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Sequence spaces constructed by using $\mathfrak{q}$-Pell-Lucas matrix and its geometric properties

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Abstract. In this paper, the $\mathfrak{q}$-Pell-Lucas matrix, denoted by $\grave{\mathrm{Q}}(\mathfrak{q}) = (\grave{\mathrm{Q}}^{(\mathfrak{q})}_{\mathrm{nk}})_{\mathrm{n,k}\in\mathbb{N}_0}$ is defined by

$$\grave{\mathrm{Q}}^{(\mathfrak{q})}_{\mathrm{nk}} = \begin{cases} \dfrac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}, & 0 \leq \mathrm{k} \leq \mathrm{n}, \\ 0, & \mathrm{k} > \mathrm{n}, \end{cases}$$

and $\{\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})\}$ corresponds to terms of the $\mathfrak{q}$-Pell-Lucas sequence. The sequence $\{\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})\}$ is defined by

$$\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) = \left\{2\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-2}(\mathfrak{q})\right\} \text{ for } \mathrm{k} \leq 2,\ \grave{\mathrm{Q}}_0(\mathfrak{q}) = 2, \grave{\mathrm{Q}}_1(\mathfrak{q}) = 2.$$

The $\mathfrak{q}$-Pell-Lucas matrix serves as the foundation for the construction of matrix domains known as the $\mathfrak{q}$-Pell-Lucas sequence spaces. Within these spaces, we develop a Schauder basis, carry out a detailed investigation of operator ideals, and provide a comprehensive study of the geometric properties of $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$ and $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$, particularly addressing the Dunford–Pettis property, and the solidity property.

Keywords. q-Pell-Lucas numbers, sequence space, Schauder basis, operator ideal, geometric property.

1 Introduction

The investigation of matrix operators represents a fundamental area of functional analysis and operator theory, particularly in the context of sequence spaces. Such operators offer a rigorous analytical framework for examining the transformation properties of sequences. A Banach space V is classified as a BK-space if, for every index $i \in \mathbb{N}$, the mapping $\pi_i : V \to \mathbb{C}$, where $\pi_i(v) = v_i$, acts continuously. Here, $\mathbb{C}$ denotes the complex number field. Notable examples of such spaces include ℓ_p (for $1 \leq p < \infty$) and ℓ_∞, each defined under corresponding norm conditions

$$\|v\|_{\ell_p} = \left(\sum_i |v_i|^p\right)^{\frac{1}{p}}, \quad \|v\|_{\ell_\infty} = \sup_{i\in\mathbb{N}} |v_i|, \text{ respectively.}$$

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Let $\mathfrak{V}$ and $\mathfrak{W}$ be two sequence spaces, and let $\Upsilon = (n_{ij})$ represent an infinite matrix of real entries. For simplicity, we will denote the matrix as $\Upsilon = (n_{ij})$ throughout this paper, omitting the limits ∞ for clarity. The i-th row of Υ is referred to as Υ_i.

The matrix Υ induces a transformation from $\mathfrak{V}$ to $\mathfrak{W}$ if, for every sequence $\mathfrak{v} = (\mathfrak{v}_i)$, the Υ-transform of $\mathfrak{v}$, denoted by $\Upsilon\mathfrak{v} = \{(\Upsilon\mathfrak{v})_i\}$, lies in $\mathfrak{W}$. Mathematically, the Υ-transform is defined by

$$(\Upsilon\mathfrak{v})_i = \sum_{j=1}^{\infty} \mathrm{n}_{ij}\mathfrak{v}_j, \quad i \in \mathbb{N}.$$

Sequence spaces $\mathfrak{V}_\Upsilon$ are defined as

$$\mathfrak{V}_\Upsilon = \{\mathfrak{v} = (\mathfrak{v}_i) \in \mathfrak{w} : \Upsilon\mathfrak{v} \in \mathfrak{V}\},$$

is known as the domain of the matrix Υ in the space $\mathfrak{V}$. The space ω represents the set of all scalar sequences. Within this framework, several specialized sequence spaces are defined as follows

- c_0: The set of sequences that converge to zero (null sequences).
- c: The set of all convergent sequences.
- ℓ_∞: The set of all bounded sequences.
- ℓ_p $(1 \leq p < \infty)$: The set of all sequences that are absolutely p-summable.
- cs: The set of sequences whose series are convergent.
- bs: The set of sequences with bounded series.

An $\mathfrak{q}$-analogue provides a generalization of classical structures by introducing the parameter $\mathfrak{q}$, in which the limit as $\mathfrak{q} \to 1$, yields the original formulation. The concept was first suggested by Euler, while Jackson later applied it to established $\mathfrak{q}$-differentiation and $\mathfrak{q}$-integration [14]. Recently, the utility of $\mathfrak{q}$-analogues has grown considerably, influencing many fields such as algebra, combinatorics, approximation theory, and special functions. Their relevance has also been demonstrated in summability theory and sequence spaces.

The investigation of $\mathfrak{q}$-analogues of sequence spaces has recently attracted considerable attention. Demiriz and Şahin [9], as well as Yaying *et al.* [29], studied the spaces $X(C(\mathfrak{q})) := X_{C(\mathfrak{q})}$ for $X \in \{\ell_p, c_0, c, \ell_\infty\}$. Subsequently, Yılmaz and Akdemir [32] analyzed the algebraic, topological, and geometric aspects of $(\ell_p)C(\mathfrak{q})$ and $(\ell_\infty)C(\mathfrak{q})$. Furthermore, Alotaibi *et al.* [1] expanded the field by defining new sequence spaces $\ell_p(\nabla^2_{\mathfrak{q}}) := (\ell_p)_{\nabla^2_{\mathfrak{q}}}$ and $\ell_\infty(\nabla^2_{\mathfrak{q}}) := (\ell_\infty)_{\nabla^2_{\mathfrak{q}}}$, using the operator $\nabla^2_{\mathfrak{q}}$ and established that $\ell_\infty(\nabla^2_{\mathfrak{q}})$ does not exhibit this symmetry. In the context of sequence spaces, a space X is termed symmetric [27] if the inclusion of a sequence (y_n) in X ensures that its permutation $y_{\pi(n)}$ also lies in X. For a basic understanding of the $\mathfrak{q}$-theory, readers can refer to the monograph [16].

Yaying *et al.* [29] introduced the $\mathfrak{q}$-Cesàro matrix $C(\mathfrak{q}) = (c^{\mathfrak{q}}_{\mathrm{n}v})_{n,v\in\mathbb{N}_0}$, where the entries $c^{\mathfrak{q}}_{\mathrm{n}v}$ are given by

$$c^{\mathfrak{q}}_{\mathrm{n}v} = \begin{cases} \dfrac{\mathfrak{q}^v}{[\mathrm{n}+1]_{\mathfrak{q}}}, & 0 \leq v \leq \mathrm{n}, \\ 0, & v > \mathrm{n}, \end{cases}$$

where $[\mathrm{n}+1]_{\mathfrak{q}}$ denotes the $\mathfrak{q}$-analogue of $\mathrm{n}+1$. Subsequently, in 2023, Yılmaz and Akdemir [32] examined the topological and geometric characteristics of the sequence spaces $(\ell_p)_{C(\mathfrak{q})}$ and

$(\ell_\infty)_{C(\mathfrak{q})}$. Further advances in the study of $\mathfrak{q}$-sequence spaces were made by Yaying [30], who introduced $\mathfrak{q}$-Euler (or $(p,\mathfrak{q})$-Euler) sequence spaces, represented as $K(E(\mathfrak{q})) = K_{E(\mathfrak{q})}$. The $\mathfrak{q}$-Euler matrix is given by

$$e^{\mathfrak{q}}_{\mathrm{n}v}(a,b) = \begin{cases} \dfrac{1}{(a+b)^{\mathrm{n}}_{\mathfrak{q}}} \dbinom{n}{v}_{\mathfrak{q}} \mathfrak{q}^{\binom{v}{2}} a^v b^{\mathrm{n}-v}, & 0 \le v \le \mathrm{n}, \\ 0, & v > \mathrm{n}. \end{cases}$$

Additionally, the author investigated the domains $c_0(\nabla^2_{\mathfrak{q}}) := (c_0)_{\nabla^2_{\mathfrak{q}}}$ and $c(\nabla^2_{\mathfrak{q}}) := c_{\nabla^2_{\mathfrak{q}}}$, which integrate the $\mathfrak{q}$-difference operator $\nabla^2_{\mathfrak{q}}$ within the spaces c_0 and c. The second-order $\mathfrak{q}$-difference operator is defined as

$$(\nabla^2_{\mathfrak{q}} s)_{\mathrm{n}} = s_{\mathrm{n}} - (1+\mathfrak{q})s_{\mathrm{n}-1} + \mathfrak{q}s_{\mathrm{n}-2}.$$

Yaying *et al.* [31], provided a major contribution to this field by introducing a new $\mathfrak{q}$-Fibonacci matrix and its associated sequence spaces. They analyzed matrix domains in $\ell_p(F(\mathfrak{q}))$ and $\ell_\infty(F(\mathfrak{q}))$, studying Schauder bases, dual spaces, and matrix transformations. The work also explored geometric properties, the Dunford-Pettis property of $\mathfrak{q}$-Fibonacci spaces in functional analysis. Recent research has made substantial contributions to the advancement of sequence space theory.

