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A log-convex generalization of Alzer–Fonseca–Kovačec type inequalities and applications

Đặng Võ Phúc

Abstract. We develop a multiplicative/logarithmic counterpart of the convex-analytic scheme of Phung–Huy [9] for Alzer–Fonseca–Kovačec type inequalities. At the scalar level, we establish comparison inequalities for strictly increasing log-convex functions on the logarithmic scale. We also record a derivative-weighted estimate expressed via the logarithmic derivative $r = (\log \varphi)'$, and we clarify parameter regimes where this derivative factor can improve the universal constant. Concrete applications are presented for the function t^t and the Gamma function Γ. We then extend the scalar comparison to commuting positive definite matrices under unitarily invariant norms with the universal constant $\frac{v(1-v)}{\tau(1-\tau)}$. For the non-commuting case, we obtain "envelope" bounds via spectral pinching onto finite-dimensional abelian subalgebras, and we include a Heinz-centered quantitative estimate in the pinched abelian setting, combining a Lipschitz control for $g = \log \circ \varphi$ with the Heinz norm inequality under unitarily invariant norms.

Keywords. Log-convex function, multiplicative inequality, arithmetic–geometric mean, Gamma function, spectral pinching, Heinz mean, Kubo–Ando theory

1 Introduction

For $a, b > 0$ and $v \in (0, 1)$ set

$$a\nabla_v b := (1-v)a + vb, \quad a\#_v b := a^{1-v}b^v.$$

A recent and elegant contribution of Phung and Huy [9] provides a unified convex-analytic framework to refine Young-type comparisons between $a\nabla_v b$ and $a\#_v b$. If Φ is increasing and convex, then Phung and Huy [9] proved that the following inequality holds whenever $(a-b)(v-\tau) \geq 0$, and it is reversed otherwise:

$$\frac{\Phi(a\nabla_v b) - \Phi(a\#_v b)}{\Phi(a\nabla_\tau b) - \Phi(a\#_\tau b)} \leq \frac{v(1-v)}{\tau(1-\tau)}. \tag{1.1}$$

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Corresponding author: Đặng Võ Phúc.

Remark 1 (Direction of (1.1)). Let $a, b > 0$ with $a \neq b$, and let $\Phi : (0, \infty) \to \mathbb{R}$ be increasing and convex. For $\nu \in (0,1)$, write $x_\nu := a \nabla_\nu b$ and $g_\nu := a \#_\nu b$ (equivalently, $x_\nu = (1-\nu)a + \nu b$ and $g_\nu = a^{1-\nu} b^\nu$), and set

$$D_\nu^\Phi(a,b) := \Phi(x_\nu) - \Phi(g_\nu).$$

By [9, Thm. 2.2] (take $\varphi = \mathrm{id}$ and $\psi = \Phi$), for any $v, \tau \in (0,1)$,

$$\frac{D_v^\Phi(a,b)}{D_\tau^\Phi(a,b)} \leq \frac{v(1-v)}{\tau(1-\tau)} \quad \text{whenever } (a-b)(v-\tau) \geq 0,$$

and the inequality is reversed otherwise. In particular, under $0 < v \leq \tau < 1$, (1.1) holds for $a < b$ and is reversed for $a > b$. Additionally, taking $\Phi = \mathrm{id}$ in (1.1) yields the refined AM–GM gap control

$$\frac{a \nabla_v b - a \#_v b}{a \nabla_\tau b - a \#_\tau b} \leq \frac{v(1-v)}{\tau(1-\tau)} \qquad (a < b,\ 0 < v \leq \tau < 1). \tag{1.2}$$

Inequalities of Alzer–Fonseca–Kovačec (AFK) type arise as special cases of this framework.

Question. *What changes if we replace the convex class by the log-convex class?*

This is not a cosmetic substitution: it switches the geometry from *additive* to *multiplicative*. While convexity governs linear interpolation and additive gaps $\Phi(x) - \Phi(y)$, log-convexity is naturally expressed on the logarithmic scale through ratios

$$\log \frac{\varphi(x)}{\varphi(y)} = \big(\log \circ \varphi\big)(x) - \big(\log \circ \varphi\big)(y),$$

and underlies many special functions and information-theoretic quantities (e.g. the functions Γ, t^t, entropy-like terms $t \log t$).

More recently, Huy established sharp inequalities between mean quantities in a broader convexity framework (covering in particular both convex increasing and convex decreasing functions); see [5, Theorem 3.1]. Our results pursue a different direction by focusing on strictly increasing *log-convex* functions and by developing operator extensions via pinching/envelope techniques.

In this paper, we develop analogues of (1.1)–(1.2) for a strictly increasing, log-convex φ, setting $g := \log \circ \varphi$, and give related applications. A brief summary of our main contributions follows; full details are given in Sections 2 and 3:

- *Scalar inequality (multiplicative AFK).* We prove a base inequality comparing

  $$\log \frac{\varphi(a \nabla_v b)}{\varphi(a \#_v b)} \quad \text{and} \quad \log \frac{\varphi(a \nabla_\tau b)}{\varphi(a \#_\tau b)},$$

  via the monotonicity of log-secants of g and the AM–GM normalization (1.2) (with the correct direction as in Remark 1). See Theorem 2.1.
- *A derivative-weighted estimate.* Writing $r := \varphi'/\varphi = (\log \varphi)'$ and using the standard bounds for monotone derivatives, we obtain an upper bound featuring a local factor

  $$\frac{\sup_{[g_v, x_v]} r}{\inf_{[g_\tau, x_\tau]} r}.$$

  We also identify parameter regimes (e.g. $x_v \leq g_\tau$) in which this local factor is ≤ 1 and hence truly improves the universal constant. See Theorem 2.2 and Lemma 2.1.

