is flat, co-symplectic, and locally equivalent to the product of complex Euclidean space $\mathbb{C}^n$ and the real line $\mathbb{R}$.

Example 3.1. Consider a three-dimensional $\mathtt{NC}_{10}$-manifold with vanishing structural tensors, that is, $A^{ad}_{bc}=0$, $C^{abc}=0$, and $F^{ab}=0$. This means that the manifold is QC-flat. Geometrically, it behaves similarly to $\mathbb{C}\times\mathbb{R}$, with a Hermitian metric in the complex plane $\mathbb{C}$ and an Euclidean metric in the real line . This structure can be represented as a cylinder that extends down the real line $\mathbb{R}$, as shown in Fig. 3.1.

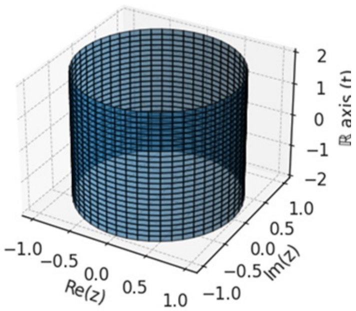

Figure 1: A cylindrical representation of $\mathbb{C}\times\mathbb{R}$, demonstrating a QC-flat $\mathtt{NC}_{10}$-manifold.

Theorem 3.4. If the manifold M of class $\mathtt{NC}_{10}$ is an η-Einstein manifold, then the rules below hold

$$\alpha=\frac{1}{n}(A^{ae}_{ae}-3C^{aed}C_{dea}-F_{ae}F^{ea}) \quad \textit{and} \quad \beta=-\frac{1}{n}(A^{ae}_{ae}-3C^{aed}C_{dea})+\frac{1-2n}{n}F_{ae}F^{ea}.$$

Proof. By comparing the relations 2.2 with the relations 2.4, we have

$$A^{be}_{ae} - 3C^{bed}C_{dea} - F_{ae}F^{eb} = \alpha\delta^b_a. \tag{3.8}$$

Contracting the equality 3.8 via the indices $(a;b)$, the following is in force:

$$\alpha = \frac{1}{n}(A^{ae}_{ae} - 3C^{aed}C_{dea} - F_{ae}F^{ea}).$$

Moreovere, since

$$-2F_{ae}F^{ea} = \alpha + \beta.$$

Therefore, we conclude

$$\beta = -\frac{1}{n}(A^{ae}_{ae} - 3C^{aed}C_{dea}) + \frac{1-2n}{n}F_{ae}F^{ea}.$$

□

4 Analogies of Gray identities of $\mathtt{NC}_{10}$-manifold

This chapter focuses on the development of novel analogues of Gray's identities within the context of $\mathtt{NC}_{10}$-manifolds. This chapter enhances our understanding of the symmetries and structural characteristics intrinsic to these geometries by adapting classical identities to the quasi-conformal curvature tensor. The insights acquired here enhance the theoretical framework of semicontinuum metric geometry and open the way for future inquiries, particularly regarding curvilinear invariants and their applications in mathematics and other related disciplines.

Definition 4.1. On the AGS-space, an $\mathtt{NC}_{10}$-manifold is called of class

1. $C\check{C}_1$ if $\check{C}_{abcd} = \check{C}_{\hat{a}bcd} = \check{C}_{\hat{a}\hat{b}cd} = 0$;
2. $C\check{C}_2$ if $\check{C}_{abcd} = \check{C}_{\hat{a}bcd} = 0$;
3. $C\check{C}_3$ if $\check{C}_{\hat{a}bcd} = 0$.

Theorem 4.1. An $\mathtt{NC}_{10}$-manifold M is a manifold of class $C\check{C}_3$.

Theorem 4.2. An $\mathtt{NC}_{10}$-manifold M is a manifold of class $C\check{C}_1$, iff M is a closely cosymplectic manifold.

Proof. Based on Definition 4.1 and Theorem 3.1; item 1, it follows that

$$\mathfrak{A}(C_{abcd} - F_{ab}F_{cd}) = 0. \tag{4.1}$$

We consider only the intersting case when $C_{abcd} - F_{ab}F_{cd} = 0$, which gives

$$C_{abcd} = F_{ab}F_{cd}. \tag{4.2}$$

Contracting 4.2 by F^{bh}, Consequently we deduce

$$C_{abcd}F^{bh} = F_{ab}F_{cd}F^{bh}. \tag{4.3}$$

Thus, for an $\mathtt{NC}_{10}$-manifold, we obtain

$$C_{abcd}F^{bh} = 0.$$

According to equation4.3, we get

$$F_{ab}F_{cd}F^{bh} = 0. \tag{4.4}$$

Reasoning in the same way as in the proof of Theorem 3.2, we obtain that from the last equality we have

$$F_{ab} = 0. \tag{4.5}$$

The converse is also true, that is, if $F_{ab} = 0$, then (4.1) is satisfied.