In a more recent development, Shah *et al.* [23] proposed the Copper-Lacus sequence spaces and examined their associated operator ideals and geometric properties. Complementing this, the authors of [32] analyzed the topological and geometric structures of $\mathfrak{q}$-Cesàro sequence spaces. In [25, 26], the $\mathfrak{q}$-Pell matrix and the $\mathfrak{q}$-Bronze Leonardo-Lucas matrix are investigated in separate studies, along with the corresponding sequence spaces generated by the $\mathfrak{q}$-Pell and $\mathfrak{q}$-Bronze Leonardo-Lucas sequences, respectively. In each case, a Schauder basis for ℓ_p $(1 \le p < \infty)$ is constructed, operator ideals are characterized, and a comprehensive analysis of the geometric properties such as the approximation property, the Dunford–Pettis property, the Hahn-Banach extension property, rotundity, and solidity.

1.1 Motivation

In recent years, $\mathfrak{q}$-calculus has become an influential framework for the development and investigation of novel sequence spaces. By providing a natural generalization of classical discrete analysis, it offers additional flexibility in extending traditional summability methods. A number of researchers have successfully employed $\mathfrak{q}$-theory to broaden the scope of matrix transformations and sequence space constructions. In particular, Demiriz and Şahin [9], along with Yaying *et al.* [29], introduced new sequence spaces generated through the $\mathfrak{q}$-analogue of the Cesàro matrix, thereby enriching the structural theory of summability and matrix domains.

A fundamental motivation for utilizing $\mathfrak{q}$-calculus arises from the distinctive behavior of $\mathfrak{q}$-integers. In the classical setting, the sequence of natural numbers $\{\mathrm{n}\}_{\mathrm{n}=1}^{\infty}$ is unbounded and diverges to infinity. In contrast, its $\mathfrak{q}$-analogue, defined by

$$[\mathrm{n}]_{\mathfrak{q}} = \frac{1-\mathfrak{q}^{\mathrm{n}}}{1-\mathfrak{q}}, \quad (0 < \mathfrak{q} < 1),$$

exhibits fundamentally different asymptotic behavior. Specifically, the sequence $\{[\mathrm{n}]_{\mathfrak{q}}\}_{\mathrm{n}=1}^{\infty}$ remains bounded and satisfies

$$\lim_{\mathrm{n}\to\infty} [\mathrm{n}]_{\mathfrak{q}} = \frac{1}{1-\mathfrak{q}}.$$

This crucial distinction between classical integers and their $\mathfrak{q}$-counterparts enables the construction of sequence spaces with bounded structural parameters, leading to properties that are unattainable in the classical framework. Consequently, the boundedness and convergence characteristics inherent in $\mathfrak{q}$-structures have stimulated substantial research activity and continue to provide new directions in the theory of summability and functional analysis.

2 Motivation for employing $\mathfrak{q}$-matrices

In quantum calculus, $\mathfrak{q}$-matrices are introduced as a deformation-based extension of classical matrices, with the parameter $\mathfrak{q} \in (0, 1)$ which naturally broaden the scope of classical matrices. In contrast to standard calculus, $\mathfrak{q}$-calculus replaces limits with difference-based frameworks regulated by $\mathfrak{q}$, there by broadening the range of algebraic and analytical exploration [11, 16]. Considering the case $\mathfrak{q} \to 1$, the proposed generalization reduces to various classical results, demonstrating its wide range of use [13]. Exploring $\mathfrak{q}$-matrices is essential as they play a crucial role in several aspects. First, they provide a foundation for broadening classical operator theory and functional analysis, thus deepening insights into approximation methods, convergence patterns, and stability issues [2, 11]. Second, introducing a $\mathfrak{q}$-parameter enables a smooth transition between discrete and continuous frameworks, allowing more precise regulation of analytic behaviour [16]. Third, the properties of $\mathfrak{q}$-matrices are intertwined with special functions, combinatorial identities, and $\mathfrak{q}$-hypergeometric series, thus creating a connection between linear algebra techniques and diverse mathematical disciplines [13]. Their involvement in quantum groups, coding theory, and various physical frameworks further highlights their importance in various disciplines. Within sequence space theory, $\mathfrak{q}$-matrices offer a robust framework for the creation and examination of new spaces, revealing intricate structural and topological aspects beyond the reach of traditional techniques [7]. Therefore, $\mathfrak{q}$-matrices serve not just as an extension of traditional mathematical frameworks, but also gain opportunities to explore innovative theoretical and practical perspectives [22].

3 Pell-Lucas sequence spaces

John Pell (1611–1685), an English mathematician, lends his name to the Pell numbers, whereas the Pell–Lucas numbers are named for the French mathematician Édouard Lucas (1842–1891). It was shown in [8, 10, 12] that Pell numbers can be expressed in matrix form. In Atabey *et al.* [3] constructed a Pell sequence space based on Pell numbers. These sequences are defined recursively, several other ways. The recurrence relation of the Pell-Lucas sequence

$$\grave{\mathrm{Q}}_{\mathrm{k}} = 2\grave{\mathrm{Q}}_{\mathrm{k}-1} + \grave{\mathrm{Q}}_{\mathrm{k}-2} \quad \text{for } \mathrm{k} \geq 2,$$

with initial terms $\grave{\mathrm{Q}}_0 = \grave{\mathrm{Q}}_1 = 2$. The sequence progresses as $2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, 6726, \ldots$.
Some basic properties of the Pell-Lucas sequence are the following

$$\sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}} = \frac{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}}{2}, \quad \mathrm{k} \geq 0,$$

and

$$\lim_{\mathrm{n}\to\infty} \frac{\grave{\mathrm{Q}}_{\mathrm{n}+1}}{\grave{\mathrm{Q}}_{\mathrm{n}}} = 2.41421\ldots \quad \text{(approximately)}.$$

In [17] authors provided the Binet's formula for the Pell-Lucas sequence. Information about Pell-Lucas numbers, along with additional details, can be accessed in various resources [4, 15]. The Pell-Lucas matrix operator $\mathfrak{Q} = (\grave{\mathrm{Q}}_{\mathrm{n,k}})_{\mathrm{n,k}\in\mathbb{N}}$ is constructed from the Pell-Lucas numbers $\{\grave{\mathrm{Q}}_{\mathrm{k}}\}_{\mathrm{k}\in\mathbb{N}_0}$ as follows:

$$\grave{\mathrm{Q}}_{\mathrm{n,k}} = \begin{cases} \dfrac{2\grave{\mathrm{Q}}_{\mathrm{k}}}{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}}, & 0 \leq \mathrm{k} \leq \mathrm{n} \\ 0, & \text{otherwise} \end{cases}$$

Or, it can be formulated as

$$\mathfrak{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{4}{8} & \frac{4}{8} & 0 & 0 & 0 & 0 & \cdots \\ \frac{4}{20} & \frac{4}{20} & \frac{12}{20} & 0 & 0 & 0 & \cdots \\ \frac{4}{48} & \frac{4}{48} & \frac{12}{48} & \frac{28}{48} & 0 & 0 & \cdots \\ \frac{4}{116} & \frac{4}{116} & \frac{12}{116} & \frac{28}{116} & \frac{68}{116} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.1}$$

It is clear that $\mathfrak{Q}$ is triangular and regular matrix. Again, $\mathfrak{Q}$-transform of a sequence $n' = (n'_{\mathrm{k}})$ is given by

$$m'_{\mathrm{n}} = (\mathfrak{Q}n')_{\mathrm{k}} = \frac{2}{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}} n'_{\mathrm{k}}. \tag{3.2}$$

Lemma 3.1 ([28]). *An infinite matrix $\mathcal{B} = (\mathfrak{b}_{\mathrm{n,k}})$ is regular if and only if*

1. $\sup_{\mathrm{n}\in\mathbb{N}} \sum_{\mathrm{k}=0}^{\infty} |\mathfrak{b}_{\mathrm{n,k}}| < \infty$,
2. $\lim_{\mathrm{n}\to\infty} \sum_{\mathrm{k}=0}^{\infty} \mathfrak{b}_{\mathrm{n,k}} = 1$,
3. $\lim_{\mathrm{n}\to\infty} \mathfrak{b}_{\mathrm{n,k}} = 0$ *for each fixed* $\mathrm{k} \in \mathbb{N}$.

Lemma 3.2 ([24]). *The matrix $\mathfrak{Q} = (\grave{Q}_{\mathrm{n,k}})_{\mathrm{n,k}\in\mathbb{N}}$ associated with the Pell-Lucas numbers is regular.*

We defined the Pell-Lucas sequence spaces, which include null, convergent, bounded and p-summable sequences as follows (see [24])

$$c_0(\mathfrak{Q}) = \left\{ n' = (n'_{\mathrm{k}}) \in \omega : \lim_{\mathrm{n}\to\infty} \sum_{\mathrm{n}=0}^{\infty} \frac{2}{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}} n'_{\mathrm{k}} = 0 \right\}.$$

$$c(\mathfrak{Q}) = \left\{ n' = (n'_{\mathrm{k}}) \in \omega : \lim_{\mathrm{n}\to\infty} \sum_{\mathrm{n}=0}^{\infty} \frac{2}{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}} n'_{\mathrm{k}} \ \text{ exists} \right\}.$$

$$\ell_\infty(\mathfrak{Q}) = \left\{ n' = (n'_{\mathrm{k}}) \in \omega : \sup_{\mathrm{n}\in\mathbf{N}} \left| \frac{2}{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}} n'_{\mathrm{k}} \right| < \infty \right\}.$$

$$\ell_p(\mathfrak{Q}) = \left\{ n' = (n'_{\mathrm{k}}) \in \omega : \sum_{\mathrm{n}=0}^{\infty} \left| \frac{2}{3\grave{\mathrm{Q}}_{\mathrm{n}} + \grave{\mathrm{Q}}_{\mathrm{n}-1}} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}} n'_{\mathrm{k}} \right|^p < \infty \right\} (1 \leq p < \infty).$$

4 Construction of the $\mathfrak{q}$-Pell–Lucas sequence and its associated matrix $\grave{Q}(\mathfrak{q})$

To establish a rigorous foundation for the $\mathfrak{q}$-Pell-Lucas matrix, we first define the classical Pell-Lucas sequence $\{\grave{Q}_n\}_{n\geq 0}$ is defined by

$$\grave{Q}_n = 2\grave{Q}_{n-1} + \grave{Q}_{n-2} \text{ for } n \geq 2 \tag{4.1}$$

$$\grave{Q}_0 = 2,\ \grave{Q}_1 = 2.$$

We extend this recurrence relation by incorporating a parameter $\mathfrak{q}$.

