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Convergence analysis of an efficient Chebyshev wavelet and its applications to differential equations via operational matrices of integration

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Abstract. In this paper, the convergence analysis of the Chebyshev wavelet of the second kind is thoroughly carried out. Operational matrices for integration and product operations of the second kind Chebyshev wavelet are constructed, and these matrices are utilized to obtain solutions to the differential equations. A theorem related to the proposed operational matrix method is established. Solutions of the differential equations considered in this paper resemble their exact solutions. The characteristics of second kind Chebyshev wavelet are utilized to transform differential equations into systems of algebraic equations, which are solved very efficiently using a suitable method.

Keywords. Chebyshev wavelet, convergence analysis, operation matrix of integration, product operation matrix, Lane–Emden type, third order singular differential equation

1 Introduction

Wavelets are compact or short-lived waves. Rather than continuing to oscillate indefinitely, they return to zero. Wavelet allows for the precise representation of various functions. They are considered basis functions $\varphi_{i,j}(t)$ in continuous time. A distinct characteristic of the wavelet basis is that each function $\varphi_{i,j}(t)$ is derived from $\varphi(t)$, which is known as the mother wavelet. Typically, a collection of linearly independent functions is generated through the translation and dilation of the mother wavelet. The theory of wavelet is relatively new and developing in the field of mathematical research. It integrates into a variety of scientific and engineering disciplines.
The second-kind Chebyshev wavelet forms an orthogonal and complete basis in $L^2[-1,1]$, and therefore any square-integrable function can be approximated by a finite number of wavelet coefficients. Owing to the smoothness and polynomial structure inherited from the Chebyshev polynomials of the second kind, the corresponding wavelet expansions exhibit faster convergence when the target function is sufficiently smooth. This wavelet provides a highly accurate and convergent framework for approximating smooth as well as moderately regular functions, making it suitable for operational matrix-based numerical solutions of differential equations.
Applications of the second-kind Chebyshev wavelet span a wide range of technological fields [31],

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especially in signal analysis [28], time-frequency analysis, and efficient algorithms for straightforward implementation [2]. Linear differential equations arise when we model a real-world phenomenon. These models appear widely in science, engineering, economics, medicine, and many other fields; for example motion with resistance, electrical systems, fluid mechanics, quantum mechanics, etc.

In numerous instances, obtaining analytical solutions for linear initial value problems is not feasible. For these situations, we employ numerical methods like the operational matrix of integration (OMI). The operational matrix of integration (OMI) is a highly effective numerical method for solving linear differential equations. This technique is based on converting differential equations into integral equations using the operational matrix of integration by eliminating the integral operator in order to reduce the problem to a system of algebraic equations, which is further solved by a suitable method.

Employing the OMI of Haar wavelet to address differential equations, a limitation arises due to a jump discontinuity at $x = 1/2$. Therefore, exploring new approaches to solve and examine differential equations has become a fascinating topic within the realm of wavelet.

Since, Chebyshev wavelets are very useful wavelet methods, in this study, we develop the operational matrix of integration (OMI) and the product operation matrix (POM) for the second-kind Chebyshev wavelet and use this technique to solve some of the most important linear differential equations. The suggested technique for solving these differential equations utilizes a limited number of bases and takes advantage of the orthogonality of second-kind Chebyshev wavelet to transform the linear differential equations into a straightforward system of algebraic equations. One can find more details on orthogonal functions and polynomials in [3, 9, 11, 19, 29].

Further, in the case of linear differential equations, we consider first-order linear, Lane-Emden type, third order linear and third-order singular differential equations, which are solved using the operational matrix of integration for $k = 3$ and $M = 4$, and the product operational matrix. Linear differential equations have applications in the field of motion with resistance; charging of capacitors in the field of electrical systems, while the Lane-Emden differential equation has applications in astrophysics and theoretical physics. They are primarily used to describe the structure of self-gravitating, spherically symmetric polytropic gas spheres, such as stars and gaseous planets. Third-order linear differential equations have applications in the field of mechanical systems involving motion with damping and higher-order inertia effects; vibration analysis and control systems in engineering, while the third-order singular differential equations have applications in models of physical systems with time-dependent or spatially varying parameters, such as unsteady fluid flow or mechanical oscillations. Their analysis provides insights into variable-coefficient dynamical systems frequently encountered in engineering and applied sciences.

