TAMKANG JOURNAL OF MATHEMATICS
Volume 57, Number 1, 51-62, 2026
DOI:10.5556/j.tkjm.57.2026.5910

Blow-up results of a time fractional heat equation with a nonlinear Neumann boundary condition

Hind Ghazi Hameed, Burhan Selcuk, Maan A. Rasheed

Abstract. The study of blow-up phenomena in fractional diffusion equations is of great interest due to its numerous applications and the fact that these types of problems are encountered in several areas of science and engineering. This article is concerned with the blow-up solutions of a one-dimensional time-fractional heat equation, where the time derivative is defined in the sense of the Caputo fractional formula, subject to a nonlinear Neumann boundary condition of a power-type function. Firstly, global existence and blow-up are studied. Under a restricted condition on the nonlinear boundary term, it is proved that every positive solution blows up in finite time; otherwise, positive solutions are continued globally. Secondly, we prove that the blow-up phenomenon can occur only on the boundary.

Keywords. Fractional heat equation, blow up, nonlinear boundary condition, maximum principles.

1 Introduction

We consider the initial-boundary value problem of the Caputo time-fractional heat equation:

$$\begin{cases} {}^cD_t^\alpha u(x,t) = u_{xx}(x,t), 0<x<1, 0<t<T, \\ u_x(0,t)=0, u_x(1,t)=u^p(1,t), 0<t<T, \\ u(x,0)=u_0(x), 0\le x\le 1. \end{cases} \tag{1.1}$$

where ${}^cD_t^\alpha u$ denotes Caputo fractional derivative operator, defined as

$${}^cD_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial \tau}d\tau$$

and Γ is Gamma function with $0<\alpha<1$. Also, we assume that $p>1$, and T is the maximal existence time. The initial function $u_0(x)$ is nonnegative, smooth, and satisfies the compatibility conditions:

$$u_0'(0)=0,\ u_0'(1)=u_0^p(1).$$

Received date: September 25, 2025; Published online: January 20, 2026.
2010 *Mathematics Subject Classification*. 35R11, 35K60, 35B50.
Corresponding author: Burhan Selcuk.

The blow-up phenomenon refers to the case where a solution to a partial differential equation becomes unbounded or infinite within a finite time interval. In some Latin-derived languages, blow-up is known as "explosion". We can generalize the concept of blowing up more broadly, as it represents the phenomenon in which solutions cannot be globally continued over time, resulting from the infinite growth of the nonlinear terms that show the evolutionary process.

The heat equation (diffusion equation) describes the physical processes that involve the transfer of a given quantity (such as heat or matter) from a high concentration area to a low concentration area due to diffusion. a physical phenomenon that appears to be widespread. Diffusion arises from Brownian motion, which describes the random motion of atoms, particles, or molecules suspended in a fluid, ultimately leading to their homogeneous mixing. At the macro-level, it is defined by the famous Fick's laws that describe diffusion or Fourier's laws, also known as the law of heat conduction [1]. This classical heat equation is linear and given by $u_t = c^2 u_{xx}$ where c is the diffusion coefficient and u is the temperature or concentration. It can be solved easily. However, the problem becomes more complicated when a source term that depends on u is added to the equation, as it becomes a semi-linear heat equation.

In physical problems [2], the time-fractional derivative presents the nonlocal nature. Therefore, the Caputo time-fractional heat equation subject to Neumann boundary conditions can be considered a partial differential equation (PDE) that incorporates fractional calculus in modeling the diffusion of heat over time. It is an extension of the classical heat equation, where the presence of the Caputo fractional derivative in the time variable introduces memory effects and nonlocal behavior. These types of problems can be used for modeling several real-life problems involving memory, delay effects and non-local descriptions [3, 4, 5, 6, 7].

Although the classical diffusion equations have been widely studied in the literature, there are still many open problems for fractional diffusion equations. The analytical complexity of the properties of fractional operators and the theoretical and numerical difficulties encountered in modeling real-world problems are some of the reasons for this limited interest in the abstract mathematical properties of fractional differential equations.