Definition 1. For $\mathfrak{q} \in (0,1)$, the $\mathfrak{q}$-Pell-Lucas sequence $\{\grave{Q}_n(\mathfrak{q})\}_{n\geq 0}$ is defined recursively by

$$\grave{Q}_n(\mathfrak{q}) = 2\grave{Q}_{n-1}(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{n-2}(\mathfrak{q}),\ \text{for } n \geq 2,$$

$$\grave{Q}_0(\mathfrak{q}) = 2,\ \grave{Q}_1(\mathfrak{q}) = 2.$$

If $\mathfrak{q} = 1$, then the above recurrence reduces to

$$\grave{Q}_n(1) = 2\grave{Q}_{n-1}(1) + \grave{Q}_{n-2}(1),$$

which coincides with the classical Pell-Lucas sequence. Hence,

$$\grave{Q}_n(1) = \grave{Q}_n,$$

and the above definition provides a genuine $\mathfrak{q}$-extension of the Pell-Lucas numbers. The first few terms of the $\mathfrak{q}$-Pell-Lucas sequence are given by

$$\begin{aligned}
&\grave{Q}_0(\mathfrak{q}) = 2,\\
&\grave{Q}_1(\mathfrak{q}) = 2,\\
&\grave{Q}_2(\mathfrak{q}) = 2\grave{Q}_1(\mathfrak{q}) + \mathfrak{q}\grave{Q}_0(\mathfrak{q}) = 4 + 2\mathfrak{q},\\
&\grave{Q}_3(\mathfrak{q}) = 2\grave{Q}_2(\mathfrak{q}) + \mathfrak{q}\grave{Q}_1(\mathfrak{q}) = 8 + 6\mathfrak{q},\\
&\grave{Q}_4(\mathfrak{q}) = 2\grave{Q}_3(\mathfrak{q}) + \mathfrak{q}\grave{Q}_2(\mathfrak{q}) = 16 + 16\mathfrak{q} + 2\mathfrak{q}^2,\\
&\grave{Q}_5(\mathfrak{q}) = 32 + 40\mathfrak{q} + 12\mathfrak{q}^2.
\end{aligned}$$

Using recurrence relation (4.1), we can derive the following important relation, which plays a key role in this paper

$$\sum_{k=0}^{n} \grave{Q}_k(\mathfrak{q}) = \frac{(2+\mathfrak{q})\grave{Q}_n(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{n-1}(\mathfrak{q})}{(\mathfrak{q}+1)}. \tag{4.2}$$

Define the matrix $\grave{Q}_{nk}^{(\mathfrak{q})} = \left(\grave{Q}_{nk}^{(\mathfrak{q})}\right)_{n,k\in\mathbb{N}}$ by

$$\grave{Q}_{nk}^{(\mathfrak{q})} = \begin{cases} \dfrac{(\mathfrak{q}+1)\ \grave{Q}_k(\mathfrak{q})}{(2+\mathfrak{q})\grave{Q}_n(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{n-1}(\mathfrak{q})}, & \text{if } 0 \leq k \leq n,\\ 0, & \text{if } k > n. \end{cases}$$

Alternatively, it can be expressed as

$$\grave{\mathrm{Q}}(\mathfrak{q}) = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \dfrac{\mathfrak{q}+1}{2+2\mathfrak{q}} & \dfrac{\mathfrak{q}+1}{2+2\mathfrak{q}} & 0 & \cdots \\ \dfrac{1+\mathfrak{q}}{\mathfrak{q}^2+5\mathfrak{q}+4} & \dfrac{1+\mathfrak{q}}{\mathfrak{q}^2+5\mathfrak{q}+4} & \dfrac{2+3\mathfrak{q}+\mathfrak{q}^2}{4+5\mathfrak{q}+\mathfrak{q}^2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Taking the limit as $\mathfrak{q} \to 1$, the entries of this matrix reduce to their classical forms, and hence the matrix becomes exactly the classical Pell–Lucas matrix given in (3.1). Using the matrix $\grave{\mathrm{Q}}(\mathfrak{q})$), we define a sequence $s = (s_{\mathrm{k}})_{\mathrm{k}\in\mathbb{N}}$ from a sequence $\upsilon = (\upsilon_{\mathrm{k}})_{\mathrm{k}\in\mathbb{N}}$ by

$$s_{\mathrm{k}} = (\grave{\mathrm{Q}}(\mathfrak{q})\upsilon)_{\mathrm{k}} = \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{c}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}\upsilon_c. \tag{4.3}$$

Thus, the sequence $s = (s_{\mathrm{k}})$ is the $\grave{\mathrm{Q}}(q)$-transform of the sequence $\upsilon = (\upsilon_{\mathrm{k}})$. We now define the sequence spaces $c_0(\grave{\mathrm{Q}}(\mathfrak{q}))$, $c(\grave{\mathrm{Q}}(\mathfrak{q}))$, $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$ and $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$, as follows

$$c_0(\grave{\mathrm{Q}}(\mathfrak{q})) := \left\{ \upsilon = (\upsilon_{\mathrm{k}}) \in \omega : \left\{ s = \grave{\mathrm{Q}}(\mathfrak{q})\upsilon = \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{c}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}\upsilon_c \right\} \in c_0 \right\};$$

$$c(\grave{\mathrm{Q}}(\mathfrak{q})) := \left\{ \upsilon = (\upsilon_{\mathrm{k}}) \in \omega : \left\{ s = \grave{\mathrm{Q}}(\mathfrak{q})\upsilon = \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{c}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}\upsilon_c \right\} \in c \right\};$$

$$\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q})) := \left\{ \upsilon = (\upsilon_{\mathrm{k}}) \in \omega : \left\{ s = \grave{\mathrm{Q}}(\mathfrak{q})\upsilon = \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{c}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}\upsilon_c \right\} \in \ell_\infty \right\};$$

$$\ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) := \left\{ \upsilon = (\upsilon_{\mathrm{k}}) \in \omega : \left\{ s = \grave{\mathrm{Q}}(\mathfrak{q})\upsilon = \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{c}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}\upsilon_c \right\} \in \ell_p \right\}.$$

The sequence spaces $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$, $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$, $c(\grave{\mathrm{Q}}(\mathfrak{q}))$, and $c_0(\grave{\mathrm{Q}}(\mathfrak{q}))$ are interpreted as the domains associated with the $\mathfrak{q}$-Pell-Lucas Matrix $\grave{\mathrm{Q}}(\mathfrak{q})$, when this matrix acts within the classical spaces c_0, c, ℓ_∞, and ℓ_p, respectively. More precisely, the following relations hold

$$\ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) = (\ell_p)_{\grave{\mathrm{Q}}(\mathfrak{q})} \quad \text{and} \quad \ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q})) = (\ell_\infty)_{\grave{\mathrm{Q}}(\mathfrak{q})}.$$

$$c(\grave{\mathrm{Q}}(\mathfrak{q})) = c_{\grave{\mathrm{Q}}(\mathfrak{q})} \quad \text{and} \quad c_0(\grave{\mathrm{Q}}(\mathfrak{q})) = (c_0)_{\grave{\mathrm{Q}}(\mathfrak{q})}.$$

It is clear that as $\mathfrak{q} \to 1^-$, the spaces $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$ and $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$, $c(\grave{\mathrm{Q}}(\mathfrak{q}))$ and $c_0(\grave{\mathrm{Q}}(\mathfrak{q}))$ become $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) = (\ell_p)_{\mathfrak{Q}}$, $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q})) = (\ell_\infty)_{\mathfrak{Q}}$ and $c(\grave{\mathrm{Q}}(\mathfrak{q})) = (c)_{\mathfrak{Q}}$ and $c_0(\grave{\mathrm{Q}}(\mathfrak{q})) = (c_0)_{\mathfrak{Q}}$, respectively.

Theorem 4.1. *The $\mathfrak{q}$ Pell-Lucas matrix $\grave{Q}(\mathfrak{q}) = (\grave{Q}^{(\mathfrak{q})}_{\mathrm{nk}})_{\mathrm{n,k}\in\mathbb{N}_0}$ is regular.*

Proof. For each n, we have

$$\sum_{k} |\grave{Q}_{nk}^{(q)}| = \sum_{k} \grave{Q}_{nk}^{(q)} = \sum_{k=0}^{n} \frac{(q+1)\grave{Q}_k(q)}{(2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q)} = \frac{(q+1)}{(2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q)} \sum_{k=0}^{n} \grave{Q}_k(q).$$

Using the summation identity (4.2) for the q-Pell-Lucas numbers, it follows that

$$\lim_{n\to\infty} \sum_{k} \grave{Q}_{nk}^{(q)} = 1,$$

and the row sums are uniformly bounded. Hence, Conditions (i) and (ii) hold. Moreover, for each fixed k,

$$\grave{Q}_{n,k}^{(q)} = \frac{(q+1)\grave{Q}_k(q)}{(2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

since $(2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q) \to \infty$. Thus, Condition (iii) is satisfied. Therefore, the matrix $\grave{Q}(q)$ is regular. □

