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Matrix Parameters and Perfect 3-Colorings of Symmetric Cubic Graphs

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Abstract. A graph G is said to be perfectly m-colorable if its vertex set can be partitioned into m subsets $V_1, V_2, \ldots, V_m$, such that the number of neighbors of a vertex in V_i that lie in V_j is constant and depends only on i and j. This constant is denoted by a_{ij}, and the corresponding $m \times m$ matrix $A = (a_{ij})$ is called the parameter matrix of the perfect m-coloring of the graph. In this paper, we classify all possible parameter matrices for perfect 3-colorings in cubic graphs and explore key properties of perfectly colorable cubic graphs. Additionally, we determine the parameter matrices for several well-known cubic graphs.

1 Introduction

This paper concerns a special type of graph coloring, called perfect coloring. As with other types of graph coloring, this concept is related to a partition of the vertex set satisfying a prescribed structural property. The concept is rooted in coding theory; in fact, it can be viewed as a generalization of completely regular codes [1]. Our definitions and terminology are taken from [2]. Let $G = (V, E)$ be a graph. A partition $(V_1, V_2, \ldots, V_m)$ of V is called a *perfect m-coloring* of G if, for every i, j with $1 \leq i, j \leq m$, each vertex $v \in V_i$ has exactly a_{ij} neighbors in V_j. The matrix $A = (a_{ij})_{1 \leq i,j \leq m}$ is called the *parameter matrix* of the perfect m-coloring.

Two parameter matrices A and A' are called *isomorphic* if there exists a permutation $\sigma \in S_m$ acting simultaneously on the rows and columns of A and transforming A into A'. Such a permutation σ is also called a *permissible permutation*. It is clear that, if A and A' are isomorphic, then the existence of a perfect coloring with parameter matrix A is equivalent to the existence of a perfect coloring with parameter matrix A'.

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A general question in this area, as in other areas related to graph coloring, is the following: Which graphs G admit a perfect m-coloring? A more specific question is: Which matrices can occur as parameter matrices of perfect m-colorings of a given graph G? Since completely regular codes have algebraic constructions and perfect m-colorings naturally arise from this framework, the initial approach to this topic was mainly algebraic. For more information, see [3, 4, 5]. Over time, graph-theoretical methods have increasingly been used to study such problems; see, for example, [6, 7].

In this paper, we focus on cubic graphs and study the existence of perfect 3-colorings for them. Our approach is based on a comparison with the possible parameter matrices. In the first section, after presenting several properties of parameter matrices, we determine all possible parameter matrices for perfect 3-colorings of cubic graphs. Moreover, for each obtained parameter matrix, we describe the graph-theoretical properties of cubic graphs that admit a perfect 3-coloring with that parameter matrix. In the second section, we investigate symmetric cubic graphs. These graphs are the tetrahedral graph, the utility graph, the cubical graph, the Petersen graph, the Heawood graph, the Möbius–Kantor graph, the Pappus graph, the dodecahedral graph, the Desargues graph, the Nauru graph, $F26A$, the Coxeter graph, the Levi graph, also known as the Tutte–Coxeter graph, and the Dyck graph.

Let $G = (V, E)$ be a graph, and let $S \subseteq V$. The *external neighborhood* of S in G is defined as the set of all vertices in $V \setminus S$ that are adjacent to at least one vertex of S. Let M be a matching in a graph $G = (V, E)$. If the subgraph induced by the vertices incident with the edges of M has edge set precisely M, then M is called an *induced matching*. Let C be a union of cycles in a graph $G = (V, E)$. If the subgraph induced by the vertices of C is precisely C, then C is called an *induced union of cycles*. Since the present paper is concerned with perfect 3-colorings, such a coloring is denoted by (W, B, R), where $|W| \leq |B| \leq |R|$. The sets W, B, and R are referred to as the white, black, and red color classes, respectively.

2 Elementary Properties of Parameter Matrices

In this section, we first review some properties of the parameter matrix of a perfect m-coloring of a k-regular connected graph. We then study perfect 3-colorings of cubic connected graphs in detail.

Theorem 2.1. Suppose that $A = [a_{ij}]$ is a parameter matrix for a perfect m-coloring of a k-regular connected graph $G = (V, E)$.

1. For every i, we have $\sum_{j=1}^{m} a_{ij} = k$.

2. If V_i denotes the set of all vertices of color i, then $G[V_i]$ is an a_{ii}-regular graph.
3. For every i, $1 \leq i \leq m$, we have $0 \leq a_{ii} < k$, and for every i, j, $1 \leq i, j \leq m$, we have $0 \leq a_{ij} \leq k$.
4. For every i, j, $1 \leq i, j \leq m$, if $a_{ij} = 0$, then $a_{ji} = 0$.
5. For every i, j, $1 \leq i, j \leq m$, we have $|V_i|a_{ij} = |V_j|a_{ji}$.
6. If $|V_1| \leq |V_2| \leq \cdots \leq |V_m|$, then, for every i, j with $1 \leq i < j \leq m$, we have $a_{ji} \leq a_{ij}$.

Proof. 1. For each i, $1 \leq i \leq m$, a_{ij} denotes the number of neighbors of $v \in V_i$ in V_j, and $\deg(v) = k$.
2. Obvious.
3. Since G is connected, at least one vertex $v \in V_i$ has a neighbor in $V \setminus V_i$. Thus, $\deg_{G[V_i]}(v) < k$.
4. The condition $a_{ij} = 0$ means that there is no edge between V_i and V_j, and hence $a_{ji} = 0$.
5. The number of edges between V_i and V_j is $|V_i|a_{ij} = |V_j|a_{ji}$.
6. Obvious. □

By Parts 2 and 3 of Theorem 2.1, the number of possible parameter matrices is at most $k^m(k+1)^{m(m-1)}$. However, the remaining parts of the theorem impose further restrictions, and therefore the actual number of admissible matrices is smaller. Since one of our objectives is to determine the exact number of non-isomorphic parameter matrices arising from perfect 3-colorings of connected cubic graphs, we apply Theorem 2.1 to establish the following proposition.

Proposition 2.1. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ be the parameter matrix of a perfect 3-coloring, (W, B, R), of a cubic connected graph $G = (V, E)$, where $|W| \leq |B| \leq |R|$. Then
1.

$$a + b + c = 3,$$
$$d + e + f = 3,$$
$$g + h + i = 3.$$

2. $a, e, i \in \{0, 1, 2\}$ and $a = 0, 1$ or 2 if and only if W is an independent set, an induced matching or a disjoint union of induced cycles, $e = 0, 1$ or 2 if and only if B is an

independent set, an induced matching or an induced cycle and $i = 0, 1$ or 2 if and only if R is an independent set, an induced matching or an induced cycle.

