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# Existence of solutions to nonlocal weighted Kirchhoff-type problems via topological degree theory

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**Abstract**. This paper investigates a weighted Kirchhoff-type equation driven by the fractional $p(x,\cdot)$-Laplacian operator. The proposed model captures both nonlocal interactions and variable exponent effects, which naturally arise in several applied contexts. Under suitable structural assumptions, we analyze the associated nonlocal boundary value problem with Dirichlet conditions. By employing topological degree methods, we prove the existence of weak solutions, thereby extending and enriching existing results on Kirchhoff-type problems involving fractional and variable exponent operators.

***Keywords.*** Fractional $p(x,\cdot)$-Laplacian operator, weighted Kirchhoff-type problem, topological degree theory

## 1 Introduction

Kirchhoff-type problems have attracted considerable attention in recent years due to their broad applications in mathematical physics and engineering. These problems are characterized by the presence of a nonlocal term, which reflects the dependence of the material tension on the global deformation energy of the system. The original idea was introduced by Gustav Kirchhoff in 1883 in his study of the transverse vibrations of elastic strings, where the tension varies with the average elongation of the string [16].

Later, Lions provided a rigorous stationary formulation of Kirchhoff-type equations [17], leading to the model

$$-\mathrm{M}\left(\int_{\mathcal{O}} |\nabla u|^2\right)\Delta u = f(x,u),$$

where the Kirchhoff function M describes the nonlocal dependence on the total gradient energy. Since then, Kirchhoff-type equations have been extensively generalized to nonlinear frameworks, including problems involving the $p$-Laplacian and variable exponent $p(x)$-Laplacian operators, to capture more complex and heterogeneous nonlinear phenomena.

Recent developments have increasingly focused on Kirchhoff-type problems involving the fractional $p(x,\cdot)$-Laplacian operator, which naturally combines the nonlocal character of Kirchhoff models arising from the dependence of the operator on global energy terms with the intrinsic

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nonlocality of fractional operators. This interaction leads to significant analytical challenges, including the lack of compactness and the presence of variable exponent growth conditions. To overcome these difficulties, most existing studies have adopted a variational approach, relying on critical point theory. In particular, tools such as the mountain pass theorem and Ekeland's variational principle have played a central role in establishing existence and multiplicity results for weak solutions under appropriate structural assumptions [14, 10, 4, 20]. In addition, several alternative approaches, including fixed point theory, monotonicity methods, topological arguments, and Nehari manifold techniques have also been employed, covering local and nonlocal models with fractional operators, variable exponent growth, and singular or anisotropic effects [9, 8, 21, 19].

However, such approaches often require relatively strong assumptions, including the coercivity and differentiability of the associated energy functional, as well as restrictive growth conditions on the nonlinear terms. These requirements significantly limit their applicability, particularly in degenerate settings or in problems that do not admit a variational structure. As a consequence, alternative analytical frameworks are needed to address Kirchhoff-type problems with weaker regularity assumptions, nonstandard growth, or non-variational features.

In this framework, topological degree theory emerges as a powerful yet relatively underexplored alternative. Unlike variational methods, it is well-suited to treating problems with non-variational structures, lack of compactness, and degenerate nonlocal terms, which naturally arise in Kirchhoff-type models. Nevertheless, despite its clear potential, topological degree theory has only been marginally employed in the analysis of Kirchhoff-type problems, particularly those involving the fractional $p(x,\cdot)$-Laplacian operator.

On the other hand, topological degree methods have proved highly effective in a broad range of nonlinear and nonlocal problems, especially in establishing the existence of weak solutions for differential and integro-differential equations. Numerous works have successfully applied this approach to various classes of problems, demonstrating both its versatility and robustness (see, for instance, [1, 3, 15, 18]). Despite these advances, its systematic application to Kirchhoff-type equations, especially in the fractional $p(x,\cdot)$-Laplacian setting remains largely unexplored.

These considerations strongly motivate our choice to employ topological degree theory as the main analytical framework of this work. Indeed, this approach allows us to overcome the intrinsic limitations of variational techniques when dealing with non-variational structures, degenerate Kirchhoff terms, and variable exponent nonlocal operators. In particular, it provides a flexible and robust tool for addressing existence issues in settings where standard compactness or differentiability assumptions may fail. Within this perspective, we investigate a weighted Kirchhoff-type problem governed by a nonlocal fractional $p(x,\cdot)$-Laplacian operator, formulated as follows:

$$(\mathcal{P}_\omega)\begin{cases} \mathrm{M}\left(\varphi^s_{p(x,y)}(u)\right)\left[(-\Delta)^s_{p(x,\cdot)}(u)+\omega|u|^{\overline{p}(x)-2}u\right]=\alpha g(x,u) & \text{in } \mathcal{O},\\ u=0 & \text{on } \mathbb{R}^N\backslash\mathcal{O}, \end{cases} \tag{1.1}$$

with

- $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^N$ where $N\geqslant 2$, with Lipschitz boundary.
- $\mathrm{M}:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is the well-known Kirchhoff function that is assumed to be continuous.
- The mapping $\varphi^s_{p(x,y)}$ is defined in this way

$$\varphi^s_{p(x,y)}(u):=\varphi_{\mathcal{O}}=\int_Q\frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy+\int_{\mathcal{O}}\omega\frac{|u|^{\overline{p}(x)}}{\overline{p}(x)}dx.$$