Lemma 4.1. *The inverse of the* q*-Pell-Lucas matrix* $\grave{Q}(q)$ *is denoted* $\mathcal{G}(q) = (g_{nk}^{(q)})_{n,k\in\mathbb{N}_0} = \{\grave{Q}(q)\}^{-1}$ *and is given by*

$$(g_{nk}^{(q)}) = \begin{cases} (-1)^{n-k}\dfrac{(2+q)\grave{Q}_k(q) + q\grave{Q}_{k-1}(q)}{(q+1)\ \grave{Q}_n(q)}, & \textit{if } n-1 \le k \le n, \\ 0, & \textit{if } k > n. \end{cases}$$

Proof. Define the q-Pell-Lucas matrix $\grave{Q}(q) = \left(\grave{Q}_{nk}^{(q)}\right)_{n,k\in\mathbb{N}}$ by

$$\grave{Q}_{nk}^{(q)} = \begin{cases} \dfrac{(q+1)\ \grave{Q}_k(q)}{(2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q)}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

Observe that for $0 \le k \le n$,

$$\grave{Q}_{nk}^{(q)} = \frac{(q+1)\ \grave{Q}_k(q)}{(2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q)} = \frac{\alpha_k}{\beta_n},$$

where

$$\alpha_k = (q+1)\ \grave{Q}_k(q), \quad \beta_n = (2+q)\grave{Q}_n(q) + q\grave{Q}_{n-1}(q).$$

Hence, $\grave{Q}_{nk}^{(q)}$ admits the factorization

$$\grave{Q}_{nk}^{(q)} = D(q)L(q),$$

where

$$D(q) = \operatorname{diag}\left(\frac{1}{\beta_n}\right),$$

and $L(q) = (\ell_{nk})$ is the lower triangular matrix defined by

$$\ell_{nk} = \begin{cases} \alpha_k, & k \le n, \\ 0, & k > n. \end{cases}$$

Since

$$\grave{Q}^{(\mathfrak{q})}_{nn} = \frac{(\mathfrak{q}+1)\ \grave{Q}_k(\mathfrak{q})}{(2+\mathfrak{q})\grave{Q}_n(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{n-1}(\mathfrak{q})} \neq 0,$$

the matrix $\grave{Q}(\mathfrak{q})$ is invertible. Since $\grave{Q}^{(\mathfrak{q})}_{nk} = D(\mathfrak{q})L(\mathfrak{q})$, we have

$$\grave{Q}(\mathfrak{q})^{-1} = L(\mathfrak{q})^{-1}D(\mathfrak{q})^{-1}.$$

The inverse of $D(\mathfrak{q})$ is

$$D(\mathfrak{q})^{-1} = \mathrm{diag}(\beta_n).$$

Next, the matrix $L(\mathfrak{q})$ is of summation type, and its inverse is given by

$$(L(\mathfrak{q})^{-1})_{nk} = \begin{cases} \dfrac{1}{\alpha_n}, & k = n, \\ -\dfrac{1}{\alpha_n}, & k = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\alpha_n = (\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})$, we obtain

$$(L(\mathfrak{q})^{-1})_{nk} = \begin{cases} \dfrac{1}{(\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})}, & k = n, \\ -\dfrac{1}{(\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})}, & k = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Finally,

$$(g^{(\mathfrak{q})}_{nk}) = (L(\mathfrak{q})^{-1})_{nk}\,\beta_k,$$

which yields

$$(g^{(\mathfrak{q})}_{nk}) = \begin{cases} \dfrac{(2+\mathfrak{q})\grave{Q}_n(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{n-1}(\mathfrak{q})}{(\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})}, & k = n, \\ -\dfrac{(2+\mathfrak{q})\grave{Q}_n(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{n-1}(\mathfrak{q})}{(\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})}, & k = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

This expression may be written equivalently as

$$(g^{(\mathfrak{q})}_{nk}) = \begin{cases} (-1)^{n-k}\dfrac{(2+\mathfrak{q})\grave{Q}_k(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{k-1}(\mathfrak{q})}{(\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})}, & \text{if } n-1 \leq k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

□

The inverse $\grave{Q}(\mathfrak{q})$-transform or $\{\grave{Q}(\mathfrak{q})\}^{-1}$-transform of the sequence $s = (s_k)$ is defined as

$$\upsilon_n = \sum_{k=n-1}^{k} (-1)^{n-k}\frac{(2+\mathfrak{q})\,\grave{Q}_k(\mathfrak{q}) + \mathfrak{q}\,\grave{Q}_{k-1}(\mathfrak{q})}{(\mathfrak{q}+1)\ \grave{Q}_n(\mathfrak{q})}s_k. \tag{4.4}$$

Therefore, equations (4.3) and (4.4) are equivalent.

Theorem 4.2. *1. For $0 < p \leq 1$, the space $\ell_p(\grave{Q}(\mathfrak{q}))$ constitutes a complete space under a p-norm, where the p-norm is defined as follows*

$$\|\upsilon\|_{\ell_p(\grave{Q}(\mathfrak{q}))} = \|s\|_{\ell_p} = \sum_{\mathrm{n}=0}^{\infty} \left| \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\ \grave{Q}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{Q}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{\mathrm{k}-1}(\mathfrak{q})} \upsilon_c \right|^p .$$

2. If $1 < p < \infty$, then $\ell_p(\grave{Q}(q))$ is a BK-space with the norm

$$\|\upsilon\|_{\ell_p(\grave{Q}(\mathfrak{q}))} = \|s\|_{\ell_p} = \left(\sum_{\mathrm{n}=0}^{\infty} \left| \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\ \grave{Q}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{Q}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{\mathrm{k}-1}(\mathfrak{q})} \upsilon_c \right|^p \right)^{1/p} .$$

3. The space $\ell_\infty(\grave{Q}(\mathfrak{q}))$ is a BK-space with the norm

$$\|\upsilon\|_{\ell_\infty(\grave{Q}(\mathfrak{q}))} = \|s\|_{\ell_\infty} = \sup_{\mathrm{k}\in\mathbb{N}} \left| \sum_{c=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\ \grave{Q}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{Q}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{\mathrm{k}-1}(\mathfrak{q})} \upsilon_c \right| .$$

Proof. This result can be confirmed through straightforward verification. □

Theorem 4.3. *$\ell_p(\grave{Q}(\mathfrak{q}))$ and $\ell_\infty(\grave{Q}(\mathfrak{q}))$ are linearly isomorphic to the space ℓ_p and ℓ_∞, respectively.*

Proof. Here we prove the result only for the space $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) \cong \ell_p$. The other one can be proved in a similar way.
Consider a mapping

$$\mathcal{H} : \ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) \to \ell_p \ \text{ s.t. } \ \upsilon \mapsto \mathcal{H}\upsilon = \grave{\mathrm{Q}}(\mathfrak{q})\upsilon.$$

From the result $\mathcal{H}(\upsilon) = 0 \implies \upsilon = 0$, it implies the injection property of $\mathcal{H}$. Additionally, for $s = (s_{\mathrm{k}}) \in \ell_p$,$1 \leq p \leq \infty$, and define the sequence $\upsilon = (\upsilon_{\mathrm{k}})$ as follows

$$\upsilon_{\mathrm{k}} = \sum_{l=\mathrm{k}-1}^{\mathrm{k}} (-1)^{\mathrm{k}-l}\ \frac{(2+\mathfrak{q})\grave{\mathrm{Q}}_l(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{l-1}(\mathfrak{q})}{(\mathfrak{q}+1)\ \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})} s_l, \quad (\mathrm{k} \in \mathbb{N}). \tag{4.5}$$

Then, for $1 \leq p < \infty$

$$\begin{aligned} \|\upsilon\|_{\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))} &= \left(\sum_{\mathrm{k}=0}^{\infty} \left| \grave{\mathrm{Q}}(\mathfrak{q})\upsilon \right|^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{\mathrm{k}=0}^{\infty} \left(\sum_{l=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_l(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})} \upsilon_l \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{\mathrm{k}=0}^{\infty} \left(\sum_{l=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_l(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})} \sum_{j=l-1}^{l} (-1)^{l-j} \frac{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}{(\mathfrak{q}+1)\grave{\mathrm{Q}}_l(\mathfrak{q})} s_j \right)^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{\mathrm{k}=0}^{\infty} |s_{\mathrm{k}}|^p \right)^{\frac{1}{p}} = \|s\|_{\ell_p} < \infty. \end{aligned}$$

When $p = \infty$, the norm in the space $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$ is calculated as

$$\|\upsilon\|_{\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))} = \sup_{\mathrm{k}\in\mathbb{N}} |\grave{\mathrm{Q}}(\mathfrak{q})_{\mathrm{k}}\upsilon| = \sup_{\mathrm{k}\in\mathbb{N}} |s_{\mathrm{k}}| = \|s\|_\infty < \infty.$$

Therefore, υ lies within the space $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$ for all $p \in [1, \infty]$, and the transformation $\mathcal{H}$ preserves both the norm and the surjectivity. Consequently,

$$\ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) \cong \ell_p \quad \text{for } 1 \le p \le \infty.$$

□

Theorem 4.4. *Sequence spaces* $c_0(\grave{Q}(\mathfrak{q})) \cong c_0$ *and* $c(\grave{Q}(\mathfrak{q})) \cong c$.