3. $b, c, d, f, g, h \in \{0, 1, 2, 3\}$.

4.

$$b = 0 \Leftrightarrow d = 0,$$
$$c = 0 \Leftrightarrow g = 0,$$
$$f = 0 \Leftrightarrow h = 0.$$

5.

$$b \geq d, c \geq g, f \geq h.$$

6.

$$cdh = bfg.$$

7.

$$gf \leq hc, dh \leq bf, bg \leq dc.$$

8.

$$b = 0 \Rightarrow c, f \neq 0,$$
$$c = 0 \Rightarrow b, f \neq 0,$$
$$f = 0 \Rightarrow b, c \neq 0.$$

Proof.

1. This follows immediately from Part 1 of Theorem 2.1.
2. By Part 3 of Theorem 2.1, we have

   $$a, e, i \in \{0, 1, 2\}.$$

   Moreover, by Part 2 of Theorem 2.1, each of the induced subgraphs $G[W]$, $G[B]$, and $G[R]$ is either 0-regular, 1-regular, or 2-regular. In other words, each color class induces either an independent set, an induced matching, or a disjoint union of induced cycles.
3. This follows immediately from Part 3 of Theorem 2.1.
4. This follows immediately from Part 4 of Theorem 2.1.

5. This follows immediately from Part 6 of Theorem 2.1.

6. By Part 5 of Theorem 2.1, we have

$$b|W| = d|B|, \qquad c|W| = g|R|, \qquad f|B| = h|R|.$$

Hence,

$$1 = \frac{|W|}{|B|} \cdot \frac{|R|}{|W|} \cdot \frac{|B|}{|R|} = \frac{d}{b} \cdot \frac{c}{g} \cdot \frac{h}{f} = \frac{cdh}{bfg}.$$

Therefore,

$$cdh = bfg.$$

7. By Part 5 of Theorem 2.1, we have

$$b|W| = d|B|, \qquad f|B| = h|R|.$$

Thus,

$$|B| = \frac{b|W|}{d} = \frac{h|R|}{f}.$$

It follows that

$$\frac{|W|}{|R|} = \frac{dh}{bf}.$$

Consequently,

$$dh \leq bf.$$

Similarly, one obtains

$$gf \leq hc, \qquad bg \leq dc.$$

8. If $b = 0$, then there are no edges between W and B. Since G is connected, there must be some edges between W and R, and also between B and R. Therefore,

$$c \neq 0 \qquad \text{and} \qquad f \neq 0.$$

The other two cases are proved in a similar way.

□

As explained above, an immediate consequence of Parts 2 and 3 of Theorem 2.1 is that the number of such matrices is at most $3^3 \times 4^6$. Using Proposition 2.1 together with the next proposition, which provides a formula for computing the sizes of the color classes, this number is reduced to 33.

Proposition 2.2. [4] Let (W, B, R) be a perfect 3-coloring of a graph G with parameter matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

1. If $f = h = 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c} + 1 + \frac{bg}{cd}}.$$

2. If $c = g = 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{dh}{bf}}.$$

3. If $b = d = 0$, then

$$|W| = \frac{|V(G)|}{\frac{c}{g} + 1 + \frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h} + 1 + \frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

4. If $b, c, f \neq 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d} + 1 + \frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b} + 1 + \frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f} + 1 + \frac{g}{c}}.$$

By combining Theorem 2.1 and Proposition 2.1, the following 33 possible parameter matrices are obtained. These matrices are derived under the assumption $|W| \leq |B| \leq |R|$. By Parts 1, 3, and 4 of Theorem 2.1, the following matrices are possible:

1) the first row of a parameter matrix can be $[0, 0, 3]$, $[0, 1, 2]$, $[0, 2, 1]$, $[0, 3, 0]$, $[1, 0, 2]$, $[1, 1, 1]$, $[1, 2, 0]$, $[2, 0, 1]$ or $[2, 1, 0]$.
2) the second row can be $[0, 0, 3]$, $[0, 1, 2]$, $[0, 2, 1]$, $[1, 0, 2]$, $[1, 1, 1]$, $[1, 2, 0]$, $[2, 0, 1]$, $[2, 1, 0]$ or $[3, 0, 0]$, and
3) the third row can be $[0, 1, 2]$, $[0, 2, 1]$, $[0, 3, 0]$, $[1, 0, 2]$, $[1, 1, 1]$, $[1, 2, 0]$, $[2, 0, 1]$, $[2, 1, 0]$ or $[3, 0, 0]$.
Suppose first that the first row is $[0, 0, 3]$. By Part 4 of Proposition 2.1, we have $[d, e, f] \in \{[0, 0, 3], [0, 1, 2], [0, 2, 1]\}$.

If $[a, b, c] = [0, 0, 3]$ and $[d, e, f] = [0, 0, 3]$, then, by Part 4 of Proposition 2.1, we have $g, h \geq 1$. Moreover, by Part 7 of Proposition 2.1, $g \leq h$. Hence, the matrices M_{16} and M_{17} are obtained.

If $[a, b, c] = [0, 0, 3]$ and $[d, e, f] = [0, 1, 2]$, then again $g, h \geq 1$. Furthermore, by Part 7 of Proposition 2.1, we have $g \neq 2$. Thus, the matrices M_{18} and M_{19} are obtained.

If $[a, b, c] = [0, 0, 3]$ and $[d, e, f] = [0, 2, 1]$, then Parts 4 and 5 of Proposition 2.1 imply that $h = 1$. Hence, the matrices M_{20} and M_{21} are obtained.

Now suppose that the first row is $[0,1,2]$. By Part 5 of Proposition 2.1, we have $[d,e,f] \in \{[1,0,2],[1,1,1],[1,2,0]\}$.

If $[a,b,c]=[0,1,2]$ and $[d,e,f]=[1,0,2]$, then, by Part 4 of Proposition 2.1, we have $g,h \geq 1$. Moreover, Part 6 of Proposition 2.1 forces $g=h$. Therefore, the matrix M_{30} is obtained.

If $[a,b,c]=[0,1,2]$ and $[d,e,f]=[1,1,1]$, then Part 6 of Proposition 2.1 forces $g=2$. Hence, the matrix M_{29} is obtained.