- $\alpha$ is a positive parameter.
- $g : \mathcal{O} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function.
- $(-\Delta)^s_{p(x,\cdot)}$ is the nonlocal $p(x,\cdot)$-Laplacian operator that is defined as follows

$$(-\Delta)^s_{p(x,\cdot)} u(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}} dy,$$

for all $x \in \mathbb{R}^N$, where P.V. stands for the Cauchy principal value.

The remainder of this paper is organized as follows. In Section 2, we provide a comprehensive overview of the functional framework required for the analysis, including the definition of the relevant fractional variable exponent spaces and the main preliminary results that will be used throughout the paper. Section 3 is devoted to the statement and rigorous proof of the main existence results, which are obtained by applying topological degree theory.

# 2 Functional setting

In this section, we review the necessary functional framework, including the fundamental properties of variable-exponent Lebesgue spaces and fractional Sobolev spaces with variable exponents, as well as their weighted counterparts. These concepts provide the mathematical foundation for the analysis of our weighted Kirchhoff-type problem [13, 11].

## 2.1 Variable exponent Lebesgue space and its weighted version

Let $\mathcal{O} \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with Lipschitz boundary. We begin by introducing the class of continuous variable exponent functions defined by

$$\mathcal{P}(\mathcal{O}) = \left\{ p \in C(\overline{\mathcal{O}}) \ : \ p(x) > 1 \text{ for all } x \in \overline{\mathcal{O}} \right\}.$$

For any $p \in \mathcal{P}(\mathcal{O})$, we denote its essential bounds by

$$p^- = \inf_{x \in \overline{\mathcal{O}}} p(x) \quad \text{and} \quad p^+ = \sup_{x \in \overline{\mathcal{O}}} p(x).$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\mathcal{O})$ is defined as the set of all measurable functions $u : \mathcal{O} \to \mathbb{R}$ such that the modular

$$\rho_{p(\cdot)}(u) = \int_{\mathcal{O}} |u(x)|^{p(x)}\, dx$$

is finite.

The space $L^{p(\cdot)}(\mathcal{O})$ is endowed with the Luxemburg norm defined by

$$\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 \ : \ \rho_{p(\cdot)}\left(\frac{u}{\mu}\right) \leq 1 \right\}.$$

**Proposition 2.1.** *[13] The space $(L^{p(\cdot)}(\mathcal{O}), \|\cdot\|_{p(\cdot)})$ is a reflexive and separable Banach space.*

**Lemma 2.1** (Generalized Hölder's inequality). *Let $p \in \mathcal{P}(\mathcal{O})$ and let $\tilde{p}(\cdot)$ denote its conjugate exponent, defined by*

$$\frac{1}{p(x)} + \frac{1}{\tilde{p}(x)} = 1 \quad \text{for all } x \in \mathcal{O}.$$

*Then, for any $u \in L^{p(\cdot)}(\mathcal{O})$ and $v \in L^{\tilde{p}(\cdot)}(\mathcal{O})$, the following inequality holds:*

$$\left| \int_{\mathcal{O}} u(x)\, v(x)\, dx \right| \le 2\, \|u\|_{p(\cdot)}\, \|v\|_{\tilde{p}(\cdot)}.$$

The relationship between the modular and the Luxemburg norm is characterized by the following assertions.

**Proposition 2.2.** *Let $u \in L^{p(\cdot)}(\mathcal{O})$ and let $\{u_n\} \subset L^{p(\cdot)}(\mathcal{O})$. Then the following properties hold:*

1. $\|u\|_{p(\cdot)} < 1$ *(resp.* $\|u\|_{p(\cdot)} > 1$, $\|u\|_{p(\cdot)} = 1$*) if and only if* $\rho_{p(\cdot)}(u) < 1$ *(resp.* $\rho_{p(\cdot)}(u) > 1$, $\rho_{p(\cdot)}(u) = 1$*).*
2. *If* $\|u\|_{p(\cdot)} < 1$*, then*
$$\|u\|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}.$$
3. *If* $\|u\|_{p(\cdot)} > 1$*, then*
$$\|u\|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}.$$
4. $\|u_n - u\|_{p(\cdot)} \to 0$ *as* $n \to \infty$ *if and only if*
$$\rho_{p(\cdot)}(u_n - u) \to 0 \quad \text{as } n \to \infty.$$