- *Concrete applications.* We specialize to $\varphi(t) = t^t$ (so $g(t) = t \log t$, $r(t) = 1 + \log t$) and to $\varphi(t) = \Gamma(t + c)$ (so $r(t) = \psi(t + c)$), obtaining explicit constants and domains. See Corollaries 3.1 and 3.2.
- *Commuting matrix extensions.* In the commuting case, simultaneous diagonalization reduces the problem entrywise to the scalar result and yields unitarily invariant norm inequalities with the universal constant $\frac{v(1-v)}{\tau(1-\tau)}$. See Theorem 3.3.
- *Non-commuting operators via spectral pinching.* For $A, B \in \mathbb{P}_n$ that are comparable in the Löwner order (in particular, after swapping if necessary so that $A \leq B$), we pinch to a finite-dimensional abelian subalgebra generated by a finite orthogonal partition of unity and obtain "envelope" bounds controlled by the same universal constant; see Definition 2 and Theorem 3.6. Finally, we derive a Heinz-centered quantitative estimate in the pinched abelian setting, combining a Lipschitz control for $g = \log \circ \varphi$ with a uniform norm-sandwich for the Heinz mean under unitarily invariant norms (Theorem 3.7); see [2, Ch. 4] (see also [6]) for Kubo–Ando/Heinz means and [2, Ch. 5, Thm. 5.4.1] for the Heinz norm inequality.

2 Scalar log-convex AFK-type inequalities and refinements

Definition 1 (see [8])**.** A function $\varphi : (0, \infty) \to (0, \infty)$ is *strictly log-convex* on an interval J if $g := \log \circ \varphi$ is strictly convex on J.

To streamline the exposition, we record three standard ingredients that will be used repeatedly.

Remark 2 (Weighted AM–GM and monotonicity of weighted means)**.** For $a, b > 0$ and $v \in (0, 1)$ set $x_v := a \nabla_v b$ and $g_v := a \#_v b$. Then $g_v \leq x_v$ (weighted AM–GM). Moreover, if $a < b$ then both $v \mapsto x_v$ and $v \mapsto g_v$ are strictly increasing and

$$a = g_0 = x_0 < g_v < x_v < g_1 = x_1 = b.$$

If $a > b$ the maps are strictly decreasing and the inequalities reverse.

Remark 3 (Monotonicity of secant slopes for convex functions)**.** If f is convex on an interval and $x < y < z$, then

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Equivalently, for $a_1 < a_2 \leq b_2$ and $a_1 \leq b_1 < b_2$ in the interval,

$$\frac{f(a_2) - f(a_1)}{a_2 - a_1} \leq \frac{f(b_2) - f(b_1)}{b_2 - b_1}.$$

This is classical; see, e.g., [8, Sec. 1.3, Thm. 1.3.1].

Remark 4 (Differential inequality for C^1 convex functions)**.** If $f \in C^1([m, M])$ is convex, then for $m \leq x \leq y \leq M$,

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x).$$

In particular, if $\varphi > 0$ and $r := (\log \varphi)'$ is nondecreasing on $[m, M]$, then $g = \log \circ \varphi$ is convex on $[m, M]$ and the above inequality applies with $f = g$.

Remark 5 (No fixed order between g_v and x_τ)**.** In general there is no fixed ordering between $g_v = a^{1-v} b^v$ and $x_\tau = (1 - \tau)a + \tau b$. For example, with $(a, b, \tau, v) = (1, 4, 0.2, 0.8)$ one has $g_v \approx 3.03 > x_\tau = 1.6$, while with $(1, 4, 0.2, 0.3)$ one has $g_v \approx 1.516 < x_\tau = 1.6$.

2.1 Scalar multiplicative AFK inequality

Theorem 2.1 (Multiplicative AFK for log-convex φ)**.** *Let $a, b > 0$ and let φ be strictly log-convex and strictly increasing on an interval containing*

$$\{a\#_\alpha b,\ a\nabla_\alpha b:\ \alpha \in \{\tau, v\}\}.$$

Then, for $0 < v, \tau < 1$:

$$\text{if } a < b \text{ and } v \le \tau: \qquad \frac{\log\big(\varphi(a\nabla_v b)/\varphi(a\#_v b)\big)}{\log\big(\varphi(a\nabla_\tau b)/\varphi(a\#_\tau b)\big)} \le \frac{v(1-v)}{\tau(1-\tau)}, \tag{2.1}$$

$$\text{if } a < b \text{ and } v \ge \tau: \qquad \frac{\log\big(\varphi(a\nabla_v b)/\varphi(a\#_v b)\big)}{\log\big(\varphi(a\nabla_\tau b)/\varphi(a\#_\tau b)\big)} \ge \frac{v(1-v)}{\tau(1-\tau)}. \tag{2.2}$$

For $a > b$ the inequalities reverse in (2.1)–(2.2).

Proof. Let $f := \log\circ\varphi$. Since φ is strictly log-convex and strictly increasing, f is strictly convex and increasing on the relevant interval.