If $[a,b,c]=[0,1,2]$ and $[d,e,f]=[1,2,0]$, then, by Part 4 of Proposition 2.1, we have $h=0$. Also, by Parts 4 and 5 of Proposition 2.1, $1 \leq g \leq 2$. Thus, the matrices M_1 and M_2 are obtained. The remaining matrices are obtained in a similar way.

$$M_1=\begin{bmatrix}0&1&2\\1&2&0\\2&0&1\end{bmatrix}\quad M_2=\begin{bmatrix}0&1&2\\1&2&0\\1&0&2\end{bmatrix}\quad M_3=\begin{bmatrix}0&2&1\\2&1&0\\1&0&2\end{bmatrix}\quad M_4=\begin{bmatrix}1&1&1\\1&2&0\\1&0&2\end{bmatrix}\quad M_5=\begin{bmatrix}0&3&0\\1&1&1\\0&1&2\end{bmatrix}$$

$$M_6=\begin{bmatrix}0&3&0\\2&0&1\\0&1&2\end{bmatrix}\quad M_7=\begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}\quad M_8=\begin{bmatrix}0&3&0\\1&0&2\\0&2&1\end{bmatrix}\quad M_9=\begin{bmatrix}1&2&0\\1&1&1\\0&1&2\end{bmatrix}\quad M_{10}=\begin{bmatrix}1&2&0\\2&0&1\\0&1&2\end{bmatrix}$$

$$M_{11}=\begin{bmatrix}1&2&0\\1&0&2\\0&1&2\end{bmatrix}\quad M_{12}=\begin{bmatrix}1&2&0\\1&0&2\\0&2&1\end{bmatrix}\quad M_{13}=\begin{bmatrix}2&1&0\\1&1&1\\0&1&2\end{bmatrix}\quad M_{14}=\begin{bmatrix}2&1&0\\1&0&2\\0&1&2\end{bmatrix}\quad M_{15}=\begin{bmatrix}2&1&0\\1&0&2\\0&2&1\end{bmatrix}$$

$$M_{16}=\begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix}\quad M_{17}=\begin{bmatrix}0&0&3\\0&0&3\\1&2&0\end{bmatrix}\quad M_{18}=\begin{bmatrix}0&0&3\\0&1&2\\1&1&1\end{bmatrix}\quad M_{19}=\begin{bmatrix}0&0&3\\0&1&2\\1&2&0\end{bmatrix}\quad M_{20}=\begin{bmatrix}0&0&3\\0&2&1\\1&1&1\end{bmatrix}$$

$$M_{21}=\begin{bmatrix}0&0&3\\0&2&1\\2&1&0\end{bmatrix}\quad M_{22}=\begin{bmatrix}1&0&2\\0&0&3\\1&2&0\end{bmatrix}\quad M_{23}=\begin{bmatrix}1&0&2\\0&1&2\\1&1&1\end{bmatrix}\quad M_{24}=\begin{bmatrix}1&0&2\\0&1&2\\1&2&0\end{bmatrix}\quad M_{25}=\begin{bmatrix}1&0&2\\0&2&1\\1&1&1\end{bmatrix}$$

$$M_{26}=\begin{bmatrix}1&0&2\\0&2&1\\2&1&0\end{bmatrix}\quad M_{27}=\begin{bmatrix}2&0&1\\0&1&2\\1&2&0\end{bmatrix}\quad M_{28}=\begin{bmatrix}2&0&1\\0&2&1\\1&1&1\end{bmatrix}\quad M_{29}=\begin{bmatrix}0&1&2\\1&1&1\\2&1&0\end{bmatrix}\quad M_{30}=\begin{bmatrix}0&1&2\\1&0&2\\1&1&1\end{bmatrix}$$

$$M_{31}=\begin{bmatrix}0&2&1\\2&0&1\\1&1&1\end{bmatrix}\quad M_{32}=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\quad M_{33}=\begin{bmatrix}1&1&1\\1&0&2\\1&2&0\end{bmatrix}$$

We are not yet satisfied with this result. Therefore, we use permissible permutations to determine the isomorphism classes of these matrices and reduce the list of 33 matrices to 18. For example, by applying the permissible permutation $\sigma=\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$, the matrix M_3 is obtained from M_1. Hence, M_1 and M_3 are isomorphic matrices, and one of them can be omitted. All such permissible permutations are listed in Table 1.

Under the isomorphism relation, the 33 parameter matrices form equivalence classes.

Table 1: Permissible permutations determine isomorphic classes

	$\sigma(1)$	$\sigma(2)$	$\sigma(3)$		$\sigma(1)$	$\sigma(2)$	$\sigma(3)$
$M_3 \to M_1$	1	3	2	$M_{10} \to M_1$	3	1	2
$M_{15} \to M_1$	2	1	3	$M_{26} \to M_1$	3	2	1
$M_{27} \to M_1$	2	3	1	$M_{14} \to M_2$	2	1	3
$M_{13} \to M_4$	2	1	3	$M_{28} \to M_4$	3	2	1
$M_{20} \to M_5$	1	3	2	$M_{21} \to M_6$	1	3	2
$M_{19} \to M_8$	1	3	2	$M_{25} \to M_9$	1	3	2
$M_{24} \to M_{10}$	1	3	2	$M_{31} \to M_{29}$	1	3	2
$M_{33} \to M_{29}$	2	1	3				

Table 1 yields the following classification into equivalence classes:

$$C_1 = \{M_1, M_3, M_{10}, M_{15}, M_{26}, M_{27}\}, \quad C_2 = \{M_2, M_{14}\}, \quad C_3 = \{M_4, M_{13}, M_{28}\},$$
$$C_4 = \{M_5, M_{20}\}, \quad C_5 = \{M_6, M_{21}\}, \quad C_6 = \{M_7\}, \quad C_7 = \{M_8, M_{19}\},$$
$$C_8 = \{M_9, M_{25}\}, \quad C_9 = \{M_{11}\}, \quad C_{10} = \{M_{12}, M_{24}\}, \quad C_{11} = \{M_{16}\},$$
$$C_{12} = \{M_{17}\}, \quad C_{13} = \{M_{18}\}, \quad C_{14} = \{M_{22}\}, \quad C_{15} = \{M_{23}\},$$
$$C_{16} = \{M_{29}, M_{31}, M_{33}\}, \quad C_{17} = \{M_{30}\}, \quad C_{18} = \{M_{32}\}.$$

From each equivalence class, we choose one matrix as its representative. For each i, $1 \leq i \leq 18$, the matrix A_i denotes the representative of the class C_i. In this paper:

$$A_1 = M_1 \quad A_2 = M_2 \quad A_3 = M_4 \quad A_4 = M_5 \quad A_5 = M_6 \quad A_6 = M_7$$
$$A_7 = M_8 \quad A_8 = M_9 \quad A_9 = M_{11} \quad A_{10} = M_{12} \quad A_{11} = M_{16} \quad A_{12} = M_{17}$$
$$A_{13} = M_{18} \quad A_{14} = M_{22} \quad A_{15} = M_{23} \quad A_{16} = M_{29} \quad A_{17} = M_{30} \quad A_{18} = M_{32}$$

The following theorem provides a useful criterion for determining whether a given matrix can be a parameter matrix.

