TAMKANG JOURNAL OF MATHEMATICS
Volume vol, Number i, xxx-xxx, 20xx
DOI:10.5556/j.tkjm.57.2026.5943

Some picture fuzzy information inequalities with applications in market behaviours and pattern recognition

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Abstract. Quantifying uncertainty in robust datasets is essential in decision analysis. Merging such data with the concepts of information theory opens various aspects of uncertainty. The additional degree of uncertainty can be addressed by picture fuzzy divergences. This study precisely handles such datasets by introducing a novel picture fuzzy divergence measure (PFDM) using generalized f-divergence. The validity of the measure has been proved, and some properties are discussed. Furthermore, some information inequalities are established for classical dissimilarity measures. These inequalities offer critical mathematical boundaries for analyzing and comparing various fuzzy information measures. Additionally, the application of the measure is discussed for market behaviour trends and pattern recognition problems. The obtained results aims to minimize the divergence and maximize the result accuracy and the measures successfully quantify the variations within fuzzy sets.

Keywords. Picture fuzzy sets, divergence measure, information inequalities, new f-divergence, pattern recognition.

1 Introduction

The growing complexity and vagueness in the decision-making problems demand advancements in uncertainty models. To handle additional degrees of uncertainties, extensions of fuzzy sets came to the picture. Building upon this progression, the concept of Picture Fuzzy Sets (PFSs) emerged as a robust generalization. It is capable of simultaneously representing positive, neutral, and negative membership degrees, while ensuring the sum of these degrees remains less than or equal to one. This framework proves particularly useful in applications involving human opinions, where neutrality plays a significant role, such as in social networks, medical diagnostics, and multi-criteria decision-making (MCDM). Parallel to this, information inequalities and divergence measures have been widely used to assess the dissimilarity between different data distributions or knowledge bases. Originally developed in classical probability theory, measures like the Kullback-Leibler divergence and the Jensen-Shannon divergence have been successfully adapted to fuzzy environments. These tools help quantify how far one piece of information deviates from another,

Received date: November 12, 2025; Published online: May 27, 2026.
2010 *Mathematics Subject Classification*. 68P30, 03E72, 68T10.
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making them essential in applications ranging from decision-making to machine learning. The information theory approach allows us to understand the complexity of behavioral sequences by data-compression techniques. Information theory focuses on entropy, uncertainty and measure of information and hence it can be widely used for solving ambiguity related problems, pattern recognition and decision-making systems. The information-theoretical approach can be applied to various fields such as quantum mechanics, communication, language structure, coding, complex sciences and many more. This approach has been applied to analyze animal communication and behavior studied in [27] . Various other studies have presented clinical diagnosis from the framework of information theory. Eiseman [14] developed information measures to examine uncertainty in diagnostic testing situations and provided quantitative measures for test effectiveness. [17] used Shannon entropy method to reduce the uncertainty criteria from clinical diagnostic situations using machine learning algorithms. Huang et al. [18] studied the significance of information theory in analysis of pattern for time series data, addressing the challenges of analyzing multivariate, multi-source and noisy time series data. In traditional information theory, foundational inequalities such as Gibbs, Jensen's, and Pinsker's inequalities play a crucial role in the theoretical study of entropy, divergence, and uncertainty. These mathematical tools are used to derive bounds, assess the behaviour of information measures, and ensure their logical consistency across various applications. Shannon's entropy laid the foundation of information theory. It developed the idea of recognizing the need to remove the uncertainty and vagueness in real life scenarios. Entropy measures the amount of randomness in a system. Shannon defined entropy as the measure of the maximum possible efficiency of any encoding scheme. Wide applications of Shannon's entropy can be visualized in data-compression, cryptography and decision-making. For a probability distribution $\mathfrak{F}$, Shannon's Entropy is given as,

$$H(\mathfrak{F}) = -\sum_{i=1}^{n} \mathfrak{x}_i \log \mathfrak{x}_i \tag{1.1}$$

Divergence measures quantify the difference between probability distributions. In information theory, divergence measures are closely related to convex functions as they provide mathematical foundation to them. For f-divergences, convexity ensures properties like non-negativity, uniqueness and stability. The extension of divergence measures in information theory also determines the value added to any event, i.e. to compare the usefulness of any event over the other. Hence, Divergence measures act as the building blocks of the various information inequalities. They can be understood as statistical distance useful in comparing two probability distributions. Interrelating these two leads to the development of new measures and their analogy has significant implications in fields like model selection and density estimation in statistics, anomaly detection and generative modelling in machine learning. These measures found numerous applications in information theoretical and statistical problems. A significant amount of work has been done by I.J. Taneja [31, 32, 33], S.S. Dragomir [10, 11, 12] and many other researchers [19, 20, 28, 29, 36] as contribution to the study of information inequalities and divergence measures. Their work mainly focuses on developing new divergence measures, establishing relationships between them, and deriving inequalities that bound these measures. Divergence Measures are mainly derived from the generalization of relative entropy. Dragomir introduced f-divergences for operator mapping [13], Bilal and Khan [4, 5] illustrated f- divergences for diamond integrals. Their work mainly focuses on developing new divergence measures, establishing relationships between them, and deriving inequalities that bound these measures. Divergence Measures are mainly derived from the generalization of relative entropy. The symmetricity of the divergence measures depends on the probability distributions involved in it. Areas such as Fuzzy Inequalities [15], Pattern Recognition, Word Alignment, approximation of probability distributions, medical diagnosis etc. find utility of divergence measures. The extension of divergence measures in information theory also determines the value added to any event, i.e. to compare the usefulness of any event over the other. Classical set theory works like binary i.e., 0,1. An element either belongs to the set or not. But in practical scenarios, we have more vague options where simply putting an element in any set is not possible. For example, if we want to create a set of tall students in a class. Now assigning someone to the set is not a clear choice. For handling such situations, Zadeh [1965] coined the term Fuzzy set. Mathematically, fuzzy sets are defined as $F = \langle \mathfrak{r}, \aleph_A(\mathfrak{r}) | \mathfrak{r} \in \mathfrak{X} \rangle$ where, $\aleph_A(\mathfrak{r}) : \mathfrak{X} \to [0, 1]$ is known as the membership function. This function demonstrates the degree of belongingness of any element in a set. Fuzziness expresses uncertainty and hence, the fuzzy sets solve the matter of partial membership. Fuzzy sets play a significant role in representing the unsure data and are considerably used by researchers in pervious many years. With the evolution of fuzzy sets, classical inequalities can also be reshaped in fuzzy environments. And hence the term Fuzzy Information Inequalities came into picture. As in general, inequalities deal with unequal relationships between mathematical quantities. The fuzzy information inequalities compare levels of uncertainty across fuzzy sets providing applications in decision-making, pattern recognition and uncertain reasoning. Deo et al. [9] proposed fuzzy relation inequalities for optimization problems and examined the application of particle swarm optimization and meta-heuristics. Gehlot and Saraswat [16] utilized fuzzy information inequalities in pattern recognition problems.