The study of blow-up phenomena first appeared in the 1940s and 1950s. More comprehensive studies on this subject were conducted in the 1960s, especially by [8, 9, 10, 11]. Many researchers have studied the behavior and blow-up of solutions of linear and semilinear heat equations with nonlinear boundary conditions (Neumann). In particular, the blow-up properties of the following two problems have been studied in detail by many researchers [12, 13, 14, 15, 16, 17, 18]:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u^p, \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = 0, \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \text{in } \Omega. \end{cases} \tag{1.2}$$

$$\begin{cases} u_t(x,t) = \Delta u(x,t), \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = u^p, \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \text{in } \Omega. \end{cases} \tag{1.3}$$

Lin and Wang [19] focus on studying the blow up properties of the semi-linear heat equation with Neumann boundary conditions;

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) + u^p(x,t), 0 < x < 1, t > 0, \\ u_x(0,t) = 0, u_x(1,t) = u^q(1,t), t > 0, \\ u(x,0) = u_0(x), 0 \le x \le 1. \end{cases} \tag{1.4}$$

The study clarified the interaction between the two nonlinear terms and the role of each in determining the properties, rate, and location of the blow-up. They used analytical methods, including the comparison principle, energy methods, and integral equations, to reach the results. They proved that the blow-up occurs in a finite time when $\max(p, q) > 1$. They reach the blow-up rates and concluded that the blow-up occurs at boundary only when $x = 1$.

Kawarada first introduced the concept of quenching in [20]. Kawarada has considered an initial-boundary problem for the parabolic equation $u_t = u_{xx} + \frac{1}{1-u}$. A solution $u(x,t)$ of the problem is said to quench if there exists a finite time T such that $\lim\limits_{t \to T^-} \max\{u(x,t) : 0 \leq x \leq 1\} \to 1$.

In the article [21], Levine focused on studying the phenomenon of quenching in solutions of linear parabolic equations with nonlinear boundary conditions, where the study focused on determining the conditions under which the solutions reach a critical value, where this condition is called quenching.

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), 0 < x < L, t > 0, \\ u(0,t) = 0, u_x(L,t) = \phi(u(L,t)), t > 0, \\ u(x,0) = u_0(x), 0 \leq x \leq L. \end{cases} \tag{1.5}$$

He used analytical methods, including the principles of maximum values, energy methods, and Green's function, to reach the results. The uniqueness of the solutions was proven, and the conditions that allow their extension after quenching were established. He also proved that quenching occurs in a finite time when $L_0 > L$ at the boundary point $x = L$ and $u_t(L,t)$ becomes unbounded, and concluded that quenching does not occur and the solutions remain globally when $L_0 < L$.

The concepts of blow-up and quenching can be transformed into each other by means of a transformation [22]. Ozalp and Selcuk studied the behavior of solutions and determined standards for blow-up and quenching to these equations. Later, they considered the heat equation with nonlinear boundary conditions:

$$\begin{cases} u_t(x,t) = u_{xx}(x,t), 0 < x < 1, t > 0, \\ u_x(0,t) = u^p(0,t), u_x(1,t) = u^q(1,t), t > 0, \\ u(x,0) = u_0(x), 0 \leq x \leq 1. \end{cases} \tag{1.6}$$

They studied this problem using both analytical and numerical tools. They developed standards for blowing up and putting out an blow up based on how similar the two phenomena used to be. The study results include criteria for blowing up and cooling in the heat equation and the non-linear parabolic equation with non-linear boundary conditions. The authors also showed that blowing up and quenching are equivalent in these equations, and they used numerical models to back up their claims.

Many studies have investigated blow-up and quenching phenomena under various initial and boundary conditions across different types of equations [23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. Furthermore, in recent years, increasing attention has been devoted to the analysis of blow-up problems formulated with fractional derivatives instead of classical ones, leading to a significant expansion of the literature in this direction [33, 34, 35, 36, 37, 38, 39, 40, 41].