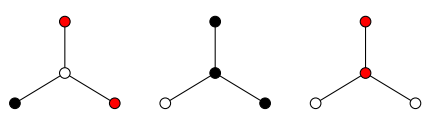
Theorem 2.2. [10] If A is the parameter matrix of a perfect m-coloring of a graph G, then each eigenvalue of A belongs to the spectrum of G.

In light of Theorem 2.2, if a matrix A has an eigenvalue that does not belong to the spectrum of G, then A cannot be the parameter matrix of a perfect m-coloring of G. Therefore, the eigenvalues of these 18 parameter matrices provide useful information.

Table 2: Spectrum of parameter matrices.

Matrix	λ_1	λ_2	λ_3	Matrix	λ_1	λ_2	λ_3
A_1	3	$\sqrt{3}$	$-\sqrt{3}$	A_2	3	2	-1
A_3	3	2	0	A_4	3	$\sqrt{2}$	$-\sqrt{2}$
A_5	3	$\frac{-1+\sqrt{17}}{2}$	$\frac{-1-\sqrt{17}}{2}$	A_6	3	1	-2
A_7	3	$-1+\sqrt{2}$	$-1-\sqrt{2}$	A_8	3	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
A_9	3	$\sqrt{2}$	$-\sqrt{2}$	A_{10}	3	1	-2
A_{11}	3	0	-2	A_{12}	3	0	-3
A_{13}	3	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	A_{14}	3	$-1+\sqrt{3}$	$-1-\sqrt{3}$
A_{15}	3	1	-1	A_{16}	3	0	-2
A_{17}	3	-1	-1	A_{18}	3	0	0

Table 2 presents the spectra of these matrices. On the other hand, for some families of cubic graphs, determining parameter matrices by the spectral method alone is not possible. In such cases, the graph-theoretical properties of parameter matrices provide a useful tool. We continue this section by describing these properties for all 18 possible parameter matrices. Let A_1 be the parameter matrix of a perfect 3-coloring of a cubic graph G. We determine the graph-theoretical properties associated with this matrix. An immediate consequence of Proposition 2.2 is $|W| = |B| = |R| = \frac{n}{3}$. Moreover, Part 2 of Proposition 2.1 implies that W, B, and R are an independent set, a union of induced cycles, and an induced matching, respectively. The neighborhoods of the vertices of G with respect to this coloring are illustrated below:



Proceeding in the same manner, for each i, $2 \leq i \leq 18$, we obtain similar graph-theoretical information, as shown in Table 3. In this table, independent sets, induced cycles, and induced matchings are denoted by "I", "C", and "M", respectively. The information presented in Table 3 leads to several useful observations about cubic graphs G of order n admitting perfect 3-colorings. We close this section with the following theorem, which collects all these observations.

Theorem 2.3. If a 3-regular graph G of order n has perfect 3-coloring with parameter matrix

1. A_1, A_3, A_{12}, A_{16} or A_{18}, then $6|n$.
2. A_2 or A_{17}, then $4|n$. (for A_2, n should be greater than 8)
3. A_4, A_7 or A_9, then $14|n$.

Table 3: Complete properties of all parameter matrices

Par. M	$\lvert W\rvert$	$\lvert B\rvert$	$\lvert R\rvert$	W	B	R			
A_1	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	I	C	M			
A_2	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	I	C	C			
A_3	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	M	C	C			
A_4	$\frac{n}{7}$	$\frac{3n}{7}$	$\frac{3n}{7}$	I	M	C			
A_5	$\frac{n}{4}$	$\frac{3n}{8}$	$\frac{3n}{8}$	I	I	C			
A_6	$\frac{n}{10}$	$\frac{3n}{10}$	$\frac{3n}{5}$	I	I	C			
A_7	$\frac{n}{7}$	$\frac{3n}{7}$	$\frac{3n}{7}$	I	I	M			
A_8	$\frac{n}{5}$	$\frac{2n}{5}$	$\frac{2n}{5}$	M	M	C			
A_9	$\frac{n}{7}$	$\frac{2n}{7}$	$\frac{4n}{7}$	M	I	C			
A_{10}	$\frac{n}{5}$	$\frac{2n}{5}$	$\frac{2n}{5}$	M	I	M			
A_{11}	$\frac{n}{5}$	$\frac{n}{5}$	$\frac{3n}{5}$	I	I	M			
A_{12}	$\frac{n}{6}$	$\frac{n}{3}$	$\frac{n}{2}$	I	I	I			
A_{13}	$\frac{2n}{11}$	$\frac{3n}{11}$	$\frac{6n}{11}$	I	M	M			
A_{14}	$\frac{3n}{13}$	$\frac{4n}{13}$	$\frac{6n}{13}$	M	I	I			
A_{15}	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	M	M	M			
A_{16}	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	I	M	I			
A_{17}	$\frac{n}{4}$	$\frac{n}{4}$	$\frac{n}{2}$	I	I	M			
A_{18}	$\frac{n}{3}$	$\frac{n}{3}$	$\frac{n}{3}$	M	M	M			

4. A_5 and A_{15}, then $8|n$.

5. A_6, A_8, A_{10} or A_{11}, then $10|n$.

6. A_{13}, then $22|n$.

7. A_{14}, then $26|n$.

Proof. 1. Suppose that G admits a perfect 3-coloring with parameter matrix A_1, A_3, A_{16}, or A_{18}. Then one of the color classes has size $\frac{n}{3}$ and induces a matching. Hence, this color class has an even number of vertices, and therefore $6 \mid n$. Moreover, if G admits a perfect 3-coloring with parameter matrix A_{12}, then it has a color class of size $\frac{n}{6}$. Hence, $6 \mid n$.

2. If G admits a perfect 3-coloring with parameter matrix A_2 or A_{17}, then it has a color class of size $\frac{n}{4}$. Therefore, $4 \mid n$.

3. Suppose that G admits a perfect 3-coloring with parameter matrix A_4, A_7, or A_9. Then it has a color class of size either $\frac{n}{7}$ or $\frac{3n}{7}$ which induces a matching. Hence, this color class has an even number of vertices, and so $14 \mid n$.

4. If G admits a perfect 3-coloring with parameter matrix A_5, then it has a color class of size $\frac{3n}{8}$. Hence, $8 \mid n$. Also, if G admits a perfect 3-coloring with parameter matrix A_{15}, then it has a color class of size $\frac{n}{4}$ which induces a matching. Therefore, $8 \mid n$.

5. If G admits a perfect 3-coloring with parameter matrix A_6, then it has a color class of size $\frac{3n}{10}$. Hence, $10 \mid n$. If G admits a perfect 3-coloring with parameter matrix A_8, A_{10}, or A_{11}, then it has a color class of size either $\frac{n}{5}$ or $\frac{3n}{5}$ which induces a matching. Therefore, $10 \mid n$.