**Corollary 2.1.** *Let $u \in L^{p(\cdot)}(\mathcal{O})$. Then the following estimates hold:*

- $\|u\|_{p(\cdot)} \le \rho_{p(\cdot)}(u) + 1,$
- $\rho_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-} + \|u\|_{p(\cdot)}^{p^+}.$

Let $\omega$ be a measurable weight function defined on $\mathcal{O}$ satisfying the following assumptions:

$$(w_1) \quad \omega(x) > 0 \quad \text{for a.e. } x \in \mathcal{O}, \tag{2.1}$$

$$(w_2) \quad \omega \in L^1(\mathcal{O}) \quad \text{and} \quad \operatorname*{ess\,inf}_{x\in\mathcal{O}} \omega(x) = \omega_0 > 0. \tag{2.2}$$

The weighted variable exponent Lebesgue space $L^{p(\cdot)}_{\omega}(\mathcal{O})$, introduced in [5], is defined as

$$L^{p(\cdot)}_{\omega}(\mathcal{O}) = \left\{ u : \mathcal{O} \to \mathbb{R} \text{ measurable} \;\middle|\; \int_{\mathcal{O}} \omega(x)\, |u(x)|^{p(x)}\, dx < \infty \right\}.$$

This space is endowed with the norm defined in this way

$$\|u\|_{p(\cdot),\omega} = \inf \left\{ \mu > 0 \;\middle|\; \int_{\mathcal{O}} \omega(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

The associated modular $\rho_{p(\cdot),\omega} : L^{p(\cdot)}_{\omega}(\mathcal{O}) \to \mathbb{R}$ is defined by

$$\rho_{p(\cdot),\omega}(u) = \int_{\mathcal{O}} \omega(x)\, |u(x)|^{p(x)}\, dx.$$

As in Proposition 2.2, the relationship between the norm $\|\cdot\|_{p(\cdot),\omega}$ and the modular $\rho_{p(\cdot),\omega}$ is characterized by the following properties.

**Proposition 2.3.** *Let $u \in L^{p(\cdot)}_{\omega}(\mathcal{O})$ and let $\{u_n\} \subset L^{p(\cdot)}_{\omega}(\mathcal{O})$. Then the following assertions hold:*

- $\|u\|_{p(\cdot),\omega} < 1$ *(resp. $> 1$, $= 1$) if and only if* $\rho_{p(\cdot),\omega}(u) < 1$ *(resp. $> 1$, $= 1$).*
- *If* $\|u\|_{p(\cdot),\omega} < 1$*, then*
$$\|u\|^{p^+}_{p(\cdot),\omega} \leq \rho_{p(\cdot),\omega}(u) \leq \|u\|^{p^-}_{p(\cdot),\omega};$$
- *If* $\|u\|_{p(\cdot),\omega} > 1$*, then*
$$\|u\|^{p^-}_{p(\cdot),\omega} \leq \rho_{p(\cdot),\omega}(u) \leq \|u\|^{p^+}_{p(\cdot),\omega};$$
- $\|u_n\|_{p(\cdot),\omega} \to 0$ *as* $n \to \infty$ *if and only if* $\rho_{p(\cdot),\omega}(u_n) \to 0$ *as* $n \to \infty$*;*
- $\|u_n\|_{p(\cdot),\omega} \to \infty$ *as* $n \to \infty$ *if and only if* $\rho_{p(\cdot),\omega}(u_n) \to \infty$ *as* $n \to \infty$*.*

## 2.2 Fractional Sobolev space with variable exponents and its weighted version

The intrinsic nonlocal nature of the fractional $p(x,\cdot)$-Laplacian operator requires a careful treatment of boundary effects. In particular, the classical domain $\mathcal{O}$ alone is not sufficient to capture the interactions induced by the nonlocal operator, since points outside $\mathcal{O}$ may still influence the dynamics inside the domain. For this reason, the analysis is carried out on the extended set

$$Q := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathcal{O}^c \times \mathcal{O}^c), \quad \text{where } \mathcal{O}^c = \mathbb{R}^N \setminus \mathcal{O}.$$

This formulation ensures that all relevant interactions involving at least one point in $\mathcal{O}$ are taken into account, thereby providing a correct and complete description of the boundary behavior in the fractional setting.

In what follows, let $p : \overline{Q} \to (1,\infty)$ be a continuous exponent function satisfying the following conditions:

$$1 < p^- := \inf_{(x,y)\in\overline{Q}} p(x,y) \leq p(x,y) \leq p^+ := \sup_{(x,y)\in\overline{Q}} p(x,y) < \infty, \tag{2.3}$$

$$p(x,y) = p(y,x) \quad \text{for all } (x,y) \in \overline{Q}. \tag{2.4}$$

For $x \in \overline{\mathcal{O}}$, we set $\overline{p}(x) := p(x,x)$. Let $s \in (0,1)$. The fractional Sobolev space with variable exponent, introduced in [7], is defined by

$$W^{s,p(x,y)}(Q) = \left\{ u \in L^{\overline{p}(x)}(\mathcal{O}) \;\middle|\; \exists \mu > 0 \text{ such that } \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty \right\}.$$

This space is endowed with the norm

$$\|u\|_{s,p(x,y)} = \|u\|_{\overline{p}(x)} + [u]_{s,p(x,y)},$$

where the Gagliardo seminorm is given by

$$[u]_{s,p(x,y)} = \inf \left\{ \mu > 0 \;\middle|\; \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy \leq 1 \right\}.$$

Since the problem under consideration involves a weight function, the analysis is carried out within the framework of weighted fractional Sobolev spaces with variable exponents. Let $\omega$ be a weight function satisfying condition $(w_1)$.