Proof. We define the mapping

$$\mathcal{T} : c_0(\grave{\mathrm{Q}}(\mathfrak{q})) \to c_0 \ \text{ s.t. } \ \upsilon \mapsto \mathcal{T}\upsilon = \grave{\mathrm{Q}}(\mathfrak{q})\upsilon.$$

From the result $\mathcal{T}(\upsilon) = 0 \implies \upsilon = 0$, it gives $\mathcal{T}$ is injection. Furthermore, let $s = (s_k) \in C_0$ and define the sequence $\upsilon = (\upsilon_{\mathrm{k}})$ by

$$\upsilon_{\mathrm{k}} = \sum_{l=\mathrm{k}-1}^{\mathrm{k}} (-1)^{\mathrm{k}-l} \, \frac{(2+\mathfrak{q})\grave{\mathrm{Q}}_l(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{l-1}(\mathfrak{q})}{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})} s_l, \quad (\mathrm{k} \in \mathbb{N}). \tag{4.6}$$

Then

$$\begin{aligned}
\lim_{\mathrm{k}\to\infty} (\grave{\mathrm{Q}}(\mathfrak{q})\upsilon)_{\mathrm{k}} &= \lim_{\mathrm{k}\to\infty} \left(\sum_{l=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_l(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})} \upsilon_l \right) \\
&= \lim_{\mathrm{k}\to\infty} \left(\sum_{l=0}^{\mathrm{k}} \frac{(\mathfrak{q}+1)\ \grave{\mathrm{Q}}_l(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})} \sum_{j=l-1}^{l} (-1)^{l-j} \frac{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})}{(\mathfrak{q}+1)\ \grave{\mathrm{Q}}_l(\mathfrak{q})} s_l \right) \\
&= \lim_{\mathrm{k}\to\infty} s_{\mathrm{k}} \\
&= 0.
\end{aligned}$$

Therefore, $\upsilon \in c_0(\grave{\mathrm{Q}}(\mathfrak{q}))$. Thus, $\mathcal{T}$ is surjective and preserves the norm. Consequently, $c_0(\grave{\mathrm{Q}}(\mathfrak{q})) \cong c_0$. One can prove in similar methods. □

Theorem 4.5. *The inclusion* $\ell_p \subset \ell_p(\grave{Q}(\mathfrak{q}))$ *is valid.*

Proof. Let $\upsilon = (\upsilon_{\mathrm{k}}) \in \ell_p$, for $1 < p < \infty$. Using Holder's inequality with $\mathrm{n} \in \mathbb{N}$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})_{\mathrm{n}}\upsilon|^p &\le \sum_{\mathrm{n}=0}^{\infty} \left(\sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} |\upsilon_{\mathrm{k}}| \right)^p \\
&\le \sum_{\mathrm{n}=0}^{\infty} \left(\sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} |\upsilon_{\mathrm{k}}|^p \right) \left(\sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p-1} \\
&= \sum_{\mathrm{n}=0}^{\infty} \frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) |\upsilon_{\mathrm{k}}|^p
\end{aligned}$$

$$= \sum_{\mathrm{k}=0}^{\infty} |\upsilon_{\mathrm{k}}|^p \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) \sum_{\mathrm{n}=\mathrm{k}}^{\infty} \frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}.$$

Hence, $\|\upsilon\|^p_{\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))} \leq D\|\upsilon\|^p_{\ell_p} \leq \infty$, where

$$D = \sup_{\mathrm{k}} \left(\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) \sum_{\mathrm{n}=\mathrm{k}}^{\infty} \frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right).$$

This indicates that $\upsilon \in \ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$. Therefore, $\ell_p \subset \ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$. Similarly, it can be demonstrated that $\ell_1 \subset \ell_1(\grave{\mathrm{Q}}(\mathfrak{q}))$, so we will omit the details. □

Theorem 4.6. *The inclusion* $\ell_\infty \subset \ell_\infty(\grave{Q}(\mathfrak{q}))$ *holds.*

Proof. Consider the sequence $\upsilon = (\upsilon_{\mathrm{k}}) \in \ell_\infty$. Then, $\exists$ a constant $M > 0$ such that $|\upsilon_{\mathrm{k}}| \leq M$, $\forall$ $\mathrm{k} \in \mathbb{N}$. Therefore, for $\mathrm{n} \in \mathbb{N}$, we get

$$\begin{aligned}
|\grave{\mathrm{Q}}(\mathfrak{q})_{\mathrm{n}}\upsilon| &\leq \frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})|\upsilon'_{\mathrm{k}}| \\
&\leq \frac{(\mathfrak{q}+1)M}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \sum_{\mathrm{k}=0}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) \\
&= M.
\end{aligned}$$

Thus, $\grave{\mathrm{Q}}(\mathfrak{q})_{\mathrm{n}}\upsilon \in \ell_\infty$ for $\mathrm{n} \in \mathbb{N}$, which means $\upsilon \in \ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$. Consequently, we have $\ell_\infty \subset \ell_\infty(\grave{\mathrm{Q}}(q))$. □

Theorem 4.7. *The inclusions* $c_0 \subset c_0(\grave{Q}(\mathfrak{q}))$ *and* $c \subset c(\grave{Q}(\mathfrak{q}))$ *are strict.*

Proof. Since the matrix $\grave{\mathrm{Q}}(\mathfrak{q})$ is regular, the inclusions are automatically valid. To demonstrate the strictness of this inclusion, consider the sequence $\upsilon = (1, 0, 1, 0, \dots)$. We now calculate

$$(\grave{\mathrm{Q}}(\mathfrak{q})\upsilon)_{\mathrm{n}} = \sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} = \frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \left(\grave{\mathrm{Q}}_0(\mathfrak{q}) + \dots + \grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) \right),$$

where $\mathrm{n} \in \mathbb{N}$. This expression converges, implying that $\upsilon \in c(\grave{\mathrm{Q}}(\mathfrak{q})) \setminus c$. A similar approach can be applied to prove the other case. □

Theorem 4.8. *The inclusion* $c_0(\grave{Q}(\mathfrak{q})) \subset c(\grave{Q}(\mathfrak{q}))$ *is strict.*

Proof. To demonstrate the inclusion $c_0(\grave{\mathrm{Q}}(\mathfrak{q})) \subset c(\grave{\mathrm{Q}}(\mathfrak{q}))$, consider the sequence $\upsilon = (\upsilon_{\mathrm{k}})$, where $\upsilon_{\mathrm{k}} = 1$ for all $\mathrm{k} \geq 2$. In this case, we have the following

$$(\grave{\mathrm{Q}}(\mathfrak{q})\upsilon)_{\mathrm{n}} = \sum_{\mathrm{k}=0}^{\mathrm{n}} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} = 1, \quad \forall \mathrm{n}.$$

Since $\grave{\mathrm{Q}}(\mathfrak{q})\upsilon \in c$ and it is not yet in c_0. So, $\upsilon \in c(\grave{\mathrm{Q}}(\mathfrak{q})) \setminus c_0(\grave{\mathrm{Q}}(\mathfrak{q}))$, which proves the result. □

Definition 2. In a normed space X with norm $\|\cdot\|$, a sequence $u = (u_j)$ forms a Schauder basis if every element $w \in \mathrm{X}$ can be uniquely expressed using a scalar sequence $a = (a_j)$ such that

$$\lim_{\mathrm{k}\to\infty} \left\| w - \sum_{\mathrm{j}=0}^{\mathrm{k}} a_{\mathrm{j}} u_{\mathrm{j}} \right\| = 0.$$

Theorem 4.9. *Define the sequence* $b^{(\mathrm{k})}(\mathfrak{q}) = \{b_{\mathrm{n}}^{(\mathrm{k})}(\mathfrak{q})\}_{\mathrm{k}\in\mathbb{N}}$ *in* $\ell_p(\grave{Q}(\mathfrak{q}))$ *by*

$$b_{\mathrm{n}}^{(\mathrm{k})}(\mathfrak{q}) = \begin{cases} (-1)^{\mathrm{n}-\mathrm{k}} \dfrac{(2+\mathfrak{q})\grave{Q}_{\mathrm{k}} + \mathfrak{q}\grave{Q}_{\mathrm{k}-1}}{(\mathfrak{q}+1)\ \grave{Q}_{\mathrm{n}}(\mathfrak{q})}, & \textit{if } \mathrm{n}-1 \le \mathrm{k} \le \mathrm{n}, \\ 0, & \textit{if } \mathrm{k} > \mathrm{n}. \end{cases}$$

Then, the following statements hold

1. *The family* $\{b^{(\mathrm{k})}(\mathfrak{q})\}_{\mathrm{k}\in\mathbb{N}}$ *forms the basis of* $\ell_p(\grave{Q}(\mathfrak{q}))$*, and each* $s \in \ell_p(\grave{Q}(\mathfrak{q}))$ *can be written uniquely as*

   $$\upsilon = \sum_{\mathrm{k}=0}^{\infty} s_{\mathrm{k}} b^{(\mathrm{k})}(\mathfrak{q}),$$

   where $s_{\mathrm{n}} = (\grave{Q}(\mathfrak{q})\upsilon)_{\mathrm{n}}$ *for each* $\mathrm{n} \in \mathbb{N}$.

2. *The space* $\ell_\infty(\grave{Q}(\mathfrak{q}))$ *does not admit a Schauder basis.*

3. *The sequence* $\{b^{(\mathrm{k})}(\mathfrak{q})\}$ *forms the basis of* $c_0(\grave{Q}(\mathfrak{q}))$*, and each* $\upsilon \in c_0(\grave{Q}(\mathfrak{q}))$ *has a unique representation given by*

   $$\upsilon = \sum_{\mathrm{k}=0}^{\infty} s_{\mathrm{k}} b^{(s)}(\mathfrak{q}).$$

4. *The set* $\{e, b^{(0)}(\mathfrak{q}), b^{(1)}(\mathfrak{q}), b^{(2)}(\mathfrak{q}), \dots\}$ *serves as a basis for the space* $c(\grave{Q}(\mathfrak{q}))$*. Consequently, each element* $\upsilon \in c(\grave{Q}(\mathfrak{q}))$ *can be uniquely represented in the form*

   $$\upsilon = ze + \sum_{\mathrm{k}=0}^{\infty} (s_{\mathrm{k}} - z) b^{(\mathrm{k})}(\mathfrak{q}),$$

   where $z = \lim_{\mathrm{k}\to\infty} s_{\mathrm{k}} = \lim_{s\to\infty} (\grave{Q}(\mathfrak{q})\upsilon)_{\mathrm{k}}$.