The objectives and motivation for this study are as follows:

- Correlating information measures with picture fuzzy environment to increase its applicability in practical problems like pattern recognition and market behaviour analysis.
- Investigating the theoretical aspects of the measure via information inequalities.
- Minimizing the divergence values so as to obtain maximum correspondence.

Remaining of the article has been arranged as follows: Section 2 talks about the available literature and some definitions of fuzzy measures along with its extensions. Section 3 proposes novel Picture fuzzy divergence measure and its properties. Section 4 comprises of fuzzy information inequalities which illustrates bounded relations among various divergences. The application of the proposed measure in market behaviour and pattern recognition has been discussed in section 5. The concluding remarks of the paper has been discussed in section 6.

2 Literature survey

Fuzzy set theory has been extended in different areas over the period. With various extensions of fuzzy sets, the scope of interlinking fuzzy sets with similarity and divergence measures has gained popularity. Zeng and Guo [39] introduced the distance measure between Interval valued fuzzy sets and discussed applications to approximate reasoning. Diaz et al. [7] introduced a richer class of similarity and dissimilarity measures and studied the behavior of proposed measures in general cases. Arthi and Mohana [1] studied divergence measure for interval valued Pythagorean fuzzy sets. The similarity and distance measure for Intuitionistic Fuzzy sets (IFS) plays a significant role in fuzzy entropy and gains wide applications in problems involving ambiguity of variables. Bhattacharya [3] studied the properties of IFS with various operators. Montes et al. [25] proposed divergence measures for IFS and studied relations among them. Ju et al. [21], Xiao [36] proposed application of divergence-based distance measure for intuitionistic fuzzy sets in decision making innovation management and pattern recognition. Deng and Wong, [8] proposed a novel distance measure for Fermatean fuzzy sets based on Hellinger distance and triangular divergence. Some researches [24, 26, 35] proved the applicability of novel picture fuzzy divergence measure in medical diagnosis, clustering and pattern recognition problems. A recent study Singh and Singh [30] presents innovative approaches for constructing picture fuzzy divergence measures, demonstrating their utility in complex decision-making scenarios such as pattern recognition, multi-attribute decision making and clustering analysis. Khan et al. [22] studied fuzziness for Jensen and Schur inequalities and presented Hermite-Hadamard inequality for convex fuzzy fractional integrals. Tomar and Ohlan [34] presented a series of fuzzy information inequalities using divergence measures and discussed its application in pattern recognition. Yi et al. [37] presented the variance for some special fuzzy numbers and obtained inequalities for their upper bounds. Others [2, 5] introduced fuzzy variational like inequalities in Hilbert spaces.

2.1 Preliminaries

This section provides the necessary background and definitions of Fuzzy sets, Intuitionistic Fuzzy sets and Picture Fuzzy sets.

2.1.1 Divergence functional

Consider a set of finite probability distributions

$$\varrho_n = \left\{ \mathfrak{Y} = (\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_n) \;\middle|\; \mathfrak{y}_i \geq 0, \; \sum_{i=1}^{n} \mathfrak{y}_i = 1 \right\}.$$

Then the new f-divergence introduced by Jain and Saraswat [19] is defined as

$$S_f(\mathfrak{X}, \mathfrak{Y}) = \sum_{i=1}^{n} \mathfrak{y}_i \, f\left(\frac{\mathfrak{x}_i + \mathfrak{y}_i}{2\mathfrak{y}_i} \right) \tag{2.1}$$

where $f : (0.5, \infty) \to \mathbb{R}$ is a convex function and $\mathfrak{X} = (\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_n)$ and $\mathfrak{Y} = (\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_n)$ are probability distributions. Convexity of the function f gives rise to the non-negativity property of the new f-divergence, i.e., the new f-divergence possesses the following properties:

$$S_f(\mathfrak{X}, \mathfrak{Y}) \geq f(1), \qquad S_f(\mathfrak{X}, \mathfrak{Y}) \geq 0,$$

with equality holding if and only if $\mathfrak{X} = \mathfrak{Y}$.

2.1.2 Fuzzy sets

(Zadeh, 1965) introduced fuzzy sets as a set where each member of the set contains its degree of belongingness to the set. The concept of an object lying or not lying in a particular set gave rise to the fuzzy set. It ends the conflict of partial members of any set.

A fuzzy set $\mathfrak{Y}$ defined on a universe $\mathfrak{Z}$ is given as

$$\mathfrak{Y} = \{ \langle \mathfrak{z}, \aleph_{\mathfrak{Y}}(\mathfrak{z}) \; / \; \mathfrak{z} \in \mathfrak{Z} \rangle \},$$

where $\aleph_{\mathfrak{Y}} : \mathfrak{Z} \to [0, 1]$ is the membership function of $\mathfrak{Y}$, which determines the degree of belongingness of $\mathfrak{z}$ in $\mathfrak{Y}$ with the constraint $0 \leq \aleph_{\mathfrak{Y}}(\mathfrak{z}) \leq 1$.

2.1.3 Intuitionistic fuzzy sets

Intuitionistic fuzzy sets (Atanassov, 1986) incorporate the degree of membership along with the degree of non-membership of any element in a set. An intuitionistic fuzzy set $\mathfrak{Y}$ defined on a universe of discourse $\mathfrak{Z}$ is given as:

$$\mathfrak{Y} = \{\langle \mathfrak{z}, \aleph_{\mathfrak{Y}}(\mathfrak{z}), \Phi_{\mathfrak{Y}}(\mathfrak{z}) \ / \ \mathfrak{z} \in \mathfrak{Z} \rangle\},$$

where $\aleph_{\mathfrak{Y}}(\mathfrak{z})$ and $\Phi_{\mathfrak{Y}}(\mathfrak{z})$ are the degree of membership and degree of non-membership respectively, with the constraints

$$\aleph_{\mathfrak{Y}}(\mathfrak{z}) \geq 0, \quad \Phi_{\mathfrak{Y}}(\mathfrak{z}) \geq 0, \quad 0 \leq \aleph_{\mathfrak{Y}}(\mathfrak{z}) + \Phi_{\mathfrak{Y}}(\mathfrak{z}) \leq 1.$$

The degree of hesitancy is given by

$$\tau_{\mathfrak{Y}}(\mathfrak{z}) = 1 - \left(\aleph_{\mathfrak{Y}}(\mathfrak{z}) + \Phi_{\mathfrak{Y}}(\mathfrak{z})\right),$$

which represents the amount of uncertainty or hesitation corresponding to the membership of any element. For $\Phi_{\mathfrak{Y}}(\mathfrak{z}) = 0$, the IFS behaves as a simple fuzzy set.