Case $a < b$, $v \le \tau$. By Remark 2, we have

$$g_v < x_v, \quad g_\tau < x_\tau, \quad g_v \le g_\tau, \quad x_v \le x_\tau.$$

Apply the secant-slope monotonicity in Remark 3 with

$$a_1 := g_v,\ a_2 := x_v, \qquad b_1 := g_\tau,\ b_2 := x_\tau,$$

to obtain

$$\frac{f(x_v) - f(g_v)}{x_v - g_v} \le \frac{f(x_\tau) - f(g_\tau)}{x_\tau - g_\tau}.$$

Equivalently,

$$\frac{\log(\varphi(x_v)/\varphi(g_v))}{\log(\varphi(x_\tau)/\varphi(g_\tau))} \le \frac{x_v - g_v}{x_\tau - g_\tau}.$$

Finally, apply Remark 1 in the regime $a < b$ and $v \le \tau$ to obtain

$$\frac{x_v - g_v}{x_\tau - g_\tau} \le \frac{v(1-v)}{\tau(1-\tau)}.$$

Combining this with the previous display yields (2.1).

Case $a < b$, $v \ge \tau$. Now $g_\tau \le g_v$ and $x_\tau \le x_v$ by Remark 2. Apply Remark 3 with

$$a_1 := g_\tau,\ a_2 := x_\tau, \qquad b_1 := g_v,\ b_2 := x_v,$$

to obtain

$$\frac{f(x_\tau) - f(g_\tau)}{x_\tau - g_\tau} \le \frac{f(x_v) - f(g_v)}{x_v - g_v},$$

i.e.

$$\frac{\log(\varphi(x_v)/\varphi(g_v))}{\log(\varphi(x_\tau)/\varphi(g_\tau))} \ge \frac{x_v - g_v}{x_\tau - g_\tau}.$$

Finally, apply Remark 1 in the regime $a < b$ and $v \ge \tau$ (hence the Young-gap ratio inequality is reversed) to obtain

$$\frac{x_v - g_v}{x_\tau - g_\tau} \ge \frac{v(1-v)}{\tau(1-\tau)}.$$

Combining this with the previous display yields (2.2).

Case $a > b$. Swap (a, b) and replace v by $1 - v$ and τ by $1 - \tau$ to reduce to the case $a < b$, which yields the stated reversals. □

2.2 A derivative-weighted estimate via $r = \varphi'/\varphi$

Let $x_\alpha = a\nabla_\alpha b$ and $g_\alpha = a\#_\alpha b$.

Theorem 2.2 (A derivative-weighted bound)**.** *Assume $0 < a < b$ and $\varphi \in C^1([a,b])$ is nondecreasing and log-convex on $[a,b]$. Set*

$$r(t) := (\log\varphi)'(t) = \frac{\varphi'(t)}{\varphi(t)},$$

which is nonnegative and nondecreasing on $[a,b]$. Fix $0 < v \le \tau < 1$ and assume

$$\inf_{t\in[g_\tau,x_\tau]} r(t) > 0 \quad \textit{(equivalently, } r(g_\tau) > 0\textit{)}. \tag{2.3}$$

Then

$$\frac{\log\big(\varphi(x_v)/\varphi(g_v)\big)}{\log\big(\varphi(x_\tau)/\varphi(g_\tau)\big)} \le \frac{\sup\limits_{t\in[g_v,x_v]} r(t)}{\inf\limits_{t\in[g_\tau,x_\tau]} r(t)} \cdot \frac{x_v - g_v}{x_\tau - g_\tau} \le \frac{r(b)}{r(g_\tau)} \cdot \frac{v(1-v)}{\tau(1-\tau)}. \tag{2.4}$$

Proof. Set $g := \log\circ\varphi$ and $r := g' = (\log\varphi)' = \varphi'/\varphi$, which is nondecreasing on $[a,b]$. By Remark 4 (applied to the convex g), for $g_v \le x_v$ and $g_\tau \le x_\tau$ we have

$$g(x_v) - g(g_v) \le \Big(\sup_{t\in[g_v,x_v]} r(t)\Big)(x_v - g_v), \qquad g(x_\tau) - g(g_\tau) \ge \Big(\inf_{t\in[g_\tau,x_\tau]} r(t)\Big)(x_\tau - g_\tau).$$

Since $g(x) - g(y) = \log(\varphi(x)/\varphi(y))$, dividing these inequalities gives the first bound in (2.4).

Because r is nondecreasing and $x_v \le b$, we have $\sup_{[g_v,x_v]} r \le r(b)$. Also, since r is nondecreasing on $[a,b]$, we have

$$\inf_{[g_\tau,x_\tau]} r = r(g_\tau),$$

and the assumption (2.3) ensures $r(g_\tau) > 0$. Therefore,

$$\frac{\sup_{[g_v,x_v]} r}{\inf_{[g_\tau,x_\tau]} r} \le \frac{r(b)}{r(g_\tau)}.$$

Combining this with (1.2), namely $\frac{x_v-g_v}{x_\tau-g_\tau} \le \frac{v(1-v)}{\tau(1-\tau)}$ for $a < b$ and $v \le \tau$, yields the second inequality in (2.4). ☐

Lemma 2.1 (When the local factor can be ≤ 1)**.** *Let $0 < a < b$ and $0 < v \le \tau < 1$, and set $t := b/a > 1$. Then*

$$x_v \le g_\tau \iff (1-v) + v\,t \le t^\tau.$$

Equivalently, there exists a unique $t_* = t_*(v,\tau) \ge 1$ solving $t^\tau = (1-v) + vt$ (and $t_* = 1$ if $v = \tau$), and $x_v \le g_\tau$ holds precisely when $1 \le t \le t_*$. In this regime, if r is nondecreasing, then

$$\sup_{[g_v,x_v]} r \le \inf_{[g_\tau,x_\tau]} r,$$

so the derivative multiplier in (2.4) is ≤ 1.

Proof. Write $b = at$ with $t > 1$. Then $x_v = a[(1-v) + vt]$ and $g_\tau = a\,t^\tau$. Since $a > 0$, the first equivalence follows.