Recently, studies have been focused on the behavior of blow-up solutions of time-fractional diffusion equations. For instance, Subedi and Vatsala have studied in [42] some blow up properties for time fractional one-dimensional semilinear reaction-diffusion equation subject to homogeneous Dirichlet boundary conditions.

$$\begin{cases} {}^cD_t^\alpha u(x,t) = u_{xx}(x,t) + u^p(x,t), 0 < x < 1, 0 < t < T, \\ u(0,t) = 0, u(L,t) = 0, 0 < t < T, \\ u(x,0) = 0, 0 \le x \le 1. \end{cases} \tag{1.7}$$

They employed various analytical techniques, including comparison principle methods, in which the solutions of fractional equations are compared with those of classical (non-fractional) equations whose blow-up behavior is already well understood. They also used the construction of the lower solution, where lower solutions of fractional equations are constructed using solutions of classical equations. Moreover, Green's function was used to achieve the results. They concluded that for ordinary fractional equations, the lower solutions extracted from classical equations blow up in a finite time. These lower solutions provide upper limits on the time in which the blow-up of fractional equations occurs. As for fractional diffusion reaction equations, if solutions of classical equations blow up, the solutions of fractional equations will also explode under suitable conditions. Blow-up time in the fractional case depends on the blow-up time in the classical equations, and it is modified using the fractional derivative. To summarize, the researchers demonstrated that the blow-up behavior of time-fractional diffusion–reaction equations is closely analogous to that of their non-fractional equations counterparts when similar conditions are satisfied.

Motivated by the studies in [19] and [42], this paper investigates the blow-up phenomenon for time-fractional heat equations with a nonlinear Neumann boundary condition. Specifically, the classical heat equation is extended to its fractional counterpart by incorporating a Caputo time-fractional derivative, and a nonlinear Neumann boundary condition is considered instead of the classical Dirichlet condition.

This paper is divided into four sections, in section two we recall some basic preliminaries. In section three it is given the auxiliary lemmas. In the four section, global existence, blow up in finite time, a limit for the blow up time and an blow up point are obtained. Finally, it is given the main conclusions, theoretical and practical contributions, and possible directions for future studies.

2 Preliminaries

In this section, we recall some basic definitions and Auxiliary lemma that we shall need to prove the main results.

Definition 1. A solution $u(x,t) \in C^{2,\alpha}([0,1] \times [0,T))$ of the equation (1.1) blows up in a finite-time, i.e, the existence of a $T = T(u_0) < \infty$ such that

$$\lim_{t \to T^-} \max\{u(x,t) : 0 \le x \le 1\} \to \infty.$$

The blow up time of the equation (1.1) is denoted as T.

Definition 2. A solution $v(x,t) \in C^{2,\alpha}([0,1] \times [0,T))$ is said to be lower solution of the equation (1.1), if

$$\begin{cases} {}^cD_t^\alpha v(x,t) - v_{xx}(x,t) \le 0, 0 < x < 1, 0 < t < T, \\ v_x(0,t) \le 0, v_x(a,t) \le (v(1,t))^p, 0 < t < T, \\ v(x,0) \le 0, 0 \le x \le 1. \end{cases} \tag{2.1}$$

and a solution $w(x,t) \in C^{2,\alpha}([0,1] \times [0,T))$ is said to be upper solution of the equation (2.1), if reverse inequalities are satisfied.

Definition 3. For two parameters, $\alpha, r > 0$, the Mittag Leffler function is defined as follows:

$$E_{\alpha,r}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + r)}.$$

Specially,

$$E_{\alpha,r}(\lambda(t - t_0)^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda(t - t_0)^{\alpha})^k}{\Gamma(\alpha k + r)}$$

Also, for $t_0 = 0$ and $r = 1$, we get,

$$E_{\alpha,1}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha k + 1)}$$

and for $t_0 = 0$ and $r = \alpha$, we get,

$$E_{\alpha,\alpha}(\lambda t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda t^{\alpha})^k}{\Gamma(\alpha(k + 1))}$$

where $0 < \alpha < 1$. Further if $\alpha = 1$, then $E_{\alpha,\alpha}(\lambda t^{\alpha}) = E_{\alpha,1}(\lambda t^{\alpha}) = e^{\lambda t}$.