2.1.4 Picture fuzzy sets

Picture fuzzy sets (Cuong & Kreinovich, 2013) deal with uncertainty through three degrees of membership, i.e., positive, negative, and neutral membership. Problems with answers such as "yes", "no", "abstain", and "refusal" can be effectively handled using picture fuzzy sets. A picture fuzzy set $\mathfrak{Y}$ defined on a universe of discourse $\mathfrak{Z}$ is given as:

$$\mathfrak{Y} = \{\langle \mathfrak{z}, \aleph_{\mathfrak{Y}}(\mathfrak{z}), \Psi_{\mathfrak{Y}}(\mathfrak{z}), \Phi_{\mathfrak{Y}}(\mathfrak{z}) \ / \ \mathfrak{z} \in \mathfrak{Z} \rangle\},$$

where $\aleph_{\mathfrak{Y}}(\mathfrak{z})$, $\Psi_{\mathfrak{Y}}(\mathfrak{z})$, and $\Phi_{\mathfrak{Y}}(\mathfrak{z})$ represent the degrees of positive, neutral, and negative membership respectively, with the constraints

$$\aleph_{\mathfrak{Y}}(\mathfrak{z}) \geq 0, \quad \Psi_{\mathfrak{Y}}(\mathfrak{z}) \geq 0, \quad \Phi_{\mathfrak{Y}}(\mathfrak{z}) \geq 0, \quad 0 \leq \aleph_{\mathfrak{Y}}(\mathfrak{z}) + \Psi_{\mathfrak{Y}}(\mathfrak{z}) + \Phi_{\mathfrak{Y}}(\mathfrak{z}) \leq 1.$$

The degree of refusal is given by

$$\tau_{\mathfrak{Y}}(\mathfrak{z}) = 1 - \left(\aleph_{\mathfrak{Y}}(\mathfrak{z}) + \Psi_{\mathfrak{Y}}(\mathfrak{z}) + \Phi_{\mathfrak{Y}}(\mathfrak{z})\right),$$

which reflects the uncertainty associated with the membership of elements.
If $\Psi_{\mathfrak{Y}}(\mathfrak{z}) = 0$, then the set behaves as an intuitionistic fuzzy set, and when $\Psi_{\mathfrak{Y}}(\mathfrak{z}) = 0 = \Phi_{\mathfrak{Y}}(\mathfrak{z})$, it behaves as a simple fuzzy set.
To understand the need for picture fuzzy sets in real-life scenarios, consider a hypothetical situation where a survey has been conducted to determine customer satisfaction for a newly launched product. The survey results can be divided into four categories: the *positive membership*, i.e., customers satisfied with the product; the *negative membership*, representing dissatisfied customers; the *neutral membership*, for those indifferent about the product's quality; and the *refusal degree*, corresponding to customers who decline to give feedback. In such cases, categorization of diverse opinions becomes easier with the four criteria provided by picture fuzzy sets.
Some basic fuzzy set operations for picture fuzzy sets are given as follows.
Let

$$\mathfrak{L} = \{\langle \mathfrak{b}_i, \aleph_{\mathfrak{L}}(\mathfrak{b}_i), \Psi_{\mathfrak{L}}(\mathfrak{b}_i), \Phi_{\mathfrak{L}}(\mathfrak{b}_i) \mid \mathfrak{b}_i \in \mathfrak{B} \rangle\},$$
$$\mathfrak{M} = \{\langle \mathfrak{b}_i, \aleph_{\mathfrak{M}}(\mathfrak{b}_i), \Psi_{\mathfrak{M}}(\mathfrak{b}_i), \Phi_{\mathfrak{M}}(\mathfrak{b}_i) \mid \mathfrak{b}_i \in \mathfrak{B} \rangle\}$$

be two picture fuzzy sets with positive membership function $\aleph$, neutral membership function Ψ, and negative membership function Φ. Then,

Union of picture fuzzy sets

For picture fuzzy sets $\mathfrak{L}$ and $\mathfrak{M}$,

$$\mathfrak{L} \cup \mathfrak{M} = \{\langle \mathfrak{z}_i, \max(\aleph_{\mathfrak{L}}(\mathfrak{z}_i), \aleph_{\mathfrak{M}}(\mathfrak{z}_i)), \min(\Psi_{\mathfrak{L}}(\mathfrak{z}_i), \Psi_{\mathfrak{M}}(\mathfrak{z}_i)), \min(\Phi_{\mathfrak{L}}(\mathfrak{z}_i), \Phi_{\mathfrak{M}}(\mathfrak{z}_i)) \mid \mathfrak{z}_i \in \mathfrak{Z}\rangle\}.$$

Intersection of picture fuzzy sets

For picture fuzzy sets $\mathfrak{L}$ and $\mathfrak{M}$,

$$\mathfrak{L} \cap \mathfrak{M} = \{\langle \mathfrak{z}_i, \min(\aleph_{\mathfrak{L}}(\mathfrak{z}_i), \aleph_{\mathfrak{M}}(\mathfrak{z}_i)), \max(\Psi_{\mathfrak{L}}(\mathfrak{z}_i), \Psi_{\mathfrak{M}}(\mathfrak{z}_i)), \max(\Phi_{\mathfrak{L}}(\mathfrak{z}_i), \Phi_{\mathfrak{M}}(\mathfrak{z}_i)) \mid \mathfrak{z}_i \in \mathfrak{Z}\rangle\}.$$

Subset

For picture fuzzy sets $\mathfrak{L}$ and $\mathfrak{M}$,

$$\mathfrak{L} \subseteq \mathfrak{M} \text{ if } \aleph_{\mathfrak{L}}(\mathfrak{z}_i) \leq \aleph_{\mathfrak{M}}(\mathfrak{z}_i), \quad \Psi_{\mathfrak{L}}(\mathfrak{z}_i) \leq \Psi_{\mathfrak{M}}(\mathfrak{z}_i), \quad \Phi_{\mathfrak{L}}(\mathfrak{z}_i) \geq \Phi_{\mathfrak{M}}(\mathfrak{z}_i).$$

Similarly,

$$\mathfrak{M} \subseteq \mathfrak{L} \text{ if } \aleph_{\mathfrak{L}}(\mathfrak{z}_i) \geq \aleph_{\mathfrak{M}}(\mathfrak{z}_i), \quad \Psi_{\mathfrak{L}}(\mathfrak{z}_i) \geq \Psi_{\mathfrak{M}}(\mathfrak{z}_i), \quad \Phi_{\mathfrak{L}}(\mathfrak{z}_i) \leq \Phi_{\mathfrak{M}}(\mathfrak{z}_i).$$

Complement

The complement of a picture fuzzy set $\mathfrak{L}$ is denoted as $\mathfrak{L}^c$ and is defined as:

$$\mathfrak{L} = \{\langle \mathfrak{z}, \aleph_{\mathfrak{L}}(\mathfrak{z}), \Psi_{\mathfrak{L}}(\mathfrak{z}), \Phi_{\mathfrak{L}}(\mathfrak{z}) \mid \mathfrak{z} \in \mathfrak{Z}\rangle\},$$

$$\mathfrak{L}^c = \{\langle \mathfrak{z}, \Phi_{\mathfrak{L}}(\mathfrak{z}), \Psi_{\mathfrak{L}}(\mathfrak{z}), \aleph_{\mathfrak{L}}(\mathfrak{z}) \mid \mathfrak{z} \in \mathfrak{Z}\rangle\}.$$

Equality of sets

For any two picture fuzzy sets $\mathfrak{L}$ and $\mathfrak{M}$,

$$\mathfrak{L} = \mathfrak{M} \text{ if and only if } \mathfrak{L} \subseteq \mathfrak{M} \text{ and } \mathfrak{M} \subseteq \mathfrak{L}.$$

In the literature of information theory, although various measures exists for fuzzy sets and its extensions, there remains a need to minimize the uncertainty in actual experiences. The following section proposes a picture fuzzy divergence based on New f-divergence, particularly designed to capture the intrinsic uncertainty.