6. If G admits a perfect 3-coloring with parameter matrix A_{13}, then it has a color class of size $\frac{3n}{11}$ which induces a matching. Hence, $22 \mid n$.

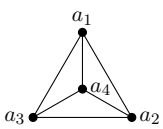
7. If G admits a perfect 3-coloring with parameter matrix A_{14}, then it has a color class of size $\frac{3n}{13}$ which induces a matching. Hence, $26 \mid n$.

□

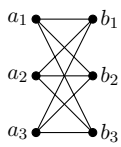
3 Small symmetric cubic graph

In this section, all symmetric cubic graphs of order at most 32 are studied, and their parameter matrices are determined based on the results of the previous section. These graphs are the tetrahedral graph, the utility graph, the cubical graph, the Petersen graph, the Heawood graph, the Möbius–Kantor graph, the Pappus graph, the dodecahedral graph, the Desargues graph, the Nauru graph, F26A, the Coxeter graph, the Levi graph, also known as the Tutte–Coxeter graph, and the Dyck graph. Some of

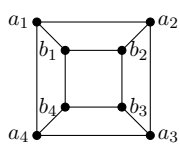
these graphs have already been studied completely, namely the tetrahedral, cubical, and dodecahedral graphs [5], the Petersen graph [4], the Heawood graph [9], and the Möbius–Kantor graph [8]. Nevertheless, these cases are reproved here in order to give a more complete and self-contained presentation. The full list of results is given below.



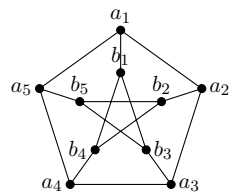
tetrahedral



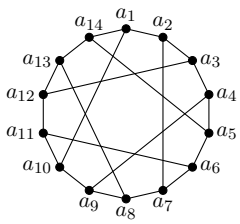
utility



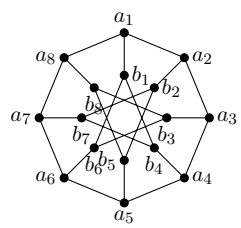
cubical



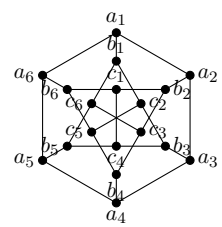
Petersen



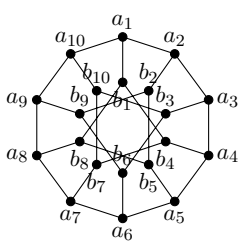
Heawood



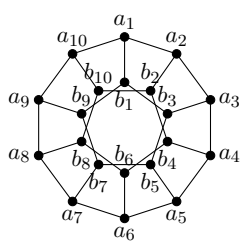
Möbius – Kantor



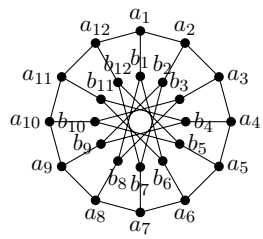
Pappus



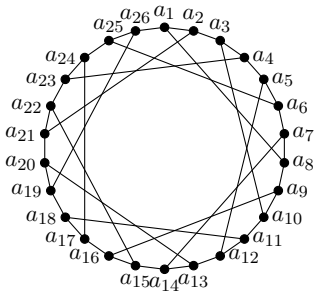
Desargues



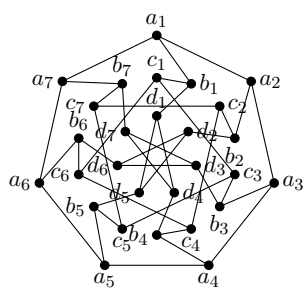
dodecahedral



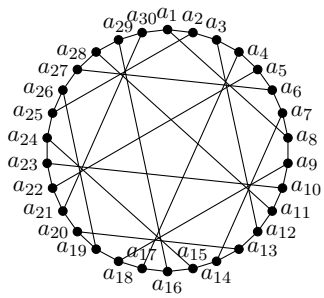
Nauru



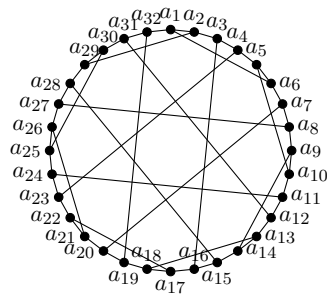
*F*26*A*



Coxeter



Levi



Dyck

Theorem 2.3 imposes restrictions on the set of possible parameter matrices for these graphs. All such matrices are listed in Table 4. Moreover, by applying Theorem 2.2 to the data in Table 4, the set of possible parameter matrices is further restricted. The resulting possible parameter matrices, obtained from Theorems 2.2 and 2.3, are presented in Table 5.

Table 4: Probable parameter matrices by Theorem 2.3

name	order	parameter matrices
tetrahedral	4	A_2 & A_{17}
utility	6	A_1, A_3, A_{12}, A_{16} & A_{18}
cubical	8	A_2, A_5, A_{15} & A_{17}
Petersen	10	A_6, A_8, A_{10} & A_{11}
Heawood	14	A_4, A_7 & A_9
Möbius-Kantor	16	A_2, A_5, A_{15} & A_{17}
Pappus	18	A_1, A_3, A_{12}, A_{16} & A_{18}
Desargues	20	A_2, A_6, A_8, A_{10}, A_{11} & A_{17}
dodecahedral	20	A_2, A_6, A_8, A_{10}, A_{11} & A_{17}
Nauru	24	A_1, A_2, A_3, A_5, A_{12}, A_{15}, A_{16}, A_{17} & A_{18}
F26A	26	A_{14}
Coxeter	28	A_2, A_4, A_7, A_9 & A_{17}
Levi	30	A_1, A_3, A_6, A_8, A_{10}, A_{11}, A_{12} A_{16} & A_{18}
Dyck	32	A_2, A_5, A_{15} & A_{17}