**Definition 1.** The weighted fractional Sobolev space with variable exponent $W_{\omega}^{s,p(x,y)}(Q)$, denoted by $\mathrm{W}_\omega$, is defined as the set of all measurable functions $u : \mathcal{O} \to \mathbb{R}$ such that

1. $u \in L_{\omega}^{\overline{p}(x)}(\mathcal{O})$,
2. there exists $\mu > 0$ for which

$$\int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\mu^{p(x,y)}|x-y|^{N+sp(x,y)}}\,dx\,dy < \infty.$$

The space $\mathrm{W}_\omega$ is endowed with the following norm defined by

$$\|u\|_{\mathrm{W}_\omega} = \inf\left\{\mu > 0 \;\middle|\; \rho_\omega\left(\frac{u}{\mu}\right) \leq 1\right\},$$

where the associated modular $\rho_\omega : \mathrm{W}_\omega \to \mathbb{R}$ is given by

$$\rho_\omega(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\mathcal{O}} \omega(x)\,|u(x)|^{\overline{p}(x)}\,dx.$$

It is known (see [6]) that the space $(\mathrm{W}_\omega, \|\cdot\|_{\mathrm{W}_\omega})$ is a reflexive and separable Banach space.

The relationship between the norm $\|\cdot\|_{\mathrm{W}_\omega}$ and the modular $\rho_\omega$ is described by the following properties.

**Proposition 2.4.** *For every $u \in \mathrm{W}_\omega$, the following assertions hold:*

*(i)* $\|u\|_{\mathrm{W}_\omega} < 1$ *(respectively* $= 1, > 1$*) if and only if* $\rho_\omega(u) < 1$ *(respectively* $= 1, > 1$*).*

*(ii) If* $\|u\|_{\mathrm{W}_\omega} < 1$*, then*

$$\|u\|_{\mathrm{W}_\omega}^{p^+} \leq \rho_\omega(u) \leq \|u\|_{\mathrm{W}_\omega}^{p^-}.$$

*(iii) If* $\|u\|_{\mathrm{W}_\omega} > 1$*, then*

$$\|u\|_{\mathrm{W}_\omega}^{p^-} \leq \rho_\omega(u) \leq \|u\|_{\mathrm{W}_\omega}^{p^+}.$$

We now state a fundamental embedding result that will play a key role in the subsequent analysis.

**Lemma 2.2.** *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary and let $s \in (0,1)$. Assume that $p : Q \to (1,\infty)$ is a continuous function satisfying conditions (2.3) and (2.4), and that $sp^- > N$. Suppose moreover that the weight function $\omega$ satisfies condition $(\omega_1)$. Then the embedding of the space $\mathrm{W}_\omega$ into $C^0(\overline{\mathcal{O}})$ is continuous and compact. Consequently, there exists a constant $c_0 > 0$ such that*

$$\|u\|_\infty \leq c_0\,\|u\|_{\mathrm{W}_\omega} \quad \textit{for all } u \in \mathrm{W}_\omega.$$

## 2.3 The fractional $p(x,\cdot)$-Laplacian operator and its properties

We begin by introducing the energy functional $\mathcal{I}_1 : \mathrm{W}_\omega \to \mathbb{R}$, defined by

$$\mathcal{I}_1(u) = \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)\,|x-y|^{N+sp(x,y)}}\,dx\,dy, \quad \text{for all } u \in \mathrm{W}_\omega.$$

The associated nonlinear operator $\mathcal{L}_1 : \mathrm{W}_\omega \to \mathrm{W}_\omega^*$ is defined via the duality pairing

$$\langle \mathcal{L}_1(u), v\rangle = \int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy, \quad \text{for all } u, v \in \mathrm{W}_\omega.$$

The operator $\mathcal{L}_1$ enjoys the following fundamental properties.