3 Picture fuzzy divergence measure

Motivated by the variational applicability and properties of New f-divergence, the following measure for picture fuzzy sets has been constructed.
Let $f : (0.5, \infty) \to \mathbb{R}$ be a normalized and convex function. Then for picture fuzzy sets $\mathfrak{L}, \mathfrak{M}$, corresponding to the new-f-divergence and using properties of picture fuzzy divergence measure, we get the new divergence which is given by

$$S_f(\mathfrak{L},\mathfrak{M}) = \sum_{i=1}^{n} \left[\begin{array}{l} \aleph_{\mathfrak{M}}(b_i)\, f\left(\dfrac{\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i)}{2\aleph_{\mathfrak{M}}(b_i)}\right) + \Psi_{\mathfrak{M}}(b_i)\, f\left(\dfrac{\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i)}{2\Psi_{\mathfrak{M}}(b_i)}\right) \\ +\Phi_{\mathfrak{M}}(b_i)\, f\left(\dfrac{\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i)}{2\Phi_{\mathfrak{M}}(b_i)}\right) \end{array} \right]. \tag{3.1}$$

The equation (3.1) may be said to define the *picture fuzzy new f-divergence measure.* The non-negativity of the measure (3.1) can be easily verified, i.e. $S_f(\mathfrak{L},\mathfrak{M}) \geq 0$, and equality holds iff $\mathfrak{L} = \mathfrak{M}$. New f-divergence leads to a series of information divergence measures by employing different convex functions. Divergence measures use the properties of convex functions in mathematical and computational contexts. For the convex function $f : (0,\infty) \to \mathbb{R}$ such that

$$f_k(t) = t\left(1-\frac{1}{t}\right)^{2k},$$

Jain and Saraswat [19] proposed the information divergence measure denoted and defined as:

$$N_k(\mathfrak{X},\mathfrak{Y}) = \frac{1}{2}\sum_{i=1}^{n} \frac{(\mathfrak{x}_i-\mathfrak{y}_i)^{2k}}{(\mathfrak{x}_i+\mathfrak{y}_i)^{2k-1}} \tag{3.2}$$

for convex function $f : (0.5,\infty) \to \mathbb{R}$ and probability distributions $\mathfrak{X} = (\mathfrak{x}_1,\mathfrak{x}_2,\ldots,\mathfrak{x}_n)$ and $\mathfrak{Y} = (\mathfrak{y}_1,\mathfrak{y}_2,\ldots,\mathfrak{y}_n)$. Extending the notion of information divergence measures, we propose a novel *picture fuzzy divergence measure* as follows:

$$SN_k(\mathfrak{L},\mathfrak{M}) = \frac{1}{2n}\sum_{i=1}^{n} \left[\begin{array}{l} \dfrac{(\aleph_{\mathfrak{L}}(b_i)-\aleph_{\mathfrak{M}}(b_i))^{2k}}{(\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i))^{2k-1}} + \dfrac{(\Psi_{\mathfrak{L}}(b_i)-\Psi_{\mathfrak{M}}(b_i))^{2k}}{(\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i))^{2k-1}} \\ +\dfrac{(\Phi_{\mathfrak{L}}(b_i)-\Phi_{\mathfrak{M}}(b_i))^{2k}}{(\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i))^{2k-1}} \end{array} \right]. \tag{3.3}$$

To prove the validity of the proposed divergence, let us examine the following axioms for a divergence measure.

Axioms:

$$\begin{aligned} A_1 &: \; 0 \leq SN_k(\mathfrak{L},\mathfrak{M}), \\ A_2 &: \; SN_k(\mathfrak{L},\mathfrak{M}) = 0 \text{ iff } \mathfrak{L} = \mathfrak{M}, \\ A_3 &: \; SN_k(\mathfrak{L},\mathfrak{M}) = SN_k(M,L), \\ A_4 &: \; SN_k(\mathfrak{L},\mathfrak{M}) = SN_k(L^c,M^c). \end{aligned}$$

Proof: For picture fuzzy sets L, M, N,

A$_1$: Since $\aleph_{\mathfrak{L}}(b_i), \Psi_{\mathfrak{L}}(b_i), \Phi_{\mathfrak{L}}(b_i), \aleph_M(b_i), \Psi_M(b_i), \Phi_M(b_i) \geq 0$,

$$\begin{aligned} SN_k(\mathfrak{L},\mathfrak{M}) = \frac{1}{2n}\sum_{i=1}^{n} \Bigg[& \frac{(\aleph_{\mathfrak{L}}(b_i)-\aleph_M(b_i))^{2k}}{(\aleph_{\mathfrak{L}}(b_i)+\aleph_M(b_i))^{2k-1}} \\ & + \frac{(\Psi_{\mathfrak{L}}(b_i)-\Psi_M(b_i))^{2k}}{(\Psi_{\mathfrak{L}}(b_i)+\Psi_M(b_i))^{2k-1}} \\ & + \frac{(\Phi_{\mathfrak{L}}(b_i)-\Phi_M(b_i))^{2k}}{(\Phi_{\mathfrak{L}}(b_i)+\phi_M(b_i))^{2k-1}} \Bigg] \geq 0. \end{aligned} \tag{3.4}$$

A$_2$: Consider two picture fuzzy sets $\mathfrak{L}, \mathfrak{M}$. Firstly, let $\mathfrak{L} = \mathfrak{M}$, i.e.

$$\aleph_{\mathfrak{L}}(b_i) = \aleph_M(b_i),\ \Psi_{\mathfrak{L}}(b_i) = \Psi_M(b_i),\ \Phi_{\mathfrak{L}}(b_i) = \Phi_M(b_i), \quad \forall b_i \in B.$$

Then,

$$\begin{aligned} SN_k(\mathfrak{L}, \mathfrak{M}) = \frac{1}{2n} \sum_{i=1}^{n} \Bigg[& \frac{(\aleph_{\mathfrak{L}}(b_i) - \aleph_M(b_i))^{2k}}{(\aleph_{\mathfrak{L}}(b_i) + \aleph_M(b_i))^{2k-1}} \\ & + \frac{(\Psi_{\mathfrak{L}}(b_i) - \Psi_M(b_i))^{2k}}{(\Psi_{\mathfrak{L}}(b_i) + \Psi_M(b_i))^{2k-1}} \\ & + \frac{(\Phi_{\mathfrak{L}}(b_i) - \Phi_M(b_i))^{2k}}{(\Phi_{\mathfrak{L}}(b_i) + \phi_M(b_i))^{2k-1}} \Bigg] = 0. \end{aligned} \tag{3.5}$$