Table 5: Probable parameter matrices by Theorems 2.2 and 2.3

name	Spectrum	parameter matrices
tetrahedral	$3,(-1)^3$	A_{17}
utility	$3,0^4,-3$	A_{12} & A_{18}
cubical	$3,1^3,(-1)^3,-3$	A_{15} & A_{17}
Petersen	$3,1^5,(-2)^4$	A_6 & A_{10}
Heawood	$3,\sqrt{2}^6,(-\sqrt{2})^6,-3$	A_4 & A_9
Möbius-Kantor	$3,\sqrt{3}^4,1^3,(-1)^3,-\sqrt{3}^4,-3$	A_{15} & A_{17}
Pappus	$3,\sqrt{3}^6,0^4,(-\sqrt{3})^6,-3$	A_1, A_{12} & A_{18}
Desargues	$3,2^4,1^5,(-1)^5,(-2)^4,-3$	A_2, A_6, A_{10}, A_{11} & A_{17}
dodecahedral	$3,\sqrt{5}^3,1^5,0^4,(-2)^4,(-\sqrt{5})^3$	A_6, A_{10} & A_{11},
Nauru	$3,2^6,1^3,0^4,(-1)^3,(-2)^6,-3$	A_2, A_3, A_{12}, A_{15}, A_{16}, A_{17} & A_{18}
Coxeter	$3,2^8,(-1)^7,(-1+\sqrt{2})^6,(-1-\sqrt{2})^6$	A_2, A_7, A_{17}
F26A	$3,(\sqrt{\frac{5+\sqrt{13}}{2}})^6,(\sqrt{\frac{5-\sqrt{13}}{2}})^6,$ $(-\sqrt{\frac{5-\sqrt{13}}{2}})^6,(-\sqrt{\frac{5+\sqrt{13}}{2}})^6,-3$	nothing!
Levi	$3,2^9,0^{10},(-2)^9,-3$	A_3, A_{11}, A_{12}, A_{16} & A_{18}
Dyck	$3,\sqrt{5}^6,1^9,(-1)^9,(-\sqrt{5})^6,-3$	A_{15} & A_{17}

From Table 5, we observe that $F26A$ is the only graph that is not perfectly 3-colorable. In what follows, we show that some of the possible parameter matrices cannot actually occur as parameter matrices for the corresponding graphs. In Table 6, the parameter matrices of each graph are presented together with a corresponding perfect 3-coloring. At the end of the paper, all colorings are illustrated.

Proposition 3.1. The Heawood graph does not admit a perfect 3-coloring with parameter matrix A_9.

Proof. Let (W,B,R) be a perfect 3-coloring for Heawood graph, H, with parameter matrix A_9. Table 3 implies that $H[W]$ is an edge, e, and all four vertices in its external neighborhood belong to B. Since $|B|=4$, $R=V\setminus N_H[W]$ and $H[V\setminus N_H[W]]$ should be a 2-regular graph and since H is symmetric, there is no such edge. □

Lemma 3.1. If a cubic graph G has a perfect 3-coloring (W,B,R) with parameter matrix A_{17}, then for $u,v\in W$, the distance between u and v is at least 3.

Proposition 3.2. *The matrices A_2, A_{11} and A_{17} are not parameter matrices of the Desargues graph.*

Proof. By Table 3, if there exists a perfect 3-coloring of D with parameter matrix A_2, then the color class B must induce a 5-cycle. However, D contains no 5-cycle. Therefore, A_2 is not possible. Let (W, B, R) be a perfect 3-coloring with parameter matrix A_{11}. Since D is symmetric, we may assume, without loss of generality, that $a_1, b_1 \in R$, $a_2 \in W$, $a_{10} \in B$. Considering Table 3 at the vertices a_2 and a_{10}, we obtain $a_3, b_2, a_9, b_{10} \in R$. Now, by the previous lemma, no color can be assigned to b_3. Hence, such a coloring does not exist. Let (W, B, R) be a perfect 3-coloring with parameter matrix A_{17}. Since $|W| = 5$, we may assume that $|\{a_1, \ldots, a_{10}\} \cap W| \geq 3$. By the previous lemma, we must have $|\{a_1, \ldots, a_{10}\} \cap W| = 3$. Moreover, by the symmetry of the graph, we may assume that $a_1, a_4, a_7 \in W$. Again, by the previous lemma, all vertices b_i, except possibly b_9, are forbidden to belong to W. This contradicts $|W| = 5$. Therefore, no such coloring exists. □

Lemma 3.2. If a cubic graph G admits a perfect 3-coloring with parameter matrix A_{18}, then, in this coloring, every vertex has exactly three neighbors, one of each color.

Lemma 3.3. If a cubic graph G admits a perfect 3-coloring with parameter matrix A_{18}, then no two vertices of the same color can have a common neighbor.

Proposition 3.3. The matrix A_{18} is not a parameter matrix of the Levi graph, also known as the Tutte–Coxeter graph.

Proof. Let (W, B, R) be a coloring with parameter matrix A_{18}. Since the graph is symmetric, we may assume, without loss of generality, that $a_{11}, a_{28} \in B$. It remains to consider two essentially distinct cases; all other cases follow from these by symmetry.

Case 1. $a_1 \in W$.

By Lemma 3.3, we have $a_{29} \notin W$ since $a_1 \in W$. Hence, applying Lemma 3.2 at a_{28}, we obtain $a_{27} \in W$ and $a_{29} \in R$. Next, by Lemma 3.2 at a_{30}, we get $a_{21} \in B$. Moreover, Lemma 3.3 implies that $a_3 \in R$, since $a_1 \in W$ and $a_{11} \in B$. Now, applying Lemma 3.3, there is no possible color for a_5, because $a_3 \in R$, $a_{27} \in W$, and $a_{21} \in B$. This is a contradiction.

Case 2. $a_1, a_8 \in B$.

By Lemma 3.2, we may assume that $a_{10} \in W$ and $a_{12} \in R$. Since $a_{12} \in R$, Lemma 3.3 gives $a_2 \notin R$. Therefore, applying Lemma 3.2 at a_1, we obtain $a_2 \in W$ and $a_{30} \in R$. Furthermore, Lemma 3.2 at a_{29} implies that $a_{16} \in W$. Now, applying Lemma 3.3, there

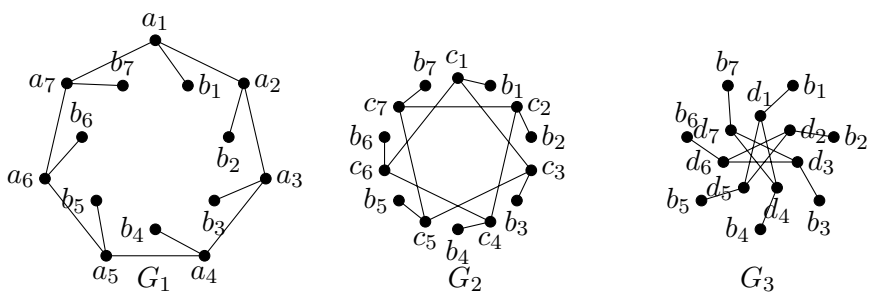
is no possible color for a_{14}, because $a_{12} \in R$, $a_{16} \in W$, and $a_8 \in B$. This is again a contradiction. □

Proposition 3.4. The Coxeter graph does not admit a perfect 3-coloring with parameter matrix A_{17}.