**Lemma 2.3.** *[11, Lemma 4.2] The operator $\mathcal{L}_1$ satisfies the following assertions:*

1. *$\mathcal{L}_1$ is bounded and strictly monotone.*
2. *$\mathcal{L}_1$ satisfies the $(S_+)$ property; that is, if a sequence $(u_n) \subset \mathrm{W}_\omega$ converges weakly to $u$ in $\mathrm{W}_\omega$ and*
$$\limsup_{n\to\infty} \langle \mathcal{L}_1(u_n) - \mathcal{L}_1(u), u_n - u\rangle \le 0,$$
*then $u_n \to u$ strongly in $\mathrm{W}_\omega$.*
3. *The mapping $\mathcal{L}_1 : \mathrm{W}_\omega \to \mathrm{W}_\omega^*$ is a homeomorphism.*

The following degree-theoretic result will be used in the proof of the main theorem. This result provides an abstract existence criterion for operator equations of Hammerstein type.

**Proposition 2.5** ([2, Proposition 2.5])**.** *Let $X$ be a real Banach space, and let $S : X \to X^*$ and $T : X^* \to X$ be two bounded and continuous operators. Assume that $S$ is quasimonotone and that $T$ is a homeomorphism, strictly monotone, and of class $(S_+)$. If the set*

$$\Lambda := \{v \in X^* \ : \ v + t\, S \circ T(v) = 0 \text{ for some } t \in [0,1]\}$$

*is bounded in $X^*$, then the equation*

$$v + S \circ T(v) = 0$$

*admits at least one solution in $X^*$.*

# 3 Main results

This section is devoted to the statement and proof of the main existence results. Throughout this section, the Kirchhoff function M and the nonlinear term $g$ are assumed to satisfy the following hypotheses.

(H1) The Kirchhoff function $\mathrm{M} : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies

$$m_0\, t^{\beta(x)-1} \le \mathrm{M}(t) \le m_1\, t^{\beta(x)-1}, \quad \text{for all } t \ge 0, \tag{3.1}$$

where $m_1 \ge m_0 > 0$ and $\beta \in \mathcal{P}(\Omega)$.

(H2) The nonlinear term $g : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|g(x,u)| \le \widetilde{C}\left(|h(x)| + |u|^{r(x)-1}\right), \quad \text{for a.e. } x \in \mathcal{O}, \tag{3.2}$$

where $h \in L^{\widetilde{r}(x)}(\mathcal{O})$ and the exponent $r$ fulfills

$$r^+ < p_s^*(x) = \frac{N\,\overline{p}(x)}{N - s\,\overline{p}(x)}, \quad \text{for all } x \in \overline{\mathcal{O}}.$$

As a consequence of the above analysis, the main result of this work is stated as follows.

**Theorem 3.1.** *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Assume that the exponent function $p$ satisfies conditions (2.3) and (2.4). Suppose further that hypotheses (3.1) and (3.2) are fulfilled, and that the weight function $\omega$ satisfies condition $(w_1)$. Then the problem $(\mathcal{P}_\omega)$ admits at least one weak solution in the space $\mathrm{W}_\omega$.*

A function $u \in \mathrm{W}_\omega$ is said to be a weak solution of problem $(\mathcal{P}_\omega)$ if

$$\mathrm{M}\Big(\varphi^s_{p(x,y)}(u)\Big)\left[\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\mathcal{O}} \omega(x)\,|u(x)|^{\overline{p}(x)-2}u(x)\,v(x)\,dx\right] = \alpha\int_{\mathcal{O}} g(x,u(x))\,v(x)\,dx,$$

for all test functions $v \in \mathrm{W}_\omega$.

The proof of Theorem 3.1 is based on several auxiliary results. To this end, define the functionals $\Phi, \Psi : \mathrm{W}_\omega \to \mathbb{R}$ by

$$\Phi(u) = \widehat{\mathrm{M}}\Big(\varphi^s_{p(x,y)}(u)\Big), \qquad \Psi(u) = -\int_{\mathcal{O}} G(x,u(x))\,dx, \quad \forall u \in \mathrm{W}_\omega,$$

where

$$\widehat{\mathrm{M}}(t) = \int_0^t \mathrm{M}(\tau)\,d\tau, \qquad G(x,t) = \int_0^t g(x,\tau)\,d\tau.$$

Since $\widehat{\mathrm{M}}$ is a primitive of the Kirchhoff function M, it follows that $\widehat{\mathrm{M}} \in C^1(\mathbb{R},\mathbb{R})$, and consequently $\Phi \in C^1(\mathrm{W}_\omega,\mathbb{R})$. Moreover, the Carathéodory nature of the nonlinearity $g$ implies that $\Psi \in C^1(\mathrm{W}_\omega,\mathbb{R})$.

The derivatives of $\Phi$ and $\Psi$ are given by

$$\langle \Phi'(u), v\rangle = \mathrm{M}\Big(\varphi^s_{p(x,y)}(u)\Big)\left[\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\mathcal{O}} \omega(x)\,|u(x)|^{\overline{p}(x)-2}u(x)\,v(x)\,dx\right],$$

$$\langle \Psi'(u), v\rangle = -\int_{\mathcal{O}} g(x,u(x))\,v(x)\,dx, \qquad \text{for all } u, v \in \mathrm{W}_\omega.$$

**Lemma 3.1.** *Under the assumptions of Theorem 3.1, the following properties hold:*

1. *The operator $\Phi' : \mathrm{W}_\omega \to \mathrm{W}^*_\omega$ is bounded, continuous, strictly monotone, and coercive.*
2. *The operator $\Phi'$ satisfies the $(S_+)$ property.*

*Proof.* The proof is divided into two steps.