It can be noted that in the recent past, particular emphasis has been placed on the use of Legendre wavelet [11, 10, 15, 30] and hybrid functions [14, 13, 16, 17]. Some of the most recent work in the context of the present paper can also be found in [22, 24, 25, 26, 27, 21, 23, 4, 11, 30, 1, 5, 6, 7, 8, 23, 32, 33].

The organization of this paper is outlined as follows: section 2 contains key definitions pertinent to the current study. In section 3, convergence of the proposed method is thoroughly analyzed. In section 4, we present a format for operational matrices related to integration and the product operation matrix (POM) for second-kind Chebyshev wavelet. Moreover, we establish a theorem related to the proposed operational matrix method. In section 5, we find solutions to first order linear, Lane–Emden type, third order linear and third order singular differential equations using the proposed method and compare these solutions with their exact solutions. In section 6, a conclusion is given.

2 Preliminaries

2.1 Wavelet and Chebyshev wavelet (CW)

Wavelets are made up of a collection of functions that are produced by shifting and dilating the graph of wavelet. Specifically, a wavelet family with mother wavelet $\varphi(t)$ consists of functions $\varphi_{a,b}(t)$ of the form

$$\varphi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\varphi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R},$$

where a and b are scale and shift parameters respectively. Here, $\varphi_{a,b}(t)$ forms a wavelet basis for $L^2(\mathbb{R})$ provided $a \neq 0$[18]. We consider the parameters a and b as discrete values $a = a_0^{-n}$, $b = mb_0a_0^{-n}, a_0 > 1, b_0 > 0$, where n and m are positive integers. Specifically, when $a_0 = 2$ and $b_0 = 1$, the set $\{\varphi_{n,m}(t)\}$ constitutes an orthonormal basis.

The Chebyshev wavelet denoted as $\varphi_{n,m}(t) = \varphi(k, n, m, t)$ rely on four parameters. Here, n takes the values $1, 2, 3, ..., 2^{k-1}$, where k belongs to positive integer, t is time and $m = 0, 1, ..., M-1$, which is the degree of Chebyshev polynomials.

Now, we define $\psi_{n,m}(t)$ as

$$\varphi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{U}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}; \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where m and n are as defined above and $\tilde{U}_m$ is given as follows:

$$\tilde{U}_m = \sqrt{\frac{2}{\pi}}U_m(t). \tag{2.2}$$

In above definition, orthonormality is preserved by using the factor $\sqrt{\frac{2}{\pi}}$. The Chebyshev polynomials of the second kind with degree m are represented by the polynomials $U_m(t)$. They follow the recursive relation (2.3) and show orthogonality with respect to the weight function $w(t) = \sqrt{1-t^2}$ over the interval [0,1]:

$$U_m(t) = 2tU_m(t) - U_{m-1}(t); \qquad m = 1, 2, ... \tag{2.3}$$

We note that

$$U_0(t) = 1, U_1(t) = 2t, U_2 = 4t^2 - 1, U_3(t) = 8t^3 - 4t, ...$$

It is noted that for Chebyshev wavelets, the weight function $w(t)$ must undergo dilation and translation, given by $w_n(t) = w(2^k t - 2n + 1)$, in order to obtain orthogonal wavelet.