Conversely, if $SN_k(\mathfrak{L}, \mathfrak{M}) = 0$, since each term is non-negative, we must have

$$\frac{(\aleph_{\mathfrak{L}}(b_i) - \aleph_M(b_i))^{2k}}{(\aleph_{\mathfrak{L}}(b_i) + \aleph_M(b_i))^{2k-1}} = 0, \quad \frac{(\Psi_{\mathfrak{L}}(b_i) - \Psi_M(b_i))^{2k}}{(\Psi_{\mathfrak{L}}(b_i) + \Psi_M(b_i))^{2k-1}} = 0, \quad \frac{(\Phi_{\mathfrak{L}}(b_i) - \Phi_M(b_i))^{2k}}{(\Phi_{\mathfrak{L}}(b_i) + \phi_M(b_i))^{2k-1}} = 0,$$

which implies

$$\aleph_{\mathfrak{L}}(b_i) = \aleph_M(b_i),\ \Psi_{\mathfrak{L}}(b_i) = \Psi_M(b_i),\ \Phi_{\mathfrak{L}}(b_i) = \Phi_M(b_i).$$

Hence, $\mathfrak{L} = \mathfrak{M}$.
A$_3$: Since

$$(\aleph_{\mathfrak{L}}(b_i) - \aleph_M(b_i))^{2k} = (\aleph_M(b_i) - \aleph_{\mathfrak{L}}(b_i))^{2k},$$

$$(\aleph_{\mathfrak{L}}(b_i) + \aleph_M(b_i))^{2k-1} = (\aleph_M(b_i) + \aleph_{\mathfrak{L}}(b_i))^{2k-1},$$

and similarly for Ψ and Φ, it follows that

$$SN_k(\mathfrak{L}, \mathfrak{M}) = SN_k(\mathfrak{M}, \mathfrak{L}).$$

A$_4$:

$$\mathfrak{L} = \{\langle b_i, \aleph_{\mathfrak{L}}(b_i), \Psi_{\mathfrak{L}}(b_i), \Phi_{\mathfrak{L}}(b_i) \mid b_i \in B\rangle\},$$

$$\mathfrak{L}^c = \{\langle b_i, \Phi_{\mathfrak{L}}(b_i), \Psi_{\mathfrak{L}}(b_i), \aleph_{\mathfrak{L}}(b_i) \mid b_i \in B\rangle\}.$$

Hence,

$$\begin{aligned} SN_k(\mathfrak{L}^c, \mathfrak{M}^c) = \frac{1}{2n} \sum_{i=1}^{n} \Bigg[& \frac{(\Phi_{\mathfrak{L}}(b_i) - \Phi_M(b_i))^{2k}}{(\Phi_{\mathfrak{L}}(b_i) + \phi_M(b_i))^{2k-1}} \\ & + \frac{(\Psi_{\mathfrak{L}}(b_i) - \Psi_M(b_i))^{2k}}{(\Psi_{\mathfrak{L}}(b_i) + \Psi_M(b_i))^{2k-1}} \\ & + \frac{(\aleph_{\mathfrak{L}}(b_i) - \aleph_M(b_i))^{2k}}{(\aleph_{\mathfrak{L}}(b_i) + \aleph_M(b_i))^{2k-1}} \Bigg] \end{aligned} \tag{3.6}$$

$$\Rightarrow\ SN_k(\mathfrak{L}^c, \mathfrak{M}^c) = SN_k(\mathfrak{L}, \mathfrak{M}).$$

Particular case: For $k=1$, $SN_k(\mathfrak{L},\mathfrak{M})$ corresponds to the picture fuzzy triangular discrimination:

$$SN_k(\mathfrak{L},\mathfrak{M}) = \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{(\aleph_{\mathfrak{L}}(b_i)-\aleph_M(b_i))^2}{(\aleph_{\mathfrak{L}}(b_i)+\aleph_M(b_i))} + \frac{(\Psi_{\mathfrak{L}}(b_i)-\Psi_M(b_i))^2}{(\Psi_{\mathfrak{L}}(b_i)+\Psi_M(b_i))} + \frac{(\Phi_{\mathfrak{L}}(b_i)-\Phi_M(b_i))^2}{(\Phi_{\mathfrak{L}}(b_i)+\phi_M(b_i))}\right] = SN_\Delta(\mathfrak{L},\mathfrak{M}). \tag{3.7}$$

4 Fuzzy information inequalities

This section presents some information inequalities for$SN_\Delta(\mathfrak{L},\mathfrak{M})$ utilizing picture fuzzy new f-divergence and other well-known divergence measures. Such fuzzy inequalities are useful in comparative study of information measures. The inequalities discussed by Jain and Saraswat [19] form the basis of the following fuzzy information inequality.

Theorem 4.1. *Consider a normalized mapping $f:(0,\infty)\to\mathbb{R}$ which is twice differentiable on $(\mathfrak{r},\mathfrak{R})$ for $0\le\mathfrak{r}\le 1\le\mathfrak{R}<\infty$. Then there exist constants $\mathfrak{m},\mathfrak{M}$ such that*

$$\mathfrak{m}\le t\,f''(t)\le\mathfrak{M}\quad \textit{for } t\in(\mathfrak{r},\mathfrak{R}).$$

For the membership, non-membership and neutrality functions $\aleph,\Phi,\Psi$ respectively, satisfying the assumptions

$$\mathfrak{r}\le\tfrac{1}{2}\le\frac{\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i)}{2\aleph_{\mathfrak{M}}(b_i)}\le\mathfrak{R},$$

$$\mathfrak{r}\le\tfrac{1}{2}\le\frac{\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i)}{2\Psi_{\mathfrak{M}}(b_i)}\le\mathfrak{R},$$

$$\mathfrak{r}\le\tfrac{1}{2}\le\frac{\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i)}{2\Phi_{\mathfrak{M}}(b_i)}\le\mathfrak{R},$$

Then we have the inequality:

$$\frac{\mathfrak{m}}{2}SN_\Delta(\mathfrak{L},\mathfrak{M})\le S_f(\mathfrak{L},\mathfrak{M})\le\frac{\mathfrak{M}}{2}SN_\Delta(\mathfrak{L},\mathfrak{M}). \tag{4.1}$$

Proof. Let us define a mapping $G(t)=f(t)-\mathfrak{M}\,\dfrac{(t-1)^2}{t}$. Then $G(t)$ is twice differentiable and a normalized function since we have

$$G''(t)=f''(t)-\frac{\mathfrak{m}}{t^3}=\frac{1}{t^3}\left[t^3f''(t)-\mathfrak{M}\right]\ge 0. \tag{4.1}$$

Now, using the non-negativity property of divergence measures, we can conclude that

$$\begin{aligned}0&\le S_G(\mathfrak{L},\mathfrak{M})\\&=S_f(\mathfrak{L},\mathfrak{M})-\mathfrak{M}\,S_{\frac{(t-1)^2}{t}}(\mathfrak{L},\mathfrak{M}).\end{aligned} \tag{4.2}$$