It is known that the Maximum principles and Hopf's lemma are considered essential tools to prove several qualitative properties of classical diffusion equations, including blow-up and quenching problems. In this article, the Maximum principles and Hopf's lemma for time-fractional equations, obtained by some researchers, in [43, 44, 45], are employed to get the main results.

By following the logical process in section 2 of [21] proved for the quenching problem, the following auxiliary theorems for the solution of problem (1.1) for $u(x, t)$ can be easily obtained using fractional maximum principles and the fractional Hopf's lemma.

Lemma 2.1. *If $u(x, t)$ solves the problem (1.1) in $(0, 1) \times (0, T)$, then:*

- *$u(x, t) > 0$ in $(0, 1) \times (0, T)$,*
- *$u_x(x, t) > 0$ in $(0, 1) \times (0, T)$, if $u_x(x, 0) \geq 0$ in $(0, 1)$,*
- *${}^cD_t^{\alpha}(u(x, t)) > 0$ and $u_t(x, t) > 0$ in $(0, 1) \times (0, T)$, if $u_{xx}(x, 0) \geq 0$ in $(0, 1)$.*

3 Blow-up and global existence

In this section, the global existence of solutions to problem (1.1) is established, the occurrence of blow-up is proven, and an upper bound for the blow-up time is derived.

Theorem 3.1. *A solution of the problem (1.1) exists globally if $p \leq 1$.*

Proof. The sufficient condition must be proven first. We need to show that the solution exists globally. The auxiliary function is $v(x, t) = CE(Kt^{\alpha})e^{Lx^2}$ where C, K and L sufficiently large constants to be determined later and $E(Kt^{\alpha})$ is Mittag-Leffler function ${}^cD_t^{\alpha}(E(Kt^{\alpha})) = KE(Kt^{\alpha}))$. The Direct calculations show that:

$${}^cD_t^{\alpha}(v(x, t)) = CKE(Kt^{\alpha})e^{Lx^2},$$

$$\begin{aligned} v_x(x,t) &= 2CLxE(Kt^\alpha)e^{Lx^2}, \\ v_{xx}(x,t) &= (4L^2x^2+2L)CE(Kt^\alpha)e^{Lx^2}, \end{aligned}$$

It follows that

$${}^cD_t^\alpha(v(x,t)) - v_{xx}(x,t) \geq 0,$$

where $K = 4L^2 + 2L$.
Apply boundary conditions:

$$\begin{aligned} v_x(0,t) &= 0, \\ v_x(1,t) &= 2CLE(Kt^\alpha)e^L \geq (2CLE(Kt^\alpha)e^L)^p = (v(1,t))^p, \end{aligned}$$

where $p \leq 1$. Also, $v(x,0) \geq 0$ on the initial line. It is clear that $v(x,t) = CE(Kt^\alpha)e^{Lx^2}$ is an upper solution to the problem (1.1) with the help of Definition 2. So, the solutions of problem (1.1) exist globally, where $p \leq 1$. □

Theorem 3.2. *A solution of the problem (1.1) blows up in a finite time, and an upper limit for the blow-up time is*

$$\left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{\frac{1}{\alpha}} \left(\frac{1}{l(0)}\right)^{\frac{\alpha}{p-1}},$$

where $p > 1, u_x(x,0) \geq 0$ and and $l(0)(>0)$ is the initial function of the following problem

$$l_t = l^p(t).$$

Proof. Now, suppose that $p > 1$ and show that the solution of problem (1.1) blows up in finite time. Let $m(t) = \int_0^1 u(x,t)dx$.

$${}^CD_t^\alpha(m(t)) = \int_0^1 {}^cD_t^\alpha u(x,t)dx = \int_0^1 u_{xx}(x,t)dx = u^p(1,t).$$