For the existence and uniqueness of t_*, consider $h(t) = t^\tau - (1-v) - vt$ on $[1, \infty)$. One has $h(1) = 0$, $h'(1) = \tau - v \geq 0$, and $h''(t) = \tau(\tau-1)t^{\tau-2} < 0$, so h is concave. Moreover, since $\tau < 1$ we have $\lim_{t\to\infty} h(t) = -\infty$.

If $\tau = v$, then $h'(1) = 0$ and concavity implies $h(t) \leq 0$ for all $t \geq 1$; hence the unique solution in $[1, \infty)$ is $t_* = 1$.

If $\tau > v$, then $h'(1) > 0$, so $h(t) > 0$ for $t > 1$ close to 1, while $h(t) \to -\infty$ as $t \to \infty$. By continuity there exists $t_* > 1$ with $h(t_*) = 0$. Concavity rules out more than one zero in $(1, \infty)$, so this t_* is unique. Since h is concave and $h(1) = h(t_*) = 0$, we also have $h(t) \geq 0$ for all $t \in [1, t_*]$.

Finally, when $x_v \leq g_\tau$, we have

$$g_v \leq x_v \leq g_\tau \leq x_\tau,$$

because $a\#_\alpha b = at^\alpha$ and $a\nabla_\alpha b = a[(1-\alpha) + \alpha t]$ are increasing in α for $t > 1$, and $a\#_\alpha b \leq a\nabla_\alpha b$ for $\alpha \in (0,1)$. Hence $[g_v, x_v] \subseteq (-\infty, g_\tau]$ and $[g_\tau, x_\tau] \subseteq [g_\tau, \infty)$, so if r is nondecreasing,

$$\sup_{[g_v,x_v]} r \leq r(g_\tau) \leq \inf_{[g_\tau,x_\tau]} r.$$

This gives the stated bound on the derivative multiplier in (2.4). ☐

2.3 Examples with numbers (illustrating the derivative factor)

We present concrete computations that quantify the possible “gain” brought by the derivative factor in (2.4). Note that, in general, the multiplier $\big(\sup_{[g_v,x_v]} r\big)/\big(\inf_{[g_\tau,x_\tau]} r\big)$ need not be ≤ 1; it is ≤ 1 in the regime $x_v \leq g_\tau$, characterized in Lemma 2.1.

Throughout, let

$$R := \frac{\log\big(\varphi(x_v)/\varphi(g_v)\big)}{\log\big(\varphi(x_\tau)/\varphi(g_\tau)\big)}, \quad B_{\text{add}} := \frac{x_v - g_v}{x_\tau - g_\tau}, \qquad B_{\text{PH}} := \frac{v(1-v)}{\tau(1-\tau)},$$

and

$$B_{\text{der}} := \frac{\sup_{t\in[g_v,x_v]} r(t)}{\inf_{t\in[g_\tau,x_\tau]} r(t)} \cdot B_{\text{add}}.$$

Example A: $\varphi(t) = t^t$

Take $a = 1$, $b = 2$, $v = 0.3$, $\tau = 0.7$ (so $v \leq \tau$ and indeed $x_v \leq g_\tau$). Then

$$\begin{aligned} &x_v = 1.300000, \quad g_v = 1.231144, \quad x_\tau = 1.700000, \quad g_\tau = 1.624505, \\ &R = \frac{x_v \log x_v - g_v \log g_v}{x_\tau \log x_\tau - g_\tau \log g_\tau} = 0.747139, \\ &B_{\text{add}} = \frac{x_v - g_v}{x_\tau - g_\tau} = 0.912052, \quad B_{\text{PH}} = \frac{v(1-v)}{\tau(1-\tau)} = 1.000000. \end{aligned}$$

For t^t we have $r(t) = 1 + \log t$. On the two intervals,

$$\sup_{[g_v,x_v]} r = 1 + \log x_v = 1.262364, \qquad \inf_{[g_\tau,x_\tau]} r = 1 + \log g_\tau = 1.485203,$$

hence $\sup / \inf = 0.849961$ and

$$B_{\text{der}} = (0.849961) \cdot B_{\text{add}} = 0.775209.$$

We obtain the chain

$$R = 0.747139 \ \le\ B_{\text{der}} = 0.775209 \ \le\ B_{\text{add}} = 0.912052 \ \le\ B_{\text{PH}} = 1.000000.$$

Example B: $\varphi(t) = t^t$ with closer endpoints

Take $a = 1.2$, $b = 1.8$, $v = 0.3$, $\tau = 0.7$. Then

$$\begin{aligned} &x_v = 1.380000,\ g_v = 1.355216, \quad x_\tau = 1.620000,\ g_\tau = 1.593841,\\ &R = 0.843811, \quad B_{\text{add}} = 0.947442, \quad B_{\text{PH}} = 1.000000,\\ &\sup_{[g_v, x_v]} r = 1 + \log x_v = 1.322083, \quad \inf_{[g_\tau, x_\tau]} r = 1 + \log g_\tau = 1.466147,\\ &B_{\text{der}} = 0.854346. \end{aligned}$$

Example C: $\varphi(t) = \Gamma(t + c)$, $c = 2$

Take $a = 1$, $b = 2$, $v = 0.3$, $\tau = 0.7$. Then

$$\begin{aligned} &x_v = 1.300000,\ g_v = 1.231144, \quad x_\tau = 1.700000,\ g_\tau = 1.624505,\\ &R = \frac{\log\Gamma(x_v + 2) - \log\Gamma(g_v + 2)}{\log\Gamma(x_\tau + 2) - \log\Gamma(g_\tau + 2)} = 0.807214,\\ &B_{\text{add}} = 0.912052, \quad B_{\text{PH}} = 1.000000. \end{aligned}$$

Here $r(t) = \psi(t + 2)$ (digamma), which is increasing. Moreover $x_v \le g_\tau$ in this example, so $\sup / \inf \le 1$.