Since $S_{\frac{(t-1)^2}{t}}(\mathfrak{L},\mathfrak{M}) = \frac{1}{2}\,SN_\Delta(\mathfrak{L},\mathfrak{M})$, equation (4.2) becomes

$$\begin{aligned} 0 &\le S_G(\mathfrak{L},\mathfrak{M}) \\ &= S_f(\mathfrak{L},\mathfrak{M}) - \frac{\mathfrak{m}}{2}\,SN_\Delta(\mathfrak{L},\mathfrak{M}) \\ \Rightarrow \quad 0 &\le S_f(\mathfrak{L},\mathfrak{M}) - \frac{\mathfrak{m}}{2}\,SN_\Delta(\mathfrak{L},\mathfrak{M}). \end{aligned} \tag{4.3}$$

Again, define $F(t) = \mathfrak{M}\,\dfrac{(t-1)^2}{t} - f(t)$, which is normalized and twice differentiable. Using the non-negativity property of divergence measures and the property $S_{\frac{(t-1)^2}{t}}(\mathfrak{L},\mathfrak{M}) = \frac{1}{2}\,SN_\Delta(\mathfrak{L},\mathfrak{M})$, we have

$$\begin{aligned} 0 &\le S_F(\mathfrak{L},\mathfrak{M}) \\ &= \mathfrak{M}\,S_{\frac{(t-1)^2}{t}}(\mathfrak{L},\mathfrak{M}) - S_f(\mathfrak{L},\mathfrak{M}) \\ &= \frac{\mathfrak{M}}{2}\,SN_\Delta(\mathfrak{L},\mathfrak{M}) - S_f(\mathfrak{L},\mathfrak{M}) \\ \Rightarrow \quad 0 &\le \frac{\mathfrak{M}}{2}\,SN_\Delta(\mathfrak{L},\mathfrak{M}) - S_f(\mathfrak{L},\mathfrak{M}). \end{aligned} \tag{4.4}$$

Equations (4.3) and (4.4) together conclude the required inequality (4.1).

□

4.1 Bounds using classical measures

4.1.1 Relative arithmetic-geometric divergence

Redefining the convex function in $S_f(\mathfrak{L},\mathfrak{M})$, let $f(x) = x\log x$ and $g(x) = x^3 f''(x)$. Then the new f-divergence becomes

$$\begin{aligned} S_f(\mathfrak{L},\mathfrak{M}) = \sum_{i=1}^{n} \Bigg[& \frac{\aleph_{\mathfrak{L}}(b_i) + \aleph_{\mathfrak{M}}(b_i)}{2} \log\left(\frac{\aleph_{\mathfrak{L}}(b_i) + \aleph_{\mathfrak{M}}(b_i)}{2\,\aleph_{\mathfrak{M}}(b_i)} \right) \\ & + \frac{\Psi_{\mathfrak{L}}(b_i) + \Psi_{\mathfrak{M}}(b_i)}{2} \log\left(\frac{\Psi_{\mathfrak{L}}(b_i) + \Psi_{\mathfrak{M}}(b_i)}{2\,\Psi_{\mathfrak{M}}(b_i)} \right) \\ & + \frac{\Phi_{\mathfrak{L}}(b_i) + \Phi_{\mathfrak{M}}(b_i)}{2} \log\left(\frac{\Phi_{\mathfrak{L}}(b_i) + \Phi_{\mathfrak{M}}(b_i)}{2\,\Phi_{\mathfrak{M}}(b_i)} \right) \Bigg] = G(\mathfrak{L},\mathfrak{M}). \end{aligned} \tag{4.2}$$

The bounds of $G(\mathfrak{L},\mathfrak{M})$ are defined as

$$\sup_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = \mathfrak{R}^2, \qquad \inf_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = \mathfrak{r}^2.$$

Then, corresponding to the inequality given by Jain and Saraswat, the picture fuzzy inequality is given as

$$\frac{\mathfrak{r}^2}{2}\left[\frac{1}{2n}\sum_{i=1}^{n}\Big(\frac{(\aleph_{\mathfrak{L}}(b_i)-\aleph_{\mathfrak{M}}(b_i))^2}{(\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i))}+\frac{(\Psi_{\mathfrak{L}}(b_i)-\Psi_{\mathfrak{M}}(b_i))^2}{(\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i))}\right.$$
$$\left.+\frac{(\Phi_{\mathfrak{L}}(b_i)-\Phi_{\mathfrak{M}}(b_i))^2}{(\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i))}\Big)\right]$$
$$\leq \sum_{i=1}^{n}\left[\frac{\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i)}{2}\log\left(\frac{\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i)}{2\,\aleph_{\mathfrak{M}}(b_i)}\right)\right.$$
$$+\frac{\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i)}{2}\log\left(\frac{\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i)}{2\,\Psi_{\mathfrak{M}}(b_i)}\right) \tag{4.3}$$
$$\left.+\frac{\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i)}{2}\log\left(\frac{\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i)}{2\,\Phi_{\mathfrak{M}}(b_i)}\right)\right]$$
$$\leq \frac{\mathfrak{R}^2}{2}\left[\frac{1}{2n}\sum_{i=1}^{n}\Big(\frac{(\aleph_{\mathfrak{L}}(b_i)-\aleph_{\mathfrak{M}}(b_i))^2}{(\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i))}+\frac{(\Psi_{\mathfrak{L}}(b_i)-\Psi_{\mathfrak{M}}(b_i))^2}{(\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i))}\right.$$
$$\left.+\frac{(\Phi_{\mathfrak{L}}(b_i)-\Phi_{\mathfrak{M}}(b_i))^2}{(\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i))}\Big)\right].$$

Hence,

$$\frac{\mathfrak{r}^2}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}) \leq G(\mathfrak{L},\mathfrak{M}) \leq \frac{\mathfrak{R}^2}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}). \tag{4.4}$$

4.1.2 Relative Jensen-Shannon divergence

Let us define a convex function for $S_f(\mathfrak{L},\mathfrak{M})$, i.e.

$$f(x) = -\log x, \qquad g(x) = x^3 f''(x).$$

Then,

$$g(x) = x^3\left(\frac{1}{x^2}\right) = x.$$

Let the functions $\aleph, \Phi, \Psi$ satisfy condition (4.1). Thus, we define the bounds of $g(x)$ as

$$\mathfrak{M} = \sup_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = \mathfrak{R}, \qquad \mathfrak{m} = \inf_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = \mathfrak{r}.$$

Also,

$$S_f(\mathfrak{L},\mathfrak{M}) = \sum_{i=1}^{n}\left[\aleph_{\mathfrak{M}}(b_i)\log\left(\frac{2\,\aleph_{\mathfrak{M}}(b_i)}{\aleph_{\mathfrak{L}}(b_i)+\aleph_{\mathfrak{M}}(b_i)}\right)+\Psi_{\mathfrak{M}}(b_i)\log\left(\frac{2\,\Psi_{\mathfrak{M}}(b_i)}{\Psi_{\mathfrak{L}}(b_i)+\Psi_{\mathfrak{M}}(b_i)}\right)\right.$$
$$\left.+\Phi_{\mathfrak{M}}(b_i)\log\left(\frac{2\,\Phi_{\mathfrak{M}}(b_i)}{\Phi_{\mathfrak{L}}(b_i)+\Phi_{\mathfrak{M}}(b_i)}\right)\right] = F(\mathfrak{M},\mathfrak{L}). \tag{4.5}$$