From Lemma 2.1, $u(1,t) \geq u(x,t)$ where $x \in [0,1)$. Then, from the above equation and given $p > 1$:

$${}^cD_t^\alpha(m(t)) \geq m^p(t). \tag{3.1}$$

Vatsala and Subedi have investigated the solution of the following problem in [42];

$${}^cD_t^\alpha(n(t)) = n^p(t), n(0) = m(0) > 0. \tag{3.2}$$

It is clear that the solution of (3.1) is an upper solution of (3.2). Unfortunately, problem (3.2) does not have an explicit solution. The problem (3.2) with classical differentiation is expressed as follows;

$$l_t = l^p(t), l(0) > 0. \tag{3.3}$$

They showed that $l(\frac{t^\alpha}{\Gamma(\alpha+1)})$ is a lower solution of the problem (3.2) where $p > 1$. Also, they obtained that the blow-up time of the problem (3.3) is

$$t_l = \left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{\frac{1}{\alpha}} \left(\frac{1}{l(0)}\right)^{\frac{\alpha}{p-1}}.$$

As a result, the solution of problem (1.1) blows up in finite time. Since the solution of problem (3.3) serves as a lower solution to problem (3.1) and exhibits blow-up, it follows that the solution of problem (3.1) also blows up. Consequently, $u(x,t)$ becomes unbounded in finite time. Moreover, an upper bound for the blow-up time of problem (3.1) is given by:

$$\left(\frac{\Gamma(\alpha+1)}{p-1}\right)^{\frac{1}{\alpha}}\left(\frac{1}{l(0)}\right)^{\frac{\alpha}{p-1}}.$$

since $t_m \leq t_n \leq t_l$ from Definition 2. Here, t_m, t_n and t_l are the blow-up times of problems (3.1),(3.2) and (3.3), respectively. □

For problem (1.1), the following results can be concluded with the help of Theorem 3.1 and Theorem 3.2.

Corollary 3.3.

- If $p > 1$, the solution can blow up in finite time,
- If $p \leq 1$, the solution exists globally.

4 Blow-up point

In this section, we consider the blow-up set of problem (1.1) and show that it can only be achieved at a boundary point.

Theorem 4.1. *If $u_x(x,0) \geq 0$ and $u_{xx}(x,0) \geq 0$ in $[0,1]$, then $x = 1$ is the only blow-up point for problem (1.1).*

Proof. Let $d_1 \in [0,1), d_2 \in [d_1,1), \tau \in (0,T)$ and $\epsilon > 0$. Define

$$F(x,t) = u_x(x,t) - \epsilon(x-d_1)u^p(x,t) \text{ in } [d_1,d_2]\times[\tau,T]$$

where $p > 1$ and ϵ is a sufficient small constant. The 1st and 2nd derivatives of the auxiliary function $F(x,t)$ with respect to x are obtained as follows;

$$F_x(x,t) = u_{xx}(x,t) - \epsilon u^p(x,t) - \epsilon p(x-d_1)u^{p-1}(x,t)u_x(x,t),$$

$$\begin{aligned} F_{xx}(x,t) &= u_{xxx}(x,t) - 2\epsilon p u^{p-1}(x,t)u_x(x,t) - \epsilon p(x-d_1)u^{p-1}(x,t)u_{xx}(x,t) \\ &\quad -\epsilon p(p-1)(x-d_1)u^{p-2}(x,t)u_x^2(x,t). \end{aligned}$$

On the other hand:

$${}^cD_t^\alpha u^p(x,t) = \frac{p}{\Gamma(1-\alpha)}\int_0^t \frac{u^{p-1}(x,s)u_s(x,s)}{(t-\tau)^\alpha}ds.$$

By generalized Leibinz rule, we have

$${}^cD_t^\alpha u^p(x,t) = pu^{p-1}(x,t){}^cD_t^\alpha u(x,t) - \frac{p(p-1)}{\Gamma(1-\alpha)}\int_0^t \frac{u^{p-2}(x,s)(u(x,t)-u(x,s))u_s(x,s)}{(t-\tau)^\alpha}ds.$$

$$\begin{aligned} {}^cD_t^\alpha F(x,t) &= {}^cD_t^\alpha u_x(x,t) - \epsilon(x-d_1){}^cD_t^\alpha u^p(x,t) \\ &= u_{xxx}(x,t) - \epsilon(x-d_1)pu^{p-1}(x,t){}^cD_t^\alpha u(x,t) \\ &\quad + \epsilon(x-d_1)\frac{p(p-1)}{\Gamma(1-\alpha)}\int_0^t \frac{u^{p-2}(x,s)(u(x,t)-u(x,s))u_s(x,s)}{(t-\tau)^\alpha}ds \end{aligned}$$