Subsutituting (4.5) into (4.2) consequently, we obtain

$$C_{acdb} = 0. \tag{4.6}$$

From (4.6) it follows that $\nabla C_{bcd} = 0$, thus C_{abc} is constant. Therefore, the manifold is closely cosymplectic. By reasoning in reverse order, it is straightforward to demonstrate that a closely cosymplectic manifold qualified as a $\mathtt{NC}_{10}$-10-manifold of class $C\check{C}_1$. □

It follows from equality (4.6) that the first structure tensor is covariantly constant in the Riemannian connection. Therefore, directly we get the following corollary.

Corollary 4.1. If M is an $\mathtt{NC}_{10}$-manifold of class $C\check{C}_1$, then the first is covariantly constant in the Riemannian connection and second structure tensors identically vanish.

Theorem 4.3. Suppose that M is an $\mathtt{NC}_{10}$-manifold of class $C\check{C}_2$, then the first structure tensor is parallel with respect to the Riemannian connection.

Proof. Let M be an $\mathtt{NC}_{10}$-manifold of class $C\check{C}_2$,
According to Definition 4.1, we have

$$\check{C}_{abcd} = 0 \text{ and } \check{C}_{ab\hat{c}\hat{d}}.$$

Based on Theorem 3.1, it follows that

$$\mathfrak{A}(C_{abcd} - F_{ab}F^{cd}) = 0. \tag{4.7}$$

Using the same technique in Theorem 4.2, it follows that:

$$F_{ab} = 0.$$

Thus (4.7) reduces to

$$C_{abcd} = 0. \tag{4.8}$$

From (4.8) it follows that $\nabla C_{bcd} = 0$, implies that C_{abc} is constant. Therefore, the first structure tensor is parallel with respect to the Riemannian connection. □

Theorem 4.4. An $\mathtt{NC}_{10}$-manifold is classified as a manifold of class $C\check{C}_2$ if and only if it is a closely cosymplectic manifold.

Proof. Suppose that M is an $\mathtt{NC}_{10}$-manifold of class $C\check{C}_2$. According to Definition 4.1, item 2 and Theorem 3.1, item1, we have

$$\mathfrak{A}(C_{abcd} - F_{ab}F_{cd}) = 0. \tag{4.9}$$

Consider the geometrically interesting case when $\mathfrak{A} \neq 0$, it follows that:

$$C_{abcd} = F_{ab}F_{cd}. \tag{4.10}$$

Contracting (4.10) by F_{bh}, according to Lemma 2.2, implies that:

$$F_{ab}F_{cd}F_{bh} = 0. \tag{4.11}$$

Using the same argument as in Theorem 4.2, we deduce that:

$$F_{cd} = 0. \tag{4.12}$$

Therefore, M is closely cosymplectic. Furthermore, for $F_ab = 0$, the equation $C_{abcd} = F_{ab}F_{cd}$ transforms to $C_{abcd} = 0$, meaning that the first structure tensor C_{abcd} vanishes .

Conversely, suppose that M is closely cosymplectic, indicating that

$$F_{ab} = 0.$$

Based on Definition 4.1, we conclude

$$\check{C}_{acdb} = \mathfrak{A}(C_{abcd} - F_{ab}F_{cd}) = \mathfrak{A}C_{abcd}. \tag{4.13}$$

In a closely cosymplectic $\mathtt{NC}_{10}$-manifold, the first structure tensor is covariantly constant with respect to the Riemannian connection, leading to C_{abcd}. Therefore, from (4.13) we have $\check{C}_{acdb} = 0$. Based on $F_{ab} = 0$, substituting into the remaining component formulas of Theorem 3.1 provides

$$\check{C}_{\hat{a}cdb} = 0.$$

Therefore, we have both $\check{C}_{acdb} = 0$ and $\check{C}_{\hat{a}cdb} = 0$, so by Definition 4.1, item 2, we obtain that M is $\mathtt{NC}_{10}$ manifold of class $C\check{C}_2$. □

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