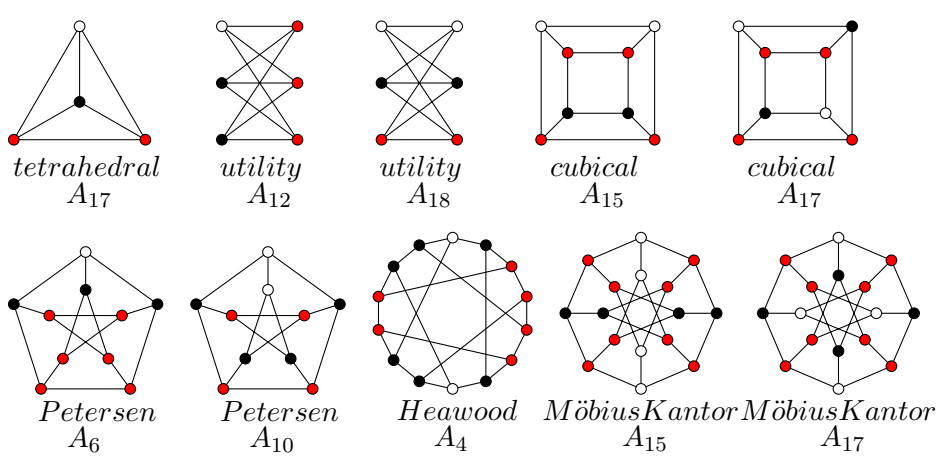
Proof. Let (W, B, R) be a coloring with parameter matrix A_{17}. By Table 3, the subgraph $G[R]$ is an induced matching consisting of seven edges. The induced subgraphs

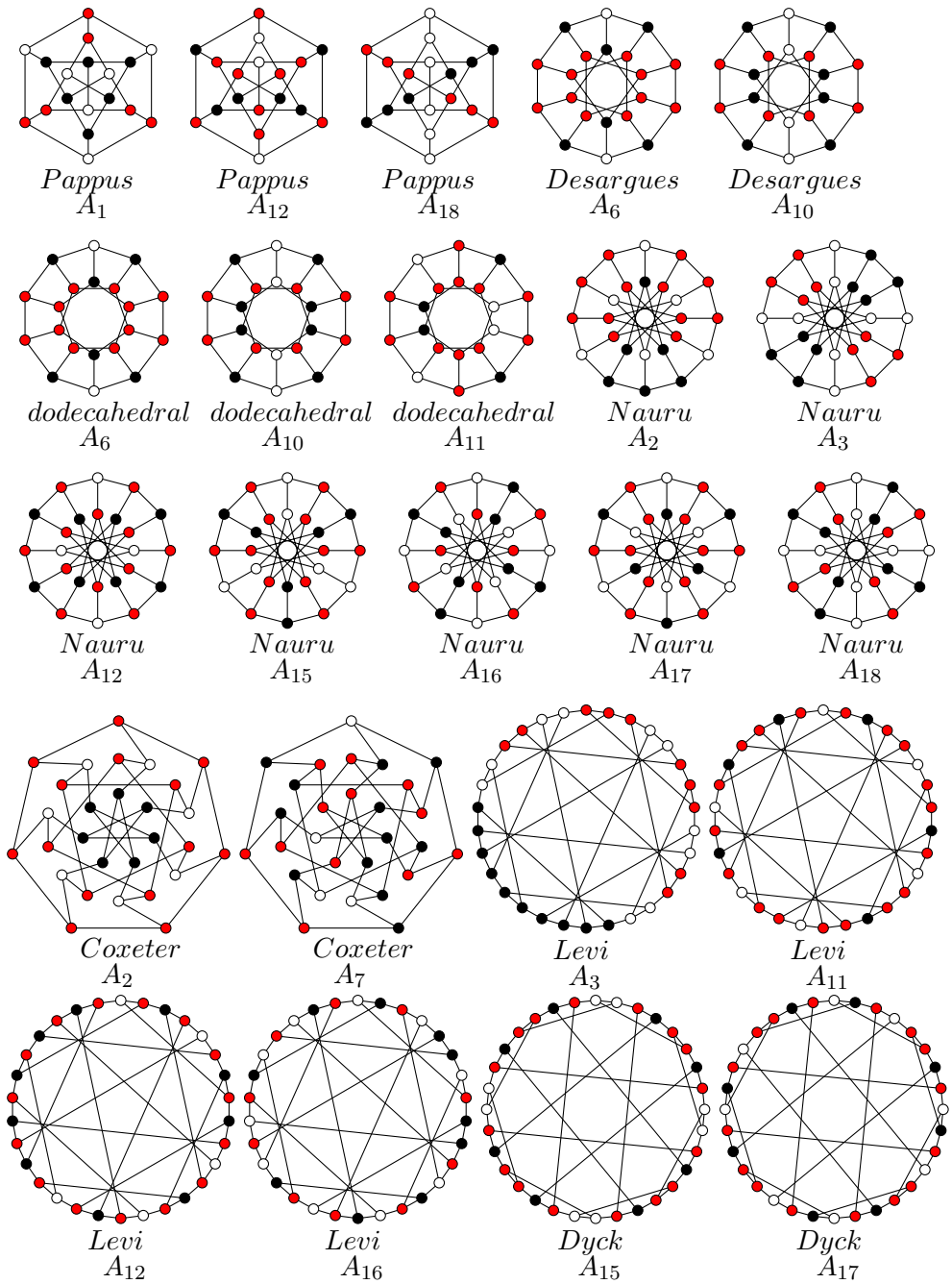
$$G_1 = G[a_1, \ldots, a_7, b_1, \ldots, b_7], \quad G_2 = G[c_1, \ldots, c_7, b_1, \ldots, b_7], \quad G_3 = G[d_1, \ldots, d_7, b_1, \ldots, b_7]$$

form an edge decomposition of the Coxeter graph. As shown below, these three subgraphs are mutually isomorphic.



By the pigeonhole principle, we may assume that $|E(R) \cap E(G_1)| = 3$. The only possible case is $E(R) \cap E(G_1) = \{a_2b_2, a_4a_5, a_7b_7\}$. We may assume that $b_4 \in W$ and $a_3 \in B$. Hence, $a_1 \in W$ and $b_1 \in B$, and consequently $c_1, d_1 \in R$. Since $b_4 \in W$, we have $d_4 \notin W$. Also, since $b_1 \in B$, we have $d_4 \notin B$. Therefore, $d_4 \in R$ and $d_5 \in W$. On the other hand, $a_6 \in B$. At this stage, there is no admissible choice for the color of b_5. Thus, no such coloring exists. □



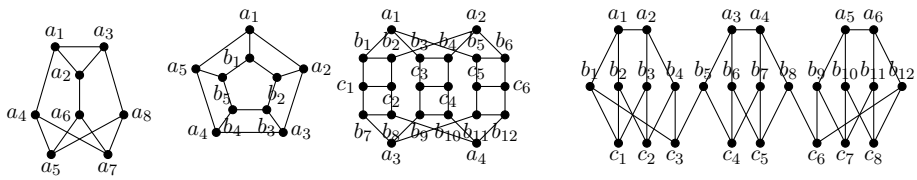


Finally, we introduce four small non-symmetric cubic graphs H_1, H_2, H_3, and H_4, whose parameter matrices are A_5, A_8, A_{13}, and A_{14}, respectively. Moreover, none of these graphs admits any other parameter matrix. For H_3 and H_4, this follows immediately from Theorem 2.3. For H_1, Theorem 2.3 leaves only the following possible parameter matrices: $A_2,\ A_5,\ A_{15},\ A_{17}$. Similarly, for H_2, the only possible candidates are $A_6,\ A_8,\ A_{10},\ A_{11}$.