By Lemma 2.1, u is positive and increases in t, it follows that:

$$ {}^cD_t^\alpha F(x,t) \geq u_{xxx}(x,t) - \epsilon(x-d_1)pu^{p-1}(x,t){}^cD_t^\alpha u(x,t)$$

and

$$ {}^cD_t^\alpha F(x,t) - F_{xx}(x,t) \geq 2\epsilon pu^{p-1}(x,t)u_x(x,t) + \epsilon p(p-1)(x-d_1)u^{p-2}(x,t)u_x^2(x,t).$$

By Lemma 2.1, $u_x(x,t) > 0$ and since $p > 1$, the following inequality is obtained:

$$ {}^cD_t^\alpha F(x,t) - F_{xx}(x,t) \geq 0 \text{ in } (d_1,d_2)\times[\tau,T).$$

By choosing ϵ to be positive and small enough, the values at the boundary are obtained as follows:

$$F(d_1,t) = u_x(d_1,t) - \epsilon(d_1-d_1)u^p(d_1,t) = u_x(d_1,t) > 0,$$

and

$$F(d_2,t) = u_x(d_2,t) - \epsilon(d_2-d_1)\cdot u^p(d_2,t) > 0.$$

Similarly, the following inequality is obtained based on the initial datum;

$$F(x,0) = u_x(x,0) - \epsilon(x-d_1)\cdot u^p(x,0) > 0.$$

With the help of the maximum principle, we obtain that $F(x,t) \geq 0$ in $[d_1,d_2]\times[\tau,T]$. So, $u_x(x,t) \geq \epsilon(x-d_1)u^p(x,t)$ in $[d_1,d_2]\times[\tau,T]$.
Integrating the last inequality with respect to x from d_1 to d_2, the following inequality is obtained;

$$u(d_1,t) \leq \left(\frac{2}{\epsilon(p-1)(d_2-d_1)^2}\right)^{\frac{1}{(p-1)}} < \infty.$$

This means the solution $u(x,t)$ remains finite for all $x \in [0,1)$ and $t \in [0,T]$. Therefore, it does not blow up in $(0,1)$ and a blow-up occurs only at $x = 1$. □

5 Conclusions

This article is concerned with the blow-up solutions of the time-fractional heat equation subject to a nonlinear Neumann boundary condition of power type. It is proven that every positive solution blows up in finite time if the power of the non-linear term is larger than one; otherwise, global existence holds. Moreover, the blow-up phenomenon can only occur at the boundary points $x = 1$. In future studies, the theoretical results obtained in this study are planned to be supported by numerical approaches. Another open problem is to study the quenching phenomenon in similar equations.

References

[1] H. Mehrer, Diffusion in solids: fundamentals, methods, materials, diffusion-controlled processes, 2007, Springer Series in Solid-State Sciences (SSSOL, volume 155)

[2] R. Herrmann, Fractional calculus: an introduction for physicists, (2011), World Scientific.

[3] Q. Wei, W. Wang, H. Zhou, R. Metzler, A. Chechkin, Time-fractional Caputo derivative versus other integrodifferential operators in generalized Fokker-Planck and generalized Langevin equations, Phys. Rev. E 108, 024125 – Published 14 August, (2023), https://doi.org/10.1103/PhysRevE.108.024125

[4] A. Ayad, A. M. Djaouti, K. Benmeriem, Nonlinear memory term for fractional diffusion equation. J Nonlinear Math Phys 31, 67 (2024). https://doi.org/10.1007/s44198-024-00236-7

[5] N. Vasylyeva, Semilinear multi-term fractional in time diusion with memory. Front. Appl. Math. Stat. 10:1388414, (2024), https://doi.org/10.3389/fams.2024.1388414

[6] B. Fan. Efficient numerical method for multi-term time-fractional diffusion equations with Caputo-Fabrizio derivatives, AIMS Mathematics, 9 (3), (2024), 7293-7320. https://doi.org/10.3934/math.2024354.