Then, from (4.2), we have

$$\frac{\mathfrak{r}}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}) \leq F(\mathfrak{M},\mathfrak{L}) \leq \frac{\mathfrak{R}}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}). \tag{4.6}$$

Interchanging the roles of $\mathfrak{L}$ and $\mathfrak{M}$,

$$\frac{\mathfrak{r}}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}) \leq F(\mathfrak{L},\mathfrak{M}) \leq \frac{\mathfrak{R}}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}). \tag{4.7}$$

4.1.3 Chi-square divergence

Let for the functions $\aleph, \Phi, \Psi$ condition (4.1) be satisfied. Then, for the convex function $f(x) = x^2 - 1$, let us define

$$g(x) = x^3 f''(x) \Rightarrow g(x) = 2x^3.$$

The bounds of $g(x)$ are given as:

$$\mathfrak{M} = \sup_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = 2\mathfrak{R}^3, \qquad \mathfrak{m} = \inf_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = 2\mathfrak{r}^3.$$

Also,

$$\begin{aligned} S_f(\mathfrak{L},\mathfrak{M}) &= \frac{1}{4}\sum_{i=1}^{n}\left[\left(\frac{\aleph_{\mathfrak{L}}(b_i)^2}{\aleph_{\mathfrak{M}}(b_i)} - 1\right) + \left(\frac{\Psi_{\mathfrak{L}}(b_i)^2}{\Psi_{\mathfrak{M}}(b_i)} - 1\right) + \left(\frac{\Phi_{\mathfrak{L}}(b_i)^2}{\Phi_{\mathfrak{M}}(b_i)} - 1\right)\right] \\ &= \frac{1}{4}\chi^2(\mathfrak{L},\mathfrak{M}). \end{aligned} \tag{4.8}$$

Then, from (4.2), we have

$$\mathfrak{r}^3\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}) \leq \frac{1}{4}\,\chi^2(\mathfrak{L},\mathfrak{M}) \leq \mathfrak{R}^3\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}). \tag{4.9}$$

4.1.4 Relative J-divergence

Let for the functions $\aleph, \Phi, \Psi$ condition (4.1) be satisfied. Then, for the convex function $f(x) = (x-1)\log x$, let us define

$$g(x) = x^3 f''(x) \Rightarrow g(x) = x(x+1).$$

The bounds of $g(x)$ are given as:

$$\mathfrak{M} = \sup_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = \mathfrak{R}(\mathfrak{R}+1), \qquad \mathfrak{m} = \inf_{x\in[\mathfrak{r},\mathfrak{R}]} g(x) = \mathfrak{r}(\mathfrak{r}+1).$$

Also,

$$\begin{aligned} S_f(\mathfrak{L},\mathfrak{M}) = \sum_{i=1}^{n}&\left[\frac{(\aleph_{\mathfrak{L}}(b_i) - \aleph_{\mathfrak{M}}(b_i))}{2}\log\left(\frac{\aleph_{\mathfrak{L}}(b_i) + \aleph_{\mathfrak{M}}(b_i)}{2\aleph_{\mathfrak{M}}(b_i)}\right)\right. \\ &+ \frac{(\Psi_{\mathfrak{L}}(b_i) - \Psi_{\mathfrak{M}}(b_i))}{2}\log\left(\frac{\Psi_{\mathfrak{L}}(b_i) + \Psi_{\mathfrak{M}}(b_i)}{2\Psi_{\mathfrak{M}}(b_i)}\right) \\ &\left.+ \frac{(\Phi_{\mathfrak{L}}(b_i) - \Phi_{\mathfrak{M}}(b_i))}{2}\log\left(\frac{\Phi_{\mathfrak{L}}(b_i) + \Phi_{\mathfrak{M}}(b_i)}{2\Phi_{\mathfrak{M}}(b_i)}\right)\right] \\ = \frac{1}{2}J_R(\mathfrak{L},\mathfrak{M}).& \end{aligned} \tag{4.10}$$

Using (4.2), the bounds are given as:

$$\frac{\mathfrak{r}(\mathfrak{r}+1)}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}) \leq \frac{1}{2}\,J_R(\mathfrak{L},\mathfrak{M}) \leq \frac{\mathfrak{R}(\mathfrak{R}+1)}{2}\,SN_{\Delta}(\mathfrak{L},\mathfrak{M}). \tag{4.11}$$

4.1.5 Bhattacharya measure and Hellinger discrimination

Let for the functions $\aleph, \Phi, \Psi$ condition (4.1) be satisfied. Then, for the convex function $f(x) = 1 - \sqrt{x}$, let us define

$$g(x) = x^3 f''(x) \Rightarrow g(x) = x^{3/2}.$$

The bounds of $g(x)$ are given as:

$$\mathfrak{M} = \sup_{x \in [\mathfrak{r}, \mathfrak{R}]} g(x) = \mathfrak{R}^{3/2}, \qquad \mathfrak{m} = \inf_{x \in [\mathfrak{r}, \mathfrak{R}]} g(x) = \mathfrak{r}^{3/2}.$$

Also,

$$S_f(\mathfrak{L}, \mathfrak{M}) = 4\big(1 - B\left(\frac{\mathfrak{L} + \mathfrak{M}}{2}, \mathfrak{M}\right)\big) = 4h\left(\frac{\mathfrak{L} + \mathfrak{M}}{2}, \mathfrak{M}\right). \tag{4.12}$$

Using (4.2), the bounds are given as:

$$\begin{aligned} \frac{\mathfrak{r}^{3/2}}{2} SN_\Delta(\mathfrak{L}, \mathfrak{M}) &\leq 4\left(1 - B\left(\frac{\mathfrak{L} + \mathfrak{M}}{2}, \mathfrak{M}\right)\right) \leq \frac{\mathfrak{R}^{3/2}}{2} SN_\Delta(\mathfrak{L}, \mathfrak{M}), \\ \frac{\mathfrak{r}^{3/2}}{2} SN_\Delta(\mathfrak{L}, \mathfrak{M}) &\leq 4h\left(\frac{\mathfrak{L} + \mathfrak{M}}{2}, \mathfrak{M}\right) \leq \frac{\mathfrak{R}^{3/2}}{2} SN_\Delta(\mathfrak{L}, \mathfrak{M}). \end{aligned} \tag{4.13}$$

5 Applications of the proposed measure

5.1 Market behaviour predictions using PFDM

Market behaviour patterns are normally vague and uncertain due to partially known data. In such scenarios, the picture fuzzy divergence models the trends having three different membership degrees. The neutral degree of membership predicts indeterminacy in the state of confusion due to mixed trends. The PFDM models can effectively capture the deviation of current market trends with historical patterns and can predict potential crashes or booms. The approach quantifies the deviation of current market indicators from known bullish patterns to assist in early trend identification.

Let us consider a problem where we need to predict the market pattern whether it is moving towards bullish, bearish or neutral trends. The market indicators we are using here are:

- Stock index movement (SIM).
- Volume of trade (VoT).
- Volatility index (VIX).
- Investors sentiments (IS).
- GDP growth (GDP).