1. **Boundedness.** Recall that

$$\langle \Phi'(u), v\rangle = \mathrm{M}\Big(\varphi^s_{p(x,y)}(u)\Big)\left[\int_Q \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy + \int_{\mathcal{O}} \omega(x)\,|u(x)|^{\overline{p}(x)-2}u(x)\,v(x)\,dx\right].$$

Since M is continuous and satisfies assumption (3.1), and since the operator associated with the fractional $p(x,\cdot)$-Laplacian is bounded and strictly monotone (see Lemma 2.3), it follows that $\Phi'$ is bounded and strictly monotone on $W_\omega$.

**Coercivity.** Let $u \in W_\omega$ with $\|u\|_{W_\omega} > 1$. By condition (3.1), one has

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|_{W_\omega}} = \frac{M\Big(\varphi^s_{p(x,y)}(u)\Big)\ \rho_\omega(u)}{\|u\|_{W_\omega}} \geq \frac{m_0\left(\varphi^s_{p(x,y)}(u)\right)^{\beta(x)-1}\rho_\omega(u)}{\|u\|_{W_\omega}}.$$

Using the inequality

$$\varphi^s_{p(x,y)}(u) \geq \frac{1}{p^+}\,\rho_\omega(u),$$

and Proposition 2.4, it follows that

$$\frac{\langle \Phi'(u), u\rangle}{\|u\|_{W_\omega}} \geq \frac{m_0}{(p^+)^{\beta^+-1}}\|u\|^{\beta^-p^--1}_{W_\omega}.$$

Since $\beta^- p^- > 1$, the right-hand side tends to $+\infty$ as $\|u\|_{W_\omega} \to +\infty$, which proves the coercivity of $\Phi'$.

**Continuity.** The functional $\Phi$ belongs to $C^1(W_\omega, \mathbb{R})$ because M is continuous and $\varphi^s_{p(x,y)}$ is continuously Fréchet differentiable. Therefore, $\Phi'$ is continuous.

2. *$(S_+)$-property of $\Phi'$.*
Let $(u_n) \subset W_\omega$ be a sequence such that

$$u_n \rightharpoonup u \quad \text{weakly in } W_\omega,$$

and

$$\limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \leq 0.$$

By the strict monotonicity of $\Phi'$, one has

$$\limsup_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle = 0 \ \text{ as } n \to \infty.$$

Since $(u_n)$ is bounded in $W_\omega$, there exists a subsequence (still denoted $(u_n)$) such that

$$\varphi^s_{p(x,y)}(u_n) \to \ell_1 \geq 0.$$

If $\ell_1 = 0$, then $u_n \to 0$ strongly in $W_\omega$ and the result follows.
Assume now that $\ell_1 > 0$. By the continuity of M, we have

$$M\Big(\varphi^s_{p(x,y)}(u_n)\Big) \to M(\ell_1) \ \text{ as } n \to \infty.$$

Hence,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \\ &= M(\ell_1)\lim_{n\to\infty}\Bigg[\int_Q \frac{|u_n(x)-u_n(y)|^{p(x,y)-2}(u_n(x)-u_n(y))((u_n-u)(x)-(u_n-u)(y))}{|x-y|^{N+sp(x,y)}}\,dx\,dy \\ &\quad + \int_{\mathcal{O}} \omega(x)|u_n|^{\overline{p}(x)-2}u_n(u_n-u)\,dx\Bigg]. \end{aligned}$$

By Lemma 2.3, the operator $\mathcal{L}_1$ associated with the fractional $p(x,\cdot)$-Laplacian is of type $(S_+)$. Consequently,

$$u_n \to u \quad \text{strongly in } \mathrm{W}_\omega.$$

This proves that $\Phi'$ satisfies the $(S_+)$-property.

□

**Lemma 3.2.** *The derivative of* $\Psi$ *is compact on* $\mathrm{W}_\omega$.

*Proof.* Consider the operator $\sigma : \mathrm{W}_\omega \to L^{\widetilde{r}(\cdot)}(\mathcal{O})$ defined by

$$\sigma(u)(x) := -\lambda\, g(x, u(x)), \quad x \in \mathcal{O}.$$

The aim is to show that $\sigma$ is bounded and continuous.