[7] T. Kosztolowicz, Subdiffusion Equation with Fractional Caputo Time Derivative with Respect to Another Function in Modeling Superdiffusion. Entropy, 27(1), 48. (2025), https://doi.org/10.3390/e27010048

[8] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. of Tokio, Sect. I, t. 13, (1966), p. 109-124.

[9] S. Kaplan, On the growth of solutions of quasi-linear parabolic equations. Communications on Pure and Applied Mathematics, 16 (3) (1963) 305-330.

[10] A. Friedman, Partial differential equations of parabolic type, (1964) Prentice-Hall.

[11] R.T. Glassey, Blow-up theorems for nonlinear wave equations, Mathematische Zeitschrift, 132 (1973), 183-203.

[12] J.L. Gomez, V. Marquez, N. Wolanski, Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition, J. Diff. Equations 92 (1991), 384-401.

[13] A. Friedman, B. Mcleod, Blow-up of Positive Solutions of Semilinear Heat Equations, Indiana University Mathematics Journal, Vol. 34, No. 2 (Summer, 1985), pp. 425-447.

[14] Y. Giga and R.V. Kohn, Asymptotic self-similar blowup of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985), 297-319.

[15] H.A. Levine, R.A. Smith, A potential well theory for the heat equation with nonlinear boundary condition, Math. Methods Appl. Sci. 9 (1987) 127-136.

[16] W. Liu, The blow-up rate of solutions of semilinear heat equations, J. of Diff. Eq. 77 (1989), 104-122.

[17] W. Walter, On existence and nonexistence in the large of solutions of parabolic differential equations with a nonlinear boundary condition, SIAM J. Math. Anal. 6 (1975), 85-90.

[18] F.B. Weissler, An L^∞ blow-up estimate for nonlinear heat equation, Comm. Pure Appl. Math. 38 (1985), 291-295.

[19] Z. Lin, M. Wang, The blow-up properties of solutions to semilinear heat equations with nonlinear boundary conditions. Z. Angew. Math. Phys. 50, 361–374 (1999).

[20] H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + \frac{1}{1-u}$, Publ. Res. Inst. Math. Sci. 10 (1975) 729-736.

[21] H. A. Levine, The quenching of solutions of linear parabolic and hyperbolic equations with nonlinear boundary conditions, Siam J. Math. Anal., Vol. 14 (6) (1983), 1139-1153.

[22] N. Ozalp, B. Selcuk, Blow up and quenching for a problem with nonlinear boundary conditions. Electron. J. Diff. Equ, Vol. 2015 (192) (2015), 1-11.

[23] B. Selcuk, Quenching behavior of a semilinear reaction-diffusion system with singular boundary condition, Turkish Journal of Mathematics, 40 (2016) 166-180.

[24] B. Selcuk, Quenching problems for a semilinear reaction-diffusion system with singular boundary outflux. Tamkang Journal of Mathematics, 47(3), (2016) 323–337. https://doi.org/10.5556/j.tkjm.47.2016.1961

[25] B. Selcuk, Quenching for Porous Medium Equations. Tamkang Journal of Mathematics, 53(2), (2022) 175–185. https://doi.org/10.5556/j.tkjm.53.2022.3853

[26] M.A. Beauregard, B. Selcuk, Quenching estimates for a non-Newtonian filtration equation with singular boundary conditions. Tamkang Journal of Mathematics, 55(1), (2024) 1–14. https://doi.org/10.5556/j.tkjm.55.2024.5009

[27] M. A. Rasheed, F. N. Ghaffoori, Numerical blow-up time and growth rate of a reaction-diffusion equation. Italian journal of pure and applied mathematics, 44 (2020), 805-813.

[28] M. A. Rasheed, Blow-up properties of a coupled system of reaction-diffusion equations, Iraqi journal of Science, 62(9), (2021), 3052-3060.

[29] B.Selcuk, Blow up for non-Newtonian equations with two nonlinear sources, Hacettepe Journal of Mathematics and Statistics, 50 (2) (2021), 541-548.