Now due to vague data for these indicators, we are representing the indicators using picture fuzzy parameters that are in the form of membership degrees. These picture fuzzy numbers represent

- $\aleph$ = Membership degree = supporting bullish trends.
- Ψ=Indeterminancy (noise or neutral).
- Φ= Negative membership = supporting bearish trends.

Table 1: Reference Bullish Pattern

Indicators	$\aleph_M$	Ψ_M	Φ_M
SIM	0.85	0.10	0.05
VoT	0.80	0.15	0.05
VIX	0.20	0.10	0.70
IS	0.75	0.15	0.10
GDP	0.85	0.10	0.05

The reference bullish market pattern ($\mathfrak{M}$) with respect to strong bullish years for the indicators are tabulated in table 1.

The current observed pattern ($\mathfrak{L}$) of the market is tabulated in table 2.

Table 2: Current Observed Pattern

Indicators	$\aleph_L$	Ψ_L	Φ_L
SIM	0.60	0.30	0.10
VoT	0.80	0.10	0.10
VIX	0.50	0.20	0.30
IS	0.65	0.25	0.10
GDP	0.70	0.20	0.10

Now we compute the deviation of the current observed pattern from the reference pattern using PFDM $SN_\Delta(\mathfrak{L}, \mathfrak{M})$.

$$SN_\Delta(L, M) = \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{(\aleph_L(b_i) - \aleph_M(b_i))^2}{\aleph_L(b_i) + \aleph_M(b_i)} + \frac{(\Psi_L(b_i) - \Psi_M(b_i))^2}{\Psi_L(b_i) + \Psi_M(b_i)} + \frac{(\Phi_L(b_i) - \Phi_M(b_i))^2}{\Phi_L(b_i) + \Phi_M(b_i)}\right].$$

$$SN_\Delta(L, M) = \frac{1}{10}\left[0.1597 + 0.0267 + 0.3219 + 0.3214 + 0.0645\right].$$

$$SN_\Delta(L, M) = 0.0605.$$

The divergence obtained is 0.0605 which indicates a low deviation from bullish conditions. The indicators VIX and Investor sentiment contribute most to the deviation. However, some indicators like GDP outlook and VoT remain aligned with bullish characteristics, suggesting a mixed or transitioning market. The overall outcome does not indicate any risk and gives clear signs of a bullish trend in the market.

5.2 Pattern recognition problem using PFDM

Consider an unknown pattern $\mathfrak{P}$ and a set of known patterns $\{\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_k\}$ in the form of picture fuzzy sets defined as:

$$\mathfrak{P} = \{\langle \mathfrak{u}_i, \aleph_{\mathfrak{P}}(\mathfrak{u}_i), \Psi_{\mathfrak{P}}(\mathfrak{u}_i), \Phi_{\mathfrak{P}}(\mathfrak{u}_i) \mid \mathfrak{u}_i \in \mathfrak{U}\rangle\},$$

$$\mathfrak{X}_k = \{\langle \mathfrak{u}_i, \aleph_{\mathfrak{X}_k}(\mathfrak{u}_i), \Psi_{\mathfrak{X}_k}(\mathfrak{u}_i), \Phi_{\mathfrak{X}_k}(\mathfrak{u}_i) \mid \mathfrak{u}_i \in \mathfrak{U}\rangle\},$$

for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$.

The objective is to recognize the similarity of the unknown pattern to the known patterns and assign the closest pattern from the set $\{\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_k\}$. To establish the similarity between the patterns, the divergence $SN_\Delta(\mathfrak{X}_k, \mathfrak{P})$ is computed between the pairs of known and unknown patterns. Let there be three known patterns $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ and an unknown pattern $\mathfrak{P}$. Table 3 defines the known patterns. Since $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ are picture fuzzy sets, the values $(\aleph_{\mathfrak{X}_k}(\mathfrak{u}_i), \Psi_{\mathfrak{X}_k}(\mathfrak{u}_i), \Phi_{\mathfrak{X}_k}(\mathfrak{u}_i))$ represent the membership, non-membership, and neutrality degrees in the given Table 3. Table 4 gives the unknown pattern $\mathfrak{P}$.

Table 3: Unknown Patterns

	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$	$\mathfrak{u}_5$
$\mathfrak{X}_1$	(0.5, 0.3, 0.15)	(0.6, 0.2, 0.15)	(0.55, 0.25, 0.1)	(0.7, 0.1, 0.2)	(0.4, 0.2, 0.3)
$\mathfrak{X}_2$	(0.7, 0.1, 0.15)	(0.5, 0.2, 0.1)	(0.2, 0.1, 0.6)	(0.15, 0.6, 0.15)	(0.2, 0.3, 0.3)
$\mathfrak{X}_3$	(0.8, 0.1, 0.05)	(0.6, 0.05, 0.3)	(0.3, 0.4, 0.2)	(0.25, 0.5, 0.25)	(0.6, 0.2, 0.1)

Table 4: Known Pattern

	$\mathfrak{u}_1$	$\mathfrak{u}_2$	$\mathfrak{u}_3$	$\mathfrak{u}_4$	$\mathfrak{u}_5$
$\mathfrak{P}$	(0.7, 0.2, 0.1)	(0.2, 0.2, 0.5)	(0.4, 0.2, 0.3)	(0.8, 0.1, 0)	(0.4, 0.3, 0.2)

Now, our objective is to assign the known patterns to any one of the unknown one with the highest correspondence. The divergence values calculated by the proposed measure is given in Table 5.

Table 5: Divergence Values with $\mathrm{SN}_\Delta(\mathfrak{L}, \mathfrak{M})$

$\mathrm{SN}_\Delta(\mathfrak{X}_k, \mathfrak{P})$	Divergence Value
$(\mathfrak{X}_1, \mathfrak{P})$	0.08277
$(\mathfrak{X}_2, \mathfrak{P})$	0.167712
$(\mathfrak{X}_3, \mathfrak{P})$	0.139571

The minimum value of the divergence gives maximum correspondence. Hence, from Table 5, the known pattern $\mathfrak{X}_1$ shows maximum similarity to the unknown pattern $\mathfrak{P}$.

6 Conclusion

This study has established a novel picture fuzzy divergence measure and validate its existence using mathematical formulations. The proposed measure deal with the uncertainty and imprecision that often come with real-world data. A new informatic-theoretic approach is established to obtain bounds and information fuzzy inequalities with different divergence measures which help make comparisons between different fuzzy measures more insightful. Altogether, these contributions add valuable tools to the field, especially for multi-criteria decision-making problems, where making the right choice often depends on handling complex, uncertain information effectively.
Although the measure successfully capture uncertainty and reduces the divergence, the symmetricity of the measure limits its effectiveness. For problems involving directional approach like preference modelling, asymmetric measures proves more appropriate. This leaves the scope for

future study and generation of asymmetric measures of this kind.

Declarations

Availability of data and materials: Not applicable
Competing interests: The authors report there are no competing interests to declare.
Funding: No Funding
Authors' contributions: N. Sharma developed the core theoretical framework, including the formulation and properties of the measure. R.N. Saraswat reviewed and supervised the overall structure of the paper. All authors have read and approved the final manuscript.

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