3 Applications

3.1 Scalar applications: the functions t^t and Γ

Throughout this subsection we keep the notation

$$x_\alpha := a\nabla_\alpha b, \quad g_\alpha := a\#_\alpha b \quad (\alpha \in \{\tau, v\}),$$

and we apply Theorems 2.1 and 2.2 with the specific choices of φ listed below.

Corollary 3.1. *Let $e^{-1} < a \le b$ and $0 < v \le \tau < 1$. Then*

$$\frac{(x_v \log x_v) - (g_v \log g_v)}{(x_\tau \log x_\tau) - (g_\tau \log g_\tau)} \le \frac{v(1-v)}{\tau(1-\tau)}. \tag{3.1}$$

$$\frac{(x_v \log x_v) - (g_v \log g_v)}{(x_\tau \log x_\tau) - (g_\tau \log g_\tau)} \le \frac{1 + \log x_v}{1 + \log g_\tau} \cdot \frac{v(1-v)}{\tau(1-\tau)} \le \frac{1 + \log b}{1 + \log a} \cdot \frac{v(1-v)}{\tau(1-\tau)}. \tag{3.2}$$

Proof. Let $\varphi(t) = t^t$. Then $\log\varphi(t) = t\log t$ is strictly convex on $(0,\infty)$, so φ is strictly log-convex. On (e^{-1},∞) one has $(\log\varphi)'(t) = 1+\log t > 0$, hence φ is strictly increasing there. Thus Theorem 2.1 applies and gives (3.1) since

$$\log\frac{\varphi(x)}{\varphi(y)} = \log\frac{x^x}{y^y} = x\log x - y\log y.$$

For (3.2), apply Theorem 2.2 with $r(t) = 1+\log t$, noting that $1+\log g_\tau \geq 1+\log a > 0$. $\square$

Corollary 3.2. *Fix $c \geq 2$ and let $0 < a \leq b$ and $0 < v \leq \tau < 1$. Then*

$$\frac{\log\bigl(\Gamma(x_v+c)/\Gamma(g_v+c)\bigr)}{\log\bigl(\Gamma(x_\tau+c)/\Gamma(g_\tau+c)\bigr)} \leq \frac{v(1-v)}{\tau(1-\tau)}. \tag{3.3}$$

$$\frac{\log\bigl(\Gamma(x_v+c)/\Gamma(g_v+c)\bigr)}{\log\bigl(\Gamma(x_\tau+c)/\Gamma(g_\tau+c)\bigr)} \leq \frac{\psi(b+c)}{\psi(a+c)}\cdot\frac{v(1-v)}{\tau(1-\tau)}. \tag{3.4}$$

Proof. Set $\varphi(t) = \Gamma(t+c)$. By the Bohr–Mollerup theorem (see also [7]), Γ is log-convex on $(0,\infty)$, hence so is φ. Its logarithmic derivative is $r(t) = (\log\Gamma)'(t+c) = \psi(t+c)$, and ψ is strictly increasing on $(0,\infty)$. For $c \geq 2$ one has $\psi(t+c) > 0$ for all $t > 0$, so φ is strictly increasing on $(0,\infty)$. Thus Theorem 2.1 applies and yields (3.3). The derivative-weighted estimate (3.4) follows from Theorem 2.2. $\square$

3.2 Matrix applications (commuting and noncommuting)

We now lift the scalar inequalities to the matrix setting. Write $\mathbb{P}_n$ for the cone of $n\times n$ Hermitian positive definite matrices.

Geometric mean notation (Kubo–Ando)

For $A,B \in \mathbb{P}_n$ and $\alpha \in [0,1]$, we use the weighted arithmetic mean

$$A\nabla_\alpha B := (1-\alpha)A + \alpha B,$$

and the weighted geometric mean (Kubo–Ando) [2, Ch. 4]

$$A\#_\alpha B := A^{1/2}\bigl(A^{-1/2}BA^{-1/2}\bigr)^\alpha A^{1/2}.$$

If A and B commute, then $A\#_\alpha B = A^{1-\alpha}B^\alpha$.

Commuting case

Let $A,B \in \mathbb{P}_n$ commute and assume, in the common eigenbasis, that $0 < mI \leq A \leq B \leq MI$ for some $0 < m < M$. Let φ be strictly increasing and strictly log-convex on $[m,M]$, and set

$$X_\alpha := A\nabla_\alpha B, \qquad G_\alpha := A\#_\alpha B, \qquad D_\alpha := \log\varphi(X_\alpha) - \log\varphi(G_\alpha).$$

Theorem 3.3 (Commuting matrices: unitarily invariant norms)**.** *Let $A,B \in \mathbb{P}_n$ commute and satisfy $0 < mI \leq A \leq B \leq MI$. Let φ be strictly increasing and strictly log-convex on $[m,M]$. Fix $0 < v \leq \tau < 1$ and set $C_0 := \frac{v(1-v)}{\tau(1-\tau)}$. Then:*

(i) In a common diagonalizing basis, D_v and D_τ are diagonal with nonnegative diagonal entries and satisfy $(D_v)_{ii} \le C_0\,(D_\tau)_{ii}$ for all i.

(ii) For every unitarily invariant norm $|||\cdot|||$, one has $|||D_v||| \le C_0\,|||D_\tau|||$.

(iii) For the trace, $\operatorname{Tr} D_v \le C_0 \operatorname{Tr} D_\tau$.