**Boundedness:** Let $u \in \mathrm{W}_\omega$. By Corollary 2.1, one has

$$\|\sigma(u)\|_{L^{\widetilde{r}(\cdot)}(\mathcal{O})} \le \rho_{\widetilde{r}(\cdot)}(\sigma(u)) + 1 = \int_{\mathcal{O}} |\lambda g(x,u)|^{\widetilde{r}(x)}\, dx + 1.$$

Using the growth condition (3.2), there exists a constant $\widetilde{C}_1 > 0$ such that

$$|g(x,u)|^{\widetilde{r}(x)} \le \widetilde{C}_1 \left( |h(x)|^{\widetilde{r}(x)} + |u|^{(r(x)-1)\widetilde{r}(x)} \right) \quad \text{a.e. in } \mathcal{O}.$$

Therefore,

$$\|\sigma(u)\|_{L^{\widetilde{r}(\cdot)}(\mathcal{O})} \le \lambda \widetilde{C}_1 \left[ \rho_{\widetilde{r}(\cdot)}(h) + \rho_{r(\cdot)}(u) \right] + 1.$$

Applying again Corollary 2.1, it follows that

$$\|\sigma(u)\|_{L^{\widetilde{r}(\cdot)}(\mathcal{O})} \le \widetilde{C}_2 \left( \|h\|^{\widetilde{r}^+}_{L^{\widetilde{r}(\cdot)}(\mathcal{O})} + \|u\|^{r^-}_{L^{r(\cdot)}(\mathcal{O})} + \|u\|^{r^+}_{L^{r(\cdot)}(\mathcal{O})} \right).$$

Since $r^+ < p_s^*(x)$ for all $x \in \overline{\mathcal{O}}$, the embedding result $\mathrm{W}_\omega \hookrightarrow L^{r(\cdot)}(\mathcal{O})$ (see Lemma 2.2) implies that $\sigma$ maps bounded subsets of $\mathrm{W}_\omega$ into bounded subsets of $L^{\widetilde{r}(\cdot)}(\mathcal{O})$. Hence, $\sigma$ is bounded.

**Continuity.** Let $\{u_k\} \subset \mathrm{W}_\omega$ be a sequence such that $u_k \to u$ strongly in $\mathrm{W}_\omega$. By the compact embedding $\mathrm{W}_\omega \hookrightarrow L^{\overline{p}(\cdot)}(\mathcal{O})$, it follows that

$$u_k \to u \quad \text{in } L^{\overline{p}(\cdot)}(\mathcal{O}) \quad \text{and a.e. in } \mathcal{O}.$$

Up to a subsequence, there exists a function $v \in L^{\overline{p}(\cdot)}(\mathcal{O})$ such that

$$|u_k(x)| \le v(x) \quad \text{a.e. in } \mathcal{O}.$$

Invoking the growth condition (3.2), one obtains

$$|g(x, u_k(x))| \le \widetilde{C} \left( |h(x)| + |v(x)|^{r(x)-1} \right),$$

which belongs to $L^{\widetilde{r}(\cdot)}(\mathcal{O})$ $(r^+ < p_s^*(x) \le \overline{p}(x))$ . Moreover,

$$|g(x, u_k(x)) - g(x, u(x))|^{\widetilde{r}(x)} \to 0 \quad \text{a.e. in } \mathcal{O}.$$

Thus, by the Dominated Convergence Theorem, it follows that

$$\int_{\mathcal{O}} |g(x,u_k) - g(x,u)|^{\widetilde{r}(x)}\, dx \to 0 \quad \text{as } k \to \infty.$$

Hence, $\sigma$ is continuous from $\mathrm{W}_\omega$ into $L^{\widetilde{r}(\cdot)}(\mathcal{O})$.

In what follows, the derivative operator $\Psi'$ can be written as the composition

$$\Psi' = I^* \circ \sigma,$$

where $I : \mathrm{W}_\omega \hookrightarrow L^{r(\cdot)}(\mathcal{O})$ denotes the compact embedding operator, and $I^* : L^{\widetilde{r}(\cdot)}(\mathcal{O}) \to \mathrm{W}_\omega^*$ is its adjoint. Since the adjoint of a compact operator is compact, it follows that $\Psi'$ is a compact operator.

□

*Proof of Theorem 3.1.* The proof relies on topological degree theory for operator equations of Hammerstein type. All functional-analytic tools required for this approach have been established in the previous results.
A function $u \in \mathrm{W}_\omega$ is a weak solution of problem $(\mathcal{P}_\omega)$ if and only if it satisfies

$$\Phi'(u) = \lambda\, \Psi'(u) \quad \text{in } \mathrm{W}_\omega^*. \tag{3.3}$$

This equation corresponds to the Euler–Lagrange formulation associated with the functionals $\Phi$ and $\Psi$ introduced earlier.