5 Operator ideal $\ell_p^{(s)}(\grave{\mathbf{Q}}(\mathfrak{q}))$

In this part, we investigate the behavior of s-type $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$ operators, with a focus on their interplay with the $\mathfrak{q}$-Pell–Lucas sequence spaces. Let $\mathcal{L}(A, B)$ denote the set of all bounded linear operators from a Banach Space A into a Banach space B. The collection of such operators between Banach spaces in general is denoted by $\mathcal{L}$. The dual space A' of A is defined as the family of continuous linear functionals on A. If $a' \in A'$ and $b \in B$, we can construct the rank-one operator $a' \otimes b : A \to B$, which acts according to $(a' \otimes b)(a) = a'(a)b, \quad \forall a \in A$.

Definition 3 ([6])**.** If a mapping $s : \mathcal{L} \to \omega^+$, where ω^+ is a class of positive real numbers, satisfies the following criteria, it is said to be an s-number sequence

1. **Monotonicity:** $\|s\| = s_1(\mathfrak{K}) \geq s_2(\mathfrak{K}) \geq \cdots \geq 0, \quad \text{for } \mathfrak{K} \in \mathcal{L}(A, B)$.
2. **Additivity:** $s_{\mathrm{n+k-1}}(\mathfrak{K} + \mathfrak{R}) \leq s_{\mathrm{n}}(\mathfrak{K}) + s_{\mathrm{k}}(\mathfrak{R}) \quad$ for $\mathfrak{K}, \mathfrak{R} \in \mathcal{L}(A, B)$ and $\mathrm{n}, \mathrm{k} \in \mathbb{N}$.
3. **Ideal property:** $s_{\mathrm{n}}(\mathfrak{K}\mathfrak{H}\mathfrak{R}) \leq \|\mathfrak{K}\| |s_{\mathrm{n}}(\mathfrak{H})| \|\mathfrak{R}\| \quad$ for $\mathfrak{K} \in \mathcal{L}(A_0, A), \mathfrak{H} \in \mathcal{L}(A, B), \mathfrak{R} \in \mathcal{L}(B, B_0)$, and $\mathrm{n} \in \mathbb{N}$.
4. **Rank property:** If rank $(\mathfrak{K}) < \mathrm{n}$, then $s_{\mathrm{n}}(\mathrm{k}) = 0$.
5. **Norming property:** $s_{\mathrm{n}}\big(I_2 : \ell_2^{(\mathrm{n})} \to \ell_2^{(\mathrm{n})}\big) = 1$,where I_2 denotes the identity operator on the 2-dimensional Hilbert space.

Definition 4 ([20])**.** If A and B are Banach spaces, write $P(A, B) = P \cap L(A, B)$ for a subset P of $\mathcal{L}$. The collection P becomes an *operator ideal* if it satisfies all of the following criteria

(i) If $a' \in A', b \in B$, then $a' \odot b \in P(A, B)$.

(ii) $\mathfrak{K} + \mathfrak{R} \in P(A, B)$ for $\mathfrak{K}, \mathfrak{R} \in P(A, B)$.

(iii) If $\mathfrak{H} \in A(A, B)$, $\mathfrak{R} \in \mathcal{L}(A_0, A)$, and $\mathfrak{K} \in P(B, B_0)$, then $\mathfrak{K}\mathfrak{H}\mathfrak{R} \in P(A_0, B_0)$.

A component of P is defined as the set $P(A, B)$, corresponding to any pair of Banach spaces A and B.

Definition 5 ([20, 21])**.** The term *ideal quasi-norm* implies a real-valued function $\mathfrak{X} : A \to \mathbb{R}_+$ that satisfy the following axioms:

(i) If $a' \in A', b \in B$, then $\mathfrak{X}(a' \odot b) = \|a'\| \|b\|$.

(ii) There exists a constant $M \geq 1$ such that

$$\mathfrak{X}(\mathfrak{K} + \mathfrak{R}) \leq M[\mathfrak{X}(\mathfrak{K}) + \mathfrak{X}(\mathfrak{R})] \quad \text{for } \mathfrak{K}, \mathfrak{R} \in P(A, B).$$

(iii) If $\mathfrak{H} \in P(A, B)$, $\mathfrak{R} \in \mathcal{L}(A_0, A)$, and $\mathfrak{K} \in \mathcal{L}(B, B_0)$, then $\mathfrak{K}\mathfrak{H}\mathfrak{R} \in P(A_0, B_0)$.

Lemma 5.1 ([19])**.** *Let* $\mathfrak{K}, \mathfrak{R} \in \mathcal{L}(A, B)$. *Then*

$$|s_{\mathrm{n}}(\mathfrak{K}) - s_{\mathrm{n}}(\mathfrak{R})| \leq \|\mathfrak{K} - \mathfrak{R}\| \quad \textit{for } \mathrm{n} \in \mathbb{N}.$$

An operator $\mathfrak{K} \in \mathcal{L}(A, B)$ *is called an s-type* $\ell_p(\grave{Q}(\mathfrak{q}))$ *operator if*

$$\sum_{\mathrm{n}=1}^{\infty} \left| \frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{Q}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{K})}{(2+\mathfrak{q}) \grave{Q}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q} \grave{Q}_{\mathrm{n}-1}(\mathfrak{q})} \right|^p < \infty, \quad (1 < p < \infty).$$

The set of all s-type operators on $\ell_p(\grave{Q}(\mathfrak{q}))$ *is denoted by* $\ell_p^{(s)}(\grave{Q}(\mathfrak{q}))$.

Theorem 5.1. *Let* $1 < p < \infty$. *Then the class* $\ell_p^{(s)}(\grave{Q}(\mathfrak{q}))$ *is an operator ideal.*

Proof. Let A and B be Banach spaces, and take $a' \in A'$, $b \in B$. The operator $a' \odot b$ has rank one, so for every $\mathrm{n} \geq 2$, $s_{\mathrm{n}}(a' \odot b) = 0$. Hence, it follows that

$$\sum_{\mathrm{n}=1}^{\infty} \left| \frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(a' \odot b)}{(2+\mathfrak{q}) \grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q} \grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right|^p = \sum_{\mathrm{n}=1}^{\infty} \left| \frac{(\mathfrak{q}+1) s_1(a' \odot b)}{(2+\mathfrak{q}) \grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q} \grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right|^p$$

$$
\begin{aligned}
&= \big[s_1(a'\odot b)\big]^p \sum_{\mathrm{n}=1}^{\infty}\left|\frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right|^p\\
&= \|a'\odot b\|^p \sum_{\mathrm{n}=1}^{\infty}\left|\frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right|^p.\\
&< \infty.
\end{aligned}
$$

Thus $a'\odot b \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$.

Let $\mathfrak{K},\mathfrak{R}\in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$ and due to the non-negativity and non-increasing properties of s-numbers, we have applied Minkowski's inequality. We get

$$
\begin{aligned}
&\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{K}+\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{1/p}\\
&\leq \left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\mathfrak{q}^{2\mathrm{k}-2}\grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})\ s_{2\mathrm{k}-1}(\mathfrak{K}+\mathfrak{R})+\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}(\mathfrak{q}+1)\ \mathfrak{q}^{2\mathrm{k}-1}\grave{\mathrm{Q}}_{2\mathrm{k}}\ s_{2\mathrm{k}}(\mathfrak{K}+\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{1/p}\\
&\leq \left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{k=1}^{\mathrm{n}}(\mathfrak{q}^{2\mathrm{k}-2}+\mathfrak{q}^{2\mathrm{k}-1})\grave{\mathrm{Q}}_{2\mathrm{k}-1}(\mathfrak{q})s_{2\mathrm{k}-1}(\mathfrak{K}+\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{1/p}\\
&\leq M\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{K})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}+\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{1/p}\\
&\leq M\left[\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{K})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{1/p}+\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{1/p}\right] < \infty.
\end{aligned}
$$

Thus, $\mathfrak{K}+\mathfrak{R}\in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$.

Let $\mathfrak{K}\in\mathcal{L}(A_0,A)$, $\mathfrak{R}\in\mathcal{L}(B,B_0)$ and $\mathfrak{H}\in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$. Using the property (3) in Definition 3, we get

$$
\begin{aligned}
&\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{R}\mathfrak{H}\mathfrak{K})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{\frac{1}{p}}\\
&\leq \|\mathfrak{R}\|\|\mathfrak{K}\|\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{H})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^p\right)^{\frac{1}{p}} < \infty.
\end{aligned}
$$

Thus, $\mathfrak{R}\mathfrak{H}\mathfrak{K} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$. Hence $\ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$ is an operator ideal. □

Theorem 5.2. *Let $1 < p \leq r < \infty$. Then $\ell_p^{(s)}(\grave{Q}(\mathfrak{q})) \subseteq \ell_r^{s}(\grave{Q}(\mathfrak{q}))$.*

Proof. The statement follows immediately from the known inclusion relationship between the spaces $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q})) \subseteq \ell_r(\grave{\mathrm{Q}}(\mathfrak{q}))$ for all p and r with $1 < p \leq r < \infty$. □

Consider the operator ideal $\ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$. For $1 < p < \infty$, we define a mapping $Q^{(s)} : \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q})) \to \omega^+$ as follows

$$Q^{(s)}(\mathfrak{T}) = \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}},$$

where $\mathfrak{T} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$.