Proof. Since A, B commute, there exists a unitary U and scalars $a_i, b_i \in [m, M]$ with $a_i \le b_i$ such that $A = U\mathrm{diag}(a_i)U^*$ and $B = U\mathrm{diag}(b_i)U^*$. Hence

$$X_\alpha = U\mathrm{diag}(x_\alpha^{(i)})U^*, \quad x_\alpha^{(i)} = (1-\alpha)a_i + \alpha b_i, \qquad G_\alpha = U\mathrm{diag}(g_\alpha^{(i)})U^*, \quad g_\alpha^{(i)} = a_i^{1-\alpha} b_i^\alpha,$$

and therefore $D_\alpha = U\,\mathrm{diag}(d_\alpha^{(i)})\,U^*$ with

$$d_\alpha^{(i)} = \log\varphi(x_\alpha^{(i)}) - \log\varphi(g_\alpha^{(i)}).$$

Since $a_i \le b_i$, the scalar Theorem 2.1 in the regime $v \le \tau$ gives $d_v^{(i)} \le C_0\, d_\tau^{(i)}$ for every i, and $d_\alpha^{(i)} \ge 0$. This proves (i). For (ii) and (iii), use that unitarily invariant norms are symmetric gauge functions of singular values (see, e.g., [1, 4]) and sum the diagonal entries for the trace. ☐

Two concrete commuting corollaries follow.

Corollary 3.4 (Commuting t^t)**.** *Assume $A, B \in \mathbb{P}_n$ commute and satisfy $e^{-1}I < mI \le A \le B \le MI$. For $0 < v \le \tau < 1$ and any unitarily invariant norm,*

$$|||X_v \log X_v - G_v \log G_v\,||| \;\le\; \frac{v(1-v)}{\tau(1-\tau)}\,|||X_\tau \log X_\tau - G_\tau \log G_\tau\,|||.$$

Proof. Take $\varphi(t) = t^t$ on (e^{-1}, ∞) so that $\log\varphi(t) = t\log t$. Then $D_\alpha = X_\alpha \log X_\alpha - G_\alpha \log G_\alpha$ by functional calculus. Apply Theorem 3.3. ☐

Corollary 3.5 (Commuting Γ)**.** *Fix $c \ge 2$ and assume $A, B \in \mathbb{P}_n$ commute and satisfy $0 < mI \le A \le B \le MI$. Then for any unitarily invariant norm,*

$$|||\log\Gamma(X_v + c) - \log\Gamma(G_v + c)||| \;\le\; \frac{v(1-v)}{\tau(1-\tau)}\,|||\log\Gamma(X_\tau + c) - \log\Gamma(G_\tau + c)|||.$$

Proof. Take $\varphi(t) = \Gamma(t+c)$ (strictly increasing and log-convex for $c \ge 2$) and apply Theorem 3.3. ☐

Noncommuting case: spectral pinching and majorization

We recall the pinching map onto a finite-dimensional abelian C^*-subalgebra; see [1].

Definition 2 (Pinching / conditional expectation)**.** Let $(P_i)_{i=1}^n$ be mutually orthogonal *rank-one* projections on $\mathbb{C}^n$ with $\sum_{i=1}^n P_i = I$. Define the pinching map

$$\mathcal{E}(X) := \sum_{i=1}^n P_i X P_i, \qquad X \in M_n(\mathbb{C}).$$

Then $\mathcal{E}$ is a trace-preserving conditional expectation onto the abelian C^*-subalgebra $\mathcal{A} := \mathrm{span}\{P_1, \ldots, P_n\}$.

Remark 6. If the projections P_i are not rank-one, then $\operatorname{Ran}(\mathcal{E}) = \bigoplus_i P_i M_n(\mathbb{C}) P_i$ is generally non-abelian; the present paper only uses rank-one pinchings so that $\operatorname{Ran}(\mathcal{E})$ is abelian.

Lemma 3.1 (Pinching inequality)**.** *For every unitarily invariant norm $|||\cdot|||$ on $M_n(\mathbb{C})$ and every X,*

$$|||\,\mathcal{E}(X)\,||| \le |||X|||.$$

Proof. This is the pinching inequality; see [1, Sec. IV.2, Eq. (IV.52)]. □

Let $A, B \in \mathbb{P}_n$ satisfy $0 < mI \le A \le B \le MI$. For $\alpha \in \{v, \tau\}$ set

$$X_\alpha := A\nabla_\alpha B, \qquad G_\alpha := A\#_\alpha B, \qquad D_\alpha := \log\varphi(X_\alpha) - \log\varphi(G_\alpha).$$

Given a pinching map $\mathcal{E}$ with range $\mathcal{A}$ as in Definition 2, set

$$\widetilde{A} := \mathcal{E}(A), \qquad \widetilde{B} := \mathcal{E}(B),$$

and for $\alpha \in \{v, \tau\}$ define

$$\widetilde{X}_\alpha := \mathcal{E}(X_\alpha) = \widetilde{A}\nabla_\alpha\widetilde{B}, \qquad \widetilde{G}_\alpha := \widetilde{A}\#_\alpha\widetilde{B}, \qquad \Delta_\alpha^{\mathcal{A}} := |||\,\log\varphi(\widetilde{X}_\alpha) - \log\varphi(\widetilde{G}_\alpha)\,|||.$$

We then define the envelope seminorm

$$\Delta_\alpha^{\mathrm{env}} := \sup_{\mathcal{A}}\ \Delta_\alpha^{\mathcal{A}},$$

where the supremum is taken over all finite-dimensional abelian subalgebras $\mathcal{A}$ arising from orthogonal rank-one partitions of unity (pinchings); see [1, 2]. See also [3, Sec. 2.4] for background on conditional expectations onto maximal abelian *-subalgebras of $M_n(\mathbb{C})$ (e.g. diagonal algebras) and the associated rank-one pinchings.