[30] B. Selcuk, Blow up for porous medium Equations, Mathematical sciences and applications e-notes, 9 (1) (2021) 22-27.

[31] T. Liu, L. Zhang, H. Liu, Blow-up and global analysis of solutions of porous medium equation with a nonlinear memory term. Z. Angew. Math. Phys. 76, 137 (2025). https://doi.org/10.1007/s00033-025-02509-7.

[32] M. A. Rasheed and M. Saeed, "Numerical Finite Difference Scheme for a Two-Dimensional Time-Fractional Semilinear Diffusion Equation," Mathematical Modelling of Engineering Problems, vol. 10, no. 4, pp. 1441-1449, 2023, doi: https://doi.org/10.18280/mmep.100440.

[33] B. Samet, Blow-up phenomena for a nonlinear time fractional heat equation in an exterior domain, Computers and Mathematics with Applications, Volume 78, Issue 5, 2019, Pages 1380-1385.

[34] S. Dipierro, B. Pellacci, E. Valdinoci, G. Verzini, Time-fractional equations with reaction terms: fundamental solutions and asymptotics. Discrete Contin. Dyn. Syst. 41 (1), (2021) 257–275.

[35] C. Cortazar, F. Quiros, N. Wolanski. A semilinear problem associated to the space-time fractional heat equation in R^N, Calc. Var. 63, 229 (2024) https://doi.org/10.1007/s00526-024-02836-z.

[36] R. Ferreira, A. de Pablo, Blow-up for a fully fractional heat equation. Discrete Contin. Dyn. Syst. 44(2), (2024) 569–584.

[37] M. A. Rasheed, M. A. Saeed, Numerical Approximations of a One-Dimensional Time-Fractional Semilinear Parabolic Equation, Iraqi Journal of Science, 2023, 6445-6459.

[38] M. A Rasheed, M. A Saeed, Numerical Finite Difference Scheme for a Two-Dimensional Time-Fractional Semilinear Diffusion Equation, Mathematical modelling of engineering problems, 10 (4) 2023, 1441-1449.

[39] M. A. Beauregard, Numerical approximations to a fractional Kawarada quenching problem, Applied Mathematics and Computation, 349 (2019), 14-22.

[40] S. Subedi, A. S Vatsala, A quenching problem of Kawarada's type for one dimensional Caputo fractional reaction diffusion equation, Nonlinear Studies, 25 (3) 2018, 505-519.

[41] S. Subedi, A. S Vatsala, Quenching Problem for Two Dimensional Caputo Time-Fractional Reaction-Diffusion Equation, Dynamic Systems and Applications, 29 (1) 2020, 26-52.

[42] S. Subedi, A. S. Vatsala, Blow-up results for one dimensional Caputo fractional reaction diffusion equation, Mathematics in Engineering, Science and Aerospace, Vol. 10, No. 1, pp. 175-190, 2019.

[43] H. Brunner, H. Han, D. Yin, The maximum principle for time-fractional diffusion equations and its application. Numerical Functional Analysis and Optimization, 36(10), (2015) 1307-1321.

[44] Y. Luchko, Maximum principle and its application for the time-fractional diffusion equations. Fractional Calculus and Applied Analysis, 14(1), (2011) 110-124.

[45] S. D. Roscani, Hopf lemma for the fractional diffusion operator and its application to a fractional free-boundary problem, Journal of Mathematical Analysis and Applications, 434 (1) (2016) 125-135.

Hind Ghazi Hameed Postgraduate education institute, Department of Mathematics, Karabuk University, Turkey,
Department of Mathematics, College of Education, Tikrit University, Iraq

E-mail: 2038179003@karabuk.edu.tr, hind.g.hameed@tu.edu.iq

Burhan Selcuk Faculty of Science, Department of Computer Science, Ankara University, Turkey

E-mail: burhanselcuk@ankara.edu.tr

Maan A. Rasheed Department of Mathematics, College of Science for Women, University of Baghdad, Iraq

E-mail: maan.a@csw.uobaghdad.edu.iq