Table 6: a coloring for the parameter matrices of each graph

Sym. C. Gr.	P. M.	W	B
tetrahedral	A_{17}	a_1	a_4
utility	A_{12}	a_1	a_2, a_3
	A_{18}	a_1, b_1	a_2, b_2
cubical	A_{15}	a_1, a_2	b_3, b_4
	A_{17}	a_1, b_3	a_2, b_4
Petersen	A_6	a_1	a_2, a_5, b_1
	A_{10}	a_1, b_1	a_2, a_5, b_3, b_4
Heawood	A_4	a_1, a_8	$a_2, a_7, a_9, a_{10}, a_{13}, a_{14}$
Möbius-Kantor	A_{15}	a_1, a_5, b_1, b_5	a_3, a_7, b_3, b_7
	A_{17}	a_1, a_5, b_3, b_7	a_3, a_7, b_1, b_5
Pappus	A_1	$a_2, a_4, a_6, c_2, c_4, c_6$	$b_2, b_4, b_6, c_1, c_3, c_5$
	A_{12}	a_4, b_1, c_1	$a_2, a_6, b_3, b_5, c_3, c_5$
	A_{18}	$a_1, a_4, b_1, b_4, c_1, c_4$	$a_2, a_5, b_2, b_5, c_2, c_5$
Desargues	A_6	a_1, a_6	$a_2, a_5, a_7, a_{10}, b_1, b_6$
	A_{10}	a_1, a_6, b_1, b_6	$a_2, a_5, a_7, a_{10}, b_3, b_4, b_8, b_9$
dodecahedral	A_6	a_1, a_6	$a_2, a_5, a_7, a_{10}, b_1, b_6$
	A_{10}	a_1, a_6, b_1, b_6	$a_2, a_5, a_7, a_{10}, b_3, b_4, b_8, b_9$
	A_{11}	a_7, a_{10}, b_3, b_4	a_2, a_5, b_8, b_9
Nauru	A_2	$a_1, a_5, a_9, b_3, b_7, b_{11}$	$a_6, a_7, a_8, b_1, b_6, b_8$
	A_3	$a_1, a_4, a_7, a_{10}, b_1, b_4, b_7, b_{10}$	$a_2, a_3, a_8, a_9, b_2, b_3, b_8, b_9$
	A_{12}	a_1, a_7, b_4, b_{10}	$a_3, a_5, a_9, a_{11}, b_2, b_6, b_8, b_{12}$
	A_{15}	$a_1, a_5, a_9, b_1, b_5, b_9$	$a_3, a_7, a_{11}, b_3, b_7, b_{11}$
	A_{16}	$a_1, a_4, a_7, a_{10}, b_3, b_6, b_9, b_{12}$	$a_2, a_5, a_8, a_{11}, b_2, b_5, b_8, b_{11}$
	A_{17}	$a_1, a_5, a_9, b_3, b_7, b_{11}$	$a_3, a_7, a_{11}, b_1, b_5, b_9$
	A_{18}	$a_1, a_4, a_7, a_{10}, b_1, b_4, b_7, b_{10}$	$a_2, a_5, a_8, a_{11}, b_2, b_5, b_8, b_{11}$
Coxeter	A_2	$b_1, \ldots, b_7$	$d_1, \ldots, d_7$
	A_7	a_1, b_4, c_5, d_6	$a_2, a_4, a_7, b_1, b_5, b_6, c_3, c_4, c_7, d_2, d_3, d_4$
Levi	A_3	$a_4, a_5, a_9, a_{10}, a_{13}, a_{14}, a_{25}, a_{26}, a_{29}, a_{30}$	$a_{15}, a_{16}, \ldots, a_{24}$
	A_{11}	$a_1, a_6, a_{11}, a_{17}, a_{20}, a_{24}$	$a_3, a_9, a_{14}, a_{22}, a_{26}, a_{29}$
	A_{12}	$a_1, a_5, a_{11}, a_{15}, a_{19}$	$a_3, a_7, a_9, a_{13}, a_{17}, a_{21}, a_{23}, a_{25}, a_{27}, a_{29}$
	A_{16}	$a_1, a_4, a_7, a_{12}, a_{15}, a_{18}, a_{21}, a_{23}, a_{26}, a_{28}$	$a_2, a_5, a_6, a_9, a_{10}, a_{13}, a_{16}, a_{20}, a_{25}, a_{29}$
Dyck	A_{15}	$a_1, a_2, a_9, a_{10}, a_{17}, a_{18}, a_{25}, a_{26}$	$a_4, a_7, a_{12}, a_{15}, a_{20}, a_{23}, a_{28}, a_{31}$
	A_{17}	$a_1, a_4, a_9, a_{12}, a_{17}, a_{20}, a_{25}, a_{28}$	$a_2, a_7, a_{10}, a_{15}, a_{18}, a_{23}, a_{26}, a_{31}$

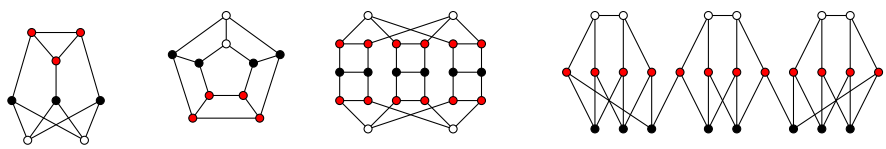
A direct verification shows that, among these candidates, only A_5 is admissible for H_1, and only A_8 is admissible for H_2.



H_1 H_2 H_3 H_4

Table 7: a coloring for some non-symmetric cubic graphs

non-Sym. C. Gr.	P. M.	W	B
H_1	A_5	a_5, a_7	a_4, a_6, a_8
H_2	A_8	a_1, b_1	a_2, b_2, a_4, a_4
H_3	A_{13}	a_1, a_2, a_3, a_4	$c_1, \ldots, c_6$
H_4	A_{14}	$a_1, \ldots, a_6$	$c_1, \ldots, c_8$



H_1 H_2 H_3 H_4

Declaration of competing interest

The authors declare that there are no conflicts of interest.

Data availability

No data were used in the research presented in this article.

Ethical approval

Not applicable.

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