Remark 7 (Order of the pinched gap)**.** Assume that $g = \log\circ\varphi$ is nondecreasing on $[m, M]$ and let $\mathcal{E}$ be a rank-one pinching map with range $\mathcal{A}$. For $\alpha \in (0, 1)$ set

$$\widetilde{A} := \mathcal{E}(A), \qquad \widetilde{B} := \mathcal{E}(B), \qquad \widetilde{X}_\alpha := \widetilde{A}\nabla_\alpha\widetilde{B}, \qquad \widetilde{G}_\alpha := \widetilde{A}\#_\alpha\widetilde{B}.$$

Since $\widetilde{A}, \widetilde{B} \in \mathcal{A}$, they commute. Hence, in $\mathcal{A}$ we have the (weighted) AM–GM inequality

$$\widetilde{G}_\alpha = \widetilde{A}\#_\alpha\widetilde{B} = \widetilde{A}^{1-\alpha}\widetilde{B}^{\alpha} \le (1-\alpha)\widetilde{A} + \alpha\widetilde{B} = \widetilde{X}_\alpha,$$

so by functional calculus and the monotonicity of g,

$$0 \le g(\widetilde{X}_\alpha) - g(\widetilde{G}_\alpha) \qquad (\text{Löwner order in } \mathcal{A}).$$

Theorem 3.6 (Envelope control (noncommuting via commuting compression))**.** *Let $A, B \in \mathbb{P}_n$ with $0 < mI \le A \le B \le MI$, let $0 < v \le \tau < 1$, and let φ be strictly increasing and strictly log-convex on $[m, M]$. Set $C_0 := \frac{v(1-v)}{\tau(1-\tau)}$. Then for every unitarily invariant norm,*

$$\Delta_v^{\mathrm{env}} \ \le\ C_0\,\Delta_\tau^{\mathrm{env}}.$$

Proof. Fix a pinching map $\mathcal{E}$ with range $\mathcal{A}$. Set $\widetilde{A} := \mathcal{E}(A)$ and $\widetilde{B} := \mathcal{E}(B)$. Then $\widetilde{A}, \widetilde{B} \in \mathcal{A}$ hence commute, and since $\mathcal{E}$ is positive we have $0 < mI \le \widetilde{A} \le \widetilde{B} \le MI$.

For $\alpha \in \{v, \tau\}$, note that

$$\widetilde{X}_\alpha = \mathcal{E}(A\nabla_\alpha B) = \widetilde{A}\nabla_\alpha \widetilde{B}, \qquad \widetilde{G}_\alpha = \widetilde{A}\#_\alpha \widetilde{B}$$

by linearity of $\mathcal{E}$ and by definition of $\widetilde{G}_\alpha$. Applying the commuting Theorem 3.3 to the commuting pair $(\widetilde{A}, \widetilde{B})$ yields

$$\Delta_v^{\mathcal{A}} = |||\, \log\varphi(\widetilde{X}_v) - \log\varphi(\widetilde{G}_v)\, ||| \le C_0\, |||\, \log\varphi(\widetilde{X}_\tau) - \log\varphi(\widetilde{G}_\tau)\, ||| = C_0\, \Delta_\tau^{\mathcal{A}}.$$

Taking the supremum over all $\mathcal{A}$ gives $\Delta_v^{\mathrm{env}} \le C_0\, \Delta_\tau^{\mathrm{env}}$. □

Remark 8 (On operator convexity of $g = \log\circ\varphi$)**.** In this paper, the envelope comparison $\Delta_v^{\mathrm{env}} \le C_0\, \Delta_\tau^{\mathrm{env}}$ follows from commuting compression (Theorem 3.6) under the scalar hypotheses (strictly increasing, strictly log-convex on $[m, M]$). If one additionally assumes that g is operator convex on $[m, M]$, then the operator Jensen inequality yields the stronger comparison $g(\mathcal{E}(Z)) \le \mathcal{E}(g(Z))$, which in turn implies bounds of the form $\Delta_\alpha^{\mathcal{A}} \le |||\, \mathcal{E}(D_\alpha)\, ||| \le |||D_\alpha|||$. This operator-convexity assumption holds, for instance, for $g(t) = t\log t$ (hence $\varphi(t) = t^t$), but it should not be presumed for $g(t) = \log\Gamma(t + c)$ on large intervals.

Lemma 3.2 (Lipschitz bound in the abelian setting)**.** *Let $\varphi : (0, \infty) \to (0, \infty)$ be C^1 on $[m, M]$, set $g := \log\circ\varphi$ and $r := g' = \varphi'/\varphi$, and write $L := \sup_{t\in[m,M]} r(t) < \infty$. If $\widetilde{X}, \widetilde{Y}$ are commuting self-adjoint matrices with spectra contained in $[m, M]$, then for every unitarily invariant norm,*

$$|||\, g(\widetilde{X}) - g(\widetilde{Y})\, ||| \;\le\; L\, |||\, \widetilde{X} - \widetilde{Y}\, |||.$$