Theorem 5.3. *For $1 < p < \infty$, the operator ideal $\ell_p^{(s)}(\grave{Q}(\mathfrak{q}))$ is equipped with a quasi-norm given by the mapping $\widetilde{Q}^{(s)}$, where*

$$\widetilde{Q}^{(s)}(\mathfrak{T}) = \frac{Q^{(s)}(\mathfrak{T})}{\left(\left(\sum_{\mathrm{n}=1}^{\infty} \frac{(\mathfrak{q}+1)\ \grave{Q}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{Q}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{Q}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}}}.$$

Proof. Let A and B be any two Banach spaces. Since $a' \odot b : A \to B$ is a rank-one operator, we have $s_{\mathrm{n}}(a' \odot b) = 0$ for all $\mathrm{n} \geq 2$. Therefore, we can express it as follows

$$\begin{aligned} Q^{(s)}(a' \odot b) &= \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(a' \odot b)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}} \\ &= \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1)\ \grave{\mathrm{Q}}_{1}(\mathfrak{q}) s_{1}(a' \odot b)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}} \\ &= \|a' \odot b\| \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}}. \end{aligned}$$

Since $\|a' \odot b\| = \|a'\|\|b\|$, we have

$$\widetilde{Q}^{(s)}(a' \odot b) = \|a'\|\|b\|.$$

Let $\mathfrak{T}, \mathfrak{R} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$. Then

$$Q^{(s)}(\mathfrak{T} + \mathfrak{R})$$

$$= \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T}+\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}}$$

$$\leq M \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}} + \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{K}}(\mathfrak{R})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right)^{\frac{1}{p}}$$

$$\leq M\Big(Q^{(s)}(\mathfrak{T}) + Q^{(s)}(\mathfrak{R}) \Big).$$

Thus,

$$\widetilde{Q}^{(s)}(\mathfrak{K}+\mathfrak{R}) \leq M \left(\widetilde{Q}^{(s)}(\mathfrak{K}) + \widetilde{Q}^{(s)}(\mathfrak{R}) \right).$$

Finally, let $\mathfrak{H} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))(A \to B)$, $\mathfrak{T} \in \mathcal{L}(B, B_0)$, and $\mathfrak{R} \in \mathcal{L}(A_0, A)$. Then

$$Q^{(s)}(\mathfrak{T}\mathfrak{H}\mathfrak{R}) = \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T}\mathfrak{H}\mathfrak{R}) \right)^{p} \right)^{\frac{1}{p}}$$

$$\leq \|\mathfrak{T}\|\|\mathfrak{R}\| \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{H}) \right)^{p} \right)^{\frac{1}{p}}$$

$$= \|\mathfrak{T}\|\|\mathfrak{R}\| \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{H}) \right)^{p} \right)^{\frac{1}{p}}.$$

Thus

$$\widetilde{Q}^{(s)}(\mathfrak{T}\mathfrak{H}\mathfrak{R}) \leq \|\mathfrak{K}\| \widetilde{Q}^{(s)}(\mathfrak{H}) \|\mathfrak{R}\|.$$

Therefore, $\widetilde{Q}^{(s)}$ serves as a quasi-norm for the operator ideal $\ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))$. □

Theorem 5.4. *For $1 < p < \infty$, the operator class $\ell_p^{(s)}(\grave{Q}(\mathfrak{q}))$ is complete with respect to the quasi-norm $\tilde{Q}^{(s)}$.*

Proof. Let $1 < p < \infty$. Then it follows that

$$\tilde{Q}^{(s)}(\mathfrak{T}) = \left[\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right]^{\frac{1}{p}}$$

$$= \left[\left(\sum_{\mathrm{n}=1}^{\infty} \frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{1}(\mathfrak{q}) s_{1}(\mathfrak{T})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^{p} \right]^{\frac{1}{p}}$$

$$\leq \|\mathfrak{T}\| \left[\left(\sum_{\mathrm{n}=1}^{\infty} \frac{(\mathfrak{q}+1)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^{p}\right]^{\frac{1}{p}}.$$

From this, we can conclude that

$$\|\mathfrak{T}\| \leq \tilde{Q}^{(s)}(\mathfrak{T}) \quad \text{for all } \mathfrak{T} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))(A \to B). \tag{5.1}$$

Now, let $(\mathfrak{T}_n)$ denote a Cauchy sequence in $\ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))(A \to B)$. For any $\epsilon > 0$, there exists $\kappa \in \mathbb{N}$ such that

$$\tilde{Q}^{(s)}(\mathfrak{T}_{\mathrm{n}} - \mathfrak{T}_{\mathrm{k}}) < \epsilon \quad \text{for all } \mathrm{n}, \mathrm{k} \geq \kappa. \tag{5.2}$$

From (5.1), we deduce that

$$\|\mathfrak{T}_n - \mathfrak{T}_m\| \leq Q(s)(\mathfrak{T}_n - \mathfrak{T}_m).$$

Applying (5.2), we obtain

$$\|\mathfrak{T}_n - \mathfrak{T}_m\| \leq Q(s)(\mathfrak{T}_n - \mathfrak{T}_m) \quad \text{for all } n, m \geq \kappa.$$

Therefore, the sequence $(\mathfrak{T}_{\mathrm{n}})$ is Cauchy in the space $\mathcal{L}(A, B)$. Since $\mathcal{L}(A, B)$ is a Banach space, we can conclude that $\mathfrak{T}_{\mathrm{n}} \to \mathfrak{T}$ as $\mathrm{n} \to \infty$ in $\mathcal{L}(A, B)$.

Utilizing Lemma 5.1, we have

$$|s_{\mathrm{n}}(\mathfrak{T}_n - \mathfrak{T}_m) - s_{\mathrm{n}}(\mathfrak{T} - \mathfrak{T}_m)| \leq \|\mathfrak{T}_n - \mathfrak{T}\|.$$

Taking the limit as $\mathrm{n} \to \infty$ gives us

$$s_{\mathrm{n}}(\mathfrak{T}_n - \mathfrak{T}_m) \to s_{\mathrm{n}}(\mathfrak{T} - \mathfrak{T}_m). \tag{5.3}$$

Now, from (5.2), we get

$$\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{T}_n - \mathfrak{T}_m)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^{p}\right)^{\frac{1}{p}} < \epsilon\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^{p}\right)^{\frac{1}{p}}.$$

Fix $m \geq \kappa$, let $\mathrm{n} \to \infty$ with $\mathrm{n} \geq \kappa$, and apply (5.3) to obtain

$$\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\sum\limits_{\mathrm{k}=1}^{\mathrm{n}}\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})s_{\mathrm{k}}(\mathfrak{T} - \mathfrak{T}_m)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^{p}\right)^{\frac{1}{p}} < \epsilon\left(\sum_{\mathrm{n}=1}^{\infty}\left(\frac{(\mathfrak{q}+1)\grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q})+\mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})}\right)^{p}\right)^{\frac{1}{p}},$$

which leads to

$$\tilde{Q}(s)(\mathfrak{T} - \mathfrak{T}_m) < \epsilon \quad \text{for all } m \geq \epsilon.$$

Hence, the sequence $(\mathfrak{T}_n)$ converges to $\mathfrak{T}$ in $\widetilde{Q}^{(s)}$ and we still need to verify that $\mathfrak{T} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))(A \to B)$.

$$\sum_{\mathfrak{T}=1}^{\mathrm{n}} s_{\mathrm{k}}(\mathfrak{T}) \leq \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}-1}(\mathfrak{q})s_{2\mathrm{k}-1}(\mathfrak{T}) + \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{2\mathrm{k}}s_{2\mathrm{k}}(\mathfrak{T})$$

$$\leq \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{2\mathrm{k}-1}(\mathfrak{q}) s_{2\mathrm{k}-1}(\mathfrak{T}).$$
$$\leq M \left(\sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T} - \mathfrak{T}_m) + \sum_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T}_m) \right).$$

Consequently

$$\left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T})}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^p \right)^{\frac{1}{p}}$$
$$\leq M \left[\left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T} - \mathfrak{T}_m)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^p \right)^{\frac{1}{p}} + \left(\sum_{\mathrm{n}=1}^{\infty} \left(\frac{(\mathfrak{q}+1) \sum\limits_{\mathrm{k}=1}^{\mathrm{n}} \grave{\mathrm{Q}}_{\mathrm{k}}(\mathfrak{q}) s_{\mathrm{k}}(\mathfrak{T}_m)}{(2+\mathfrak{q})\grave{\mathrm{Q}}_{\mathrm{n}}(\mathfrak{q}) + \mathfrak{q}\grave{\mathrm{Q}}_{\mathrm{n}-1}(\mathfrak{q})} \right)^p \right)^{\frac{1}{p}} \right],$$

which is finite. Since $\tilde{Q}(^{s)}(\mathfrak{T} - \mathfrak{T}_m) \to 0$ as $m \to \infty$ and $\mathfrak{T}_m \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))(A \to B)$. Hence, we conclude that $\mathfrak{T} \in \ell_p^{(s)}(\grave{\mathrm{Q}}(\mathfrak{q}))(A \to B)$. □

6 Geometric property

In this section, we explore various geometric properties of spaces $\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$ $(1 \leq p < \infty)$ and $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q})$.