By Lemma 3.1, the operator

$$\Phi' : \mathrm{W}_\omega \longrightarrow \mathrm{W}_\omega^*$$

is bounded, continuous, coercive, and strictly monotone, and it satisfies the $(S_+)$ property. Therefore, by the Minty–Browder Theorem, $\Phi'$ is surjective and admits a continuous inverse

$$G := (\Phi')^{-1} : \mathrm{W}_\omega^* \longrightarrow \mathrm{W}_\omega.$$

Moreover, the operator $G$ is bounded, continuous, strictly monotone, and also of class $(S_+)$.
From Lemma 3.2, the operator

$$\Psi' : \mathrm{W}_\omega \longrightarrow \mathrm{W}_\omega^*$$

is compact. In particular, $\Psi'$ is bounded and continuous. Furthermore, every compact operator is quasimonotone, and hence $\Psi'$ enjoys this property.
Equation (3.3) can be rewritten as

$$u = Gv, \quad \text{where } v = \lambda\, \Psi'(u).$$

Substituting $u = Gv$ into the second identity yields the equivalent equation

$$v - \lambda\, \Psi' \circ G(v) = 0 \quad \text{in } \mathrm{W}_\omega^*.$$

To complete the proof, topological degree theory is applied as described in [12]. The first step consists for $\mathcal{S} = -\lambda\, \Psi'$ in proving that the set

$$\mathcal{B} := \{v \in \mathrm{W}_\omega^* \ : \ v + t\mathcal{S} \circ G(v) = 0 \text{ for some } t \in [0,1]\}$$

is bounded.

Let $v \in \mathcal{B}$. Then there exists $t \in [0,1]$ such that

$$v + t\mathcal{S} \circ G(v) = 0.$$

Setting $u := G(v)$, it follows by taking the duality pairing with $u$ and using assumption (3.1) that

$$\begin{aligned}
\|u\|_{\mathrm{W}_\omega}^{\overline{p}^-} \le \rho_\omega(u) &\le \frac{(p^+)^{\frac{\beta^+-1}{\beta^-}}}{(m_0)^{\frac{1}{\beta^+}}} \langle \Phi'(u), u\rangle^{1/\beta^-} \\
&= \frac{(p^+)^{\frac{\beta^+-1}{\beta^-}}}{(m_0)^{\frac{1}{\beta^+}}} \left(\langle v, G(v)\rangle\right)^{1/\beta^-} \\
&= \frac{(p^+)^{\frac{\beta^+-1}{\beta^-}}}{(m_0)^{\frac{1}{\beta^+}}} - t^{\frac{1}{\beta^-}} \left(\langle \mathcal{S} \circ G(v), G(v)\rangle\right)^{1/\beta^-} \\
&= \frac{(p^+)^{\frac{\beta^+-1}{\beta^-}}}{(m_0)^{\frac{1}{\beta^+}}} (t\lambda)^{\frac{1}{\beta^-}} \left(\langle \Psi'(u), u\rangle\right)^{1/\beta^-}
\end{aligned}$$

Using the definition of $\Psi'$ and the growth condition (3.2), it follows that

$$\begin{aligned}
|\langle \Psi'(u), u\rangle| &= \left| \int_{\mathcal{O}} g(x, u(x))\, u(x)\, dx \right| \\
&\le C \int_{\mathcal{O}} \left( |h(x)|\, |u(x)| + |u(x)|^{r(x)} \right) dx.
\end{aligned}$$

Applying Hölder's inequality in variable exponent spaces and Proposition 2.1, one obtains

$$|\langle \Psi'(u), u\rangle| \le C_1 \Big( \|h\|_{L^{\tilde{r}(\cdot)}(\mathcal{O})} \|u\|_{L^{r(\cdot)}(\mathcal{O})} + \max\{ \|u\|_{L^{r(\cdot)}(\mathcal{O})}^{r^-}, \|u\|_{L^{r(\cdot)}(\mathcal{O})}^{r^+} \} \Big).$$

Since $r^+ < p_s^*(x)$ for all $x \in \overline{\mathcal{O}}$, the continuous embedding

$$\mathrm{W}_\omega \hookrightarrow L^{r(\cdot)}(\mathcal{O})$$

implies that there exists a constant $C_2 > 0$ such that

$$\|u\|_{\mathrm{W}_\omega}^{\overline{p}^-} \le C_2 \|u\|_{\mathrm{W}_\omega}^{\frac{r^+}{\beta^-}}.$$

Consequently,

$$\|u\|_{\mathrm{W}_\omega}^{\overline{p}^- - \frac{r^+}{\beta^-}} \le C_2.$$

This shows that the set

$$\{G(v) : v \in \mathcal{B}\}$$

is bounded. Since $\Psi'$ is bounded, it follows directly that $\mathcal{B}$ is bounded in $\mathrm{W}_\omega^*$.

Therefore, all the assumptions of Proposition 2.5 in [2] are satisfied. As a consequence, the equation

$$v + \mathcal{S} \circ G(v) = 0$$

admits at least one solution $v_1 \in \mathrm{W}_\omega^*$. Consequently, by setting $u_1 := G(v_1)$, a weak solution $u_1 \in \mathrm{W}_\omega$ of problem $(\mathcal{P}_\omega)$ is obtained, and this completes the proof. □

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