Proof. Since $\widetilde{X}$ and $\widetilde{Y}$ commute and are self-adjoint, there exists a unitary U such that $U^*\widetilde{X}U = \mathrm{diag}(x_1, \ldots, x_n)$ and $U^*\widetilde{Y}U = \mathrm{diag}(y_1, \ldots, y_n)$ with $x_i, y_i \in [m, M]$. Hence

$$U^*\big(g(\widetilde{X}) - g(\widetilde{Y})\big)U = \mathrm{diag}\big(g(x_1) - g(y_1), \ldots, g(x_n) - g(y_n)\big),$$

and similarly $U^*(\widetilde{X} - \widetilde{Y})U = \mathrm{diag}(x_1 - y_1, \ldots, x_n - y_n)$. By the scalar mean value theorem and $L = \sup_{[m,M]} |g'| = \sup_{[m,M]} r$, we have $|g(x_i) - g(y_i)| \le L|x_i - y_i|$ for each i. Therefore, for every k, the Ky Fan k-norms satisfy

$$\big\|g(\widetilde{X}) - g(\widetilde{Y})\big\|_{(k)} \le L\, \|\widetilde{X} - \widetilde{Y}\|_{(k)}.$$

The Fan dominance theorem [1, Thm. IV.2.2] then yields

$$|||\, g(\widetilde{X}) - g(\widetilde{Y})\, ||| \le L\, |||\, \widetilde{X} - \widetilde{Y}\, |||$$

for every unitarily invariant norm. □

Theorem 3.7 (Heinz-centered sandwich bound)**.** *Let $A, B \in \mathbb{P}_n$ satisfy $0 < mI \le A, B \le MI$, and let $\alpha \in [0, 1]$. Set*

$$X := A\nabla_{1/2}B = \frac{A + B}{2}, \qquad G := A\#_{1/2}B, \qquad H_\alpha := \frac{A\#_\alpha B + A\#_{1-\alpha}B}{2}.$$

Let $\varphi : (0, \infty) \to (0, \infty)$ be C^1 on $[m, M]$ and assume $g := \log\varphi$ is increasing on $[m, M]$. Write

$$L := \sup_{t\in[m,M]} g'(t) = \sup_{t\in[m,M]} \frac{\varphi'(t)}{\varphi(t)} < \infty.$$

For any rank-one pinching map $\mathcal{E}$ *with range* $\mathcal{A}$*, define* $\widetilde{Z} := \mathcal{E}(Z)$ *for* $Z \in \{A, B, X, G, H_\alpha\}$*, and set*

$$\Delta_\alpha^{\mathcal{A}} := |||\, g(\widetilde{X}) - g(\widetilde{H}_\alpha)\, |||.$$

Then, for every unitarily invariant norm,

$$\Delta_\alpha^{\mathcal{A}} \le L\, |||\, \widetilde{X} - \widetilde{H}_\alpha\, ||| \le L\, |||\, X - H_\alpha\, |||, \tag{3.5}$$

$$0 \le |||\, \widetilde{A} + \widetilde{B}\, ||| - 2|||\, \widetilde{H}_\alpha\, ||| \le |||\, \widetilde{A} + \widetilde{B}\, ||| - 2|||\, \widetilde{G}\, |||, \tag{3.6}$$

$$\frac{1}{2}\Big(|||\, \widetilde{A} + \widetilde{B}\, ||| - 2|||\, \widetilde{H}_\alpha\, |||\Big) \le |||\, \widetilde{X} - \widetilde{H}_\alpha\, |||. \tag{3.7}$$

Consequently,

$$\sup_{\mathcal{A}} \Delta_\alpha^{\mathcal{A}} \;\le\; L\, |||\, X - H_\alpha\, |||. \tag{3.8}$$

Proof. Inside the abelian algebra $\mathcal{A}$ the compressions commute. By Lemma 3.2,

$$\Delta_\alpha^{\mathcal{A}} = |||g(\widetilde{X}) - g(\widetilde{H}_\alpha)||| \le L|||\widetilde{X} - \widetilde{H}_\alpha|||.$$

The pinching map is contractive for unitarily invariant norms, hence

$$|||\widetilde{X} - \widetilde{H}_\alpha||| = |||\mathcal{E}(X - H_\alpha)||| \le |||X - H_\alpha||| \qquad \text{(by Lemma 3.1)},$$

which proves (3.5). Taking $\sup_{\mathcal{A}}$ gives (3.8).

Next, since $\widetilde{A}, \widetilde{B}$ commute in $\mathcal{A}$, we have $\widetilde{H}_\alpha = \frac{1}{2}(\widetilde{A}^{1-\alpha}\widetilde{B}^{\alpha} + \widetilde{A}^{\alpha}\widetilde{B}^{1-\alpha})$, $\widetilde{G} = \widetilde{A}^{1/2}\widetilde{B}^{1/2}$, and $\widetilde{X} = \frac{1}{2}(\widetilde{A}+\widetilde{B})$. Apply Bhatia's Heinz norm inequality ([2, Chapter 5, Theorem 5.4.1, Eq. (5.28)]) with $X = I$ to obtain

$$2|||\widetilde{G}||| \le 2|||\widetilde{H}_\alpha||| \le |||\widetilde{A} + \widetilde{B}|||,$$

which implies (3.6).

Finally, (3.7) follows from the reverse triangle inequality: for $U := \widetilde{A} + \widetilde{B}$ and $V := 2\widetilde{H}_\alpha$,

$$|||U||| - |||V||| \le |||U - V||| \quad \Rightarrow \quad |||\widetilde{A} + \widetilde{B}||| - 2|||\widetilde{H}_\alpha||| \le |||(\widetilde{A} + \widetilde{B}) - 2\widetilde{H}_\alpha||| = 2|||\widetilde{X} - \widetilde{H}_\alpha|||.$$

□

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Đặng Võ Phúc Department of Mathematics, FPT University, Quy Nhon AI Campus, An Phu Thinh New Urban Area, Vietnam. ORCID: `https://orcid.org/0000-0002-6885-3996`

E-mail: dangphuc150488@gmail.com