Definition 6 ([18]). Let $\mathcal{P}$ be a normed space. We define a function $\rho_{\mathcal{P}} : (0, \infty) \to [0, \infty)$ as follows

1. If $\mathcal{P}$ is a zero space (i.e., $\mathcal{P} \neq \{0\}$), then
$$\rho_{\mathcal{P}}(t) = \sup \left\{ \frac{1}{2} (\|p + tq\| + \|p - tq\|) - 1 : p, q \in S_{\mathcal{P}} \right\}$$
2. Also for $\mathcal{P} = \{0\}$), then
$$\rho_{\mathcal{P}}(t) = \begin{cases} 0 & \text{if } 0 < t < 1 \\ t - 1 & \text{if } t \geq 1. \end{cases}$$
Then $\rho_{\mathcal{P}}(t)$ represents the modulus of smoothness of the space $\mathcal{P}$. The space $\mathcal{P}$ is uniformly smooth if:
$$\lim_{t \to 0^+} \frac{\rho_{\mathcal{P}}(t)}{t} = 0.$$

Theorem 6.1. *The space $\ell_p(\mathcal{P}(q))$ is uniformly smooth for $1 < p < \infty$.*

Proof. Let $u, v \in \ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))$, and suppose that

$$\|u + tv\| \geq 1 \quad \text{and} \quad \|u - tv\| \geq 1 \quad \text{for all } t > 0.$$

Recall that

$$\|u+tv\|_{\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))} = \|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)\|_{\ell_p}.$$

Our aim is to compute the limit $\lim\limits_{t\to 0^+} \frac{\rho_A(t)}{t}$.

Using L'Hospital's rule,

$$\lim_{t\to 0^+} \frac{\rho_A(t)}{t} = \lim_{t\to 0^+} \frac{d}{dt}\rho_A(t).$$

Now compute $\dfrac{d}{dt}\rho_A(t)$. By the definition of the modulus of smoothness,

$$\frac{d}{dt}\rho_A(t) = \sup\left\{\frac{1}{2}\left(\frac{d}{dt}\|u+tv\| + \frac{d}{dt}\|u-tv\|\right) : x, y \in S_{\ell_p(\grave{\mathrm{Q}}(\mathfrak{q}))}\right\}.$$

Now,

$$\frac{d}{dt}\|u+tv\| = \frac{d}{dt}\left(\|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)\|_{\ell_p}\right) = \frac{d}{dt}\left(\sum_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^p\right)^{1/p}.$$

Differentiating,

$$\frac{d}{dt}\|u+tv\| = \frac{1}{p}\left(\sum_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^p\right)^{1-\frac{1}{p}} \sum_{\mathrm{n}=1}^{\infty} \frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^p.$$

Similarly,

$$\frac{d}{dt}\|u-tv\| = \frac{1}{p}\left(\sum_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u-tv)_{\mathrm{n}}|^p\right)^{1-\frac{1}{p}} \sum_{\mathrm{n}=1}^{\infty} \frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u-tv)_{\mathrm{n}}|^p.$$

In particular,

$$\frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^p = p\,|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^{p-1}\frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|.$$

But

$$\frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}| = \begin{cases} (\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & \text{if } \grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}} \geq 0, \\ -(\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & \text{if } \grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}} < 0, \end{cases}$$

because $\grave{\mathrm{Q}}(\mathfrak{q})$ is a linear operator. Eventually

$$\frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^p = p\,|\grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}}|^{p-1}\begin{cases} (\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & \text{if } \grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}} \geq 0, \\ -(\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & \text{if } \grave{\mathrm{Q}}(\mathfrak{q})(u+tv)_{\mathrm{n}} < 0. \end{cases}$$

Similarly,

$$\frac{d}{dt}|\grave{\mathrm{Q}}(\mathfrak{q})(u-tv)_{\mathrm{n}}|^p = p\,|\grave{\mathrm{Q}}(\mathfrak{q})(u-tv)_{\mathrm{n}}|^{p-1}\begin{cases} -(\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & \text{if } \grave{\mathrm{Q}}(\mathfrak{q})(u-tv)_{\mathrm{n}} \geq 0, \\ (\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & \text{if } \grave{\mathrm{Q}}(\mathfrak{q})(u-tv)_{\mathrm{n}} < 0. \end{cases}$$

Taking limit $t \to 0^+$,

$$\lim_{t \to 0^+} \frac{d}{dt}\|u + tv\| = \begin{cases} \frac{1}{p}\left(\sum\limits_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u)_{\mathrm{n}}|^p\right)^{1-\frac{1}{p}} \sum\limits_{\mathrm{n}=1}^{\infty} p(\grave{\mathrm{Q}}(\mathfrak{q})(u))_{\mathrm{n}}^{p-1}\, (\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & (\grave{\mathrm{Q}}(\mathfrak{q})u)_{\mathrm{n}} \geq 0, \\ -\frac{1}{p}\left(\sum\limits_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u)_{\mathrm{n}}|^p\right)^{1-\frac{1}{p}} \sum\limits_{\mathrm{n}=1}^{\infty} p|\grave{\mathrm{Q}}(\mathfrak{q})(u)_{\mathrm{n}}|^{p-1}\, (\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & (\grave{\mathrm{Q}}(\mathfrak{q})u)_{\mathrm{n}} < 0 \end{cases}$$

and

$$\lim_{t \to 0^+} \frac{d}{dt}\|u - tv\| = \begin{cases} -\frac{1}{p}\left(\sum\limits_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u)_{\mathrm{n}}|^p\right)^{1-\frac{1}{p}} \sum\limits_{\mathrm{n}=1}^{\infty} p(\grave{\mathrm{Q}}(\mathfrak{q})(u))_{\mathrm{n}}^{p-1}(\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & (\grave{\mathrm{Q}}(\mathfrak{q})u)_{\mathrm{n}} \geq 0, \\ \frac{1}{p}\left(\sum\limits_{\mathrm{n}=1}^{\infty} |\grave{\mathrm{Q}}(\mathfrak{q})(u)_{\mathrm{n}}|^p\right)^{1-\frac{1}{p}} \sum\limits_{\mathrm{n}=1}^{\infty} p|\grave{\mathrm{Q}}(\mathfrak{q})(u)_{\mathrm{n}}|^{p-1}(\grave{\mathrm{Q}}(\mathfrak{q})v)_{\mathrm{n}}, & (\grave{\mathrm{Q}}(\mathfrak{q})u)_{\mathrm{n}} < 0. \end{cases}$$

$$\lim_{t \to 0^+} \frac{d}{dt}\|u + tv\| + \lim_{t \to 0^+} \frac{d}{dt}\|u - tv\| = 0.$$

Hence

$$\lim_{t \to 0^+} \frac{d}{dt}\rho_A(t) = 0,$$

This completes the proof. □

Theorem 6.2. *The sequence space $\ell_p(\grave{Q}(\mathfrak{q}))$ has the Dunford-Pettis property.*

Proof. Similar procedures can be used to prove this result [18, Theorem 6]. □

Definition 7 ([5])**.** Let $\mathfrak{K}$ be a sequence space. Then $\mathfrak{K}$ is called solid if

$$\{(b_m u \in \omega : \exists (a_\mu) \in \mathfrak{K}, \forall \mu \in \mathbb{N} : |b_\mu| < |a_\mu|)\} \subset \mathfrak{K}.$$

Theorem 6.3. *$c_0(\grave{Q}(\mathfrak{q}))$ is solid.*

Proof. The solidity of $c_0(\grave{\mathrm{Q}}(\mathfrak{q})$ follows directly from Definition 7. □

Lemma 6.1 ([5])**.** *Let X be a linear subspace of ω. The space X is solid if and only if $\ell_\infty(X) \subseteq X$, where*

$$\ell_\infty(X) = \{(a_\mu b_\mu) : (a_\mu) \in \ell_\infty,\ (b_\mu) \in X\}.$$

Theorem 6.4. *$c(\grave{Q}(\mathfrak{q}))$ is not a solid sequence space.*

Proof. Consider the sequence $v = \{(-1)^\mu\} \in \ell_\infty$ and $u = \{1, 1, 1, 1, 1, 1, ...\} \in c(\grave{\mathrm{Q}}(\mathfrak{q}))$. Clearly, $uv \notin c(\grave{\mathrm{Q}}(\mathfrak{q})$. Thus, by Lemma 6.1, $c(\grave{\mathrm{Q}}(\mathfrak{q}))$ is not a solid sequence space. □

Conclusion and future work

Classical summability theory has long been shaped by the foundational roles of matrices such as the Cesàro, Zweier, Abel, and Borel types, which are distinguished by their intricate structural properties and broad applicability. In recent years, the introduction of their $\mathfrak{q}$-analogues including the $\mathfrak{q}$-Cesàro, $\mathfrak{q}$-Fibonacci, $\mathfrak{q}$-Pascal, and $\mathfrak{q}$-Catalan matrices, has significantly enriched the field, offering deeper insights and fostering the formation of novel sequence spaces that exhibit intriguing topological and geometric behaviors.

In this study, we introduced the sequence spaces $\ell_{\mathcal{P}}(\grave{\mathrm{Q}}(\mathfrak{q}))$, $\ell_\infty(\grave{\mathrm{Q}}(\mathfrak{q}))$, $c_0(\grave{\mathrm{Q}}(\mathfrak{q}))$, and $c(\grave{\mathrm{Q}}(\mathfrak{q}))$, generated by matrices constructed using the $\mathfrak{q}$-Pell Lucas numbers. A comprehensive analysis of these spaces was carried out, which included their α-, β-, and γ-duals, as well as the characterization of associated operator ideals. Additionally, we explore their geometric aspects, particularly rotundity and smoothness, which play a crucial role in the functional analytic structure of these spaces. The findings of this work not only broaden the classical summability theory within the scope of $\mathfrak{q}$-analysis but also suggest possible applications in several areas, including quantum mechanics, combinatorics, dynamical systems, functional analysis, topology, and quantum groups. The approach presented in this study offers a strong foundation for creating and studying new types of sequence space. Therefore, this research opens new directions for future studies on statistical convergence, operator boundedness, matrix transformations, and dual structures related to $\mathfrak{q}$-analogues and special numerical sequences.

Open problem: Can we completely characterize the spectrum of q-Pell Lucas matrices in the sequence spaces $c, c_0, \ell_p\ (1 \leq p \leq \infty), bv$, and cs for $0 < \mathfrak{q} < 1$?

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