2.2 Approximation of a function using CW

A function $f \in L^2_{\tilde{w}}[0, 1)$, where $\tilde{w}(t) = w(2t - 1)$, can be expanded as the following wavelet series

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(t), \tag{2.4}$$

where $c_{nm} = \langle f(t), \varphi_{nm}\rangle$. $\langle , \rangle$ denotes the inner product with respect to the weight function $w(t)$. After truncating (2.4), we have

$$f(t) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\varphi_{n,m}(t)$$

$$= C^T \varphi(t), \tag{2.5}$$

where

$$C = [c_{1,0},\ c_{1,1}\ c_{1,2}, ..., c_{1,M},\ c_{2,0}, ..., c_{2,M},\ ...c_{2^{k-1},0}, ..., c_{2^{k-1},M-1}]^T \tag{2.6}$$

and

$$\varphi_i(t) = [\varphi_{i,0}, \varphi_{i,1} ..., \varphi_{i,M}, ...\varphi_{i+1,M}, ..., \varphi_{i+2,0}, ..\varphi_{i+3,0} ..., \varphi_{2^{k-1},M-1}]^T. \tag{2.7}$$

In (2.6) and (2.7), the order of the matrices is $2^{k-1}M \times 1$ and T stands for transposition.

3 Convergence analysis

In this section, we establish the following convergence theorems for the Chebyshev wavelet of the second kind:

Theorem 3.1. *Assume that $g(t) \in L^2_w[0,1]$ possesses a bounded first derivative, that is,*

$$|g'(t)| \leq L \qquad \text{for all } t \in [0,1].$$

Suppose further that g admits the second-kind Chebyshev wavelet expansion

$$g(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\, \varphi_{nm}(t).$$

Then, each wavelet coefficient c_{nm} obeys the estimate

$$|c_{nm}| \ \leq\ \frac{L\sqrt{\pi}(m+1)}{2(n+1)m(m+1)}. \tag{3.1}$$

Consequently, the resulting second-kind Chebyshev wavelet series converges uniformly to $g(t)$.

Proof. From the definition of the coefficient c_{nm},

$$\begin{aligned} c_{nm} &= \int_0^1 g(t)\varphi_{nm}(t) w_n(t)\, dt \\ &= \frac{2^{\frac{k+1}{2}}}{\sqrt{\pi}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} g(t) U_m(2^k t - 2n + 1) w_n(t)\, dt. \end{aligned}$$

Employing the substitution $2^k t - 2n + 1 = \cos\theta$, we get

$$c_{nm} = \frac{2^{\frac{1-k}{2}}}{\sqrt{\pi}} \int_0^{\pi} g\left(\frac{\cos\theta + 2n - 1}{2^k}\right) \sin((m+1)\theta) \sin\theta\, d\theta.$$

Using integration by parts, we obtain

$$c_{nm} = \frac{1}{2^{k+1}\sqrt{\pi}} \int_0^{\pi} g'\left(\frac{\cos\theta - 2n + 1}{2^k}\right) d_m(\theta)\ d\theta,$$

where

$$d_m(\theta) = \sin\theta \left(\frac{\sin m\theta}{2m} - \frac{\sin((m+2)\theta)}{2(m+2)}\right).$$

So, we have

$$|c_{nm}| \leq \frac{L}{2^{k+1}\sqrt{\pi}} \int_0^{\pi} |d_m(\theta)|\, d\theta \leq \frac{L\sqrt{\pi}(m+1)}{2^{k+1}m(m+2)}\ .$$

Since $n \leq 2^k - 1$, we obtain

$$\begin{aligned} |c_{nm}| &\leq \frac{L\sqrt{\pi}(m+1)}{2^{k+1}m(m+2)} \\ &\leq \frac{L\sqrt{\pi}(m+1)}{2(n+1)m(m+2)}. \end{aligned}$$

Theorem 3.2. *Let $g \in L^2_w[0,1]$ be continuous, where the associated weight is*

$$w_n(t) = \sqrt{1-t^2}.$$

Assume further that the J^{th} derivative of g is uniformly bounded on $[0,1]$, i.e.,

$$\sup_{t\in[0,1]} |g^{(J)}(t)| < \infty.$$

Under these conditions, the approximation of g by the second-kind Chebyshev wavelet partial sums of order $(2^{k-1}, J)$

$$S_{2^{k-2},J}(t) = \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{J-1} c_{n,m}\, \varphi_{n,m}(t)$$

yields the following error representation:

$$E_{2^k-1,J}(g) = \|g - S_{2^{k-2},J}\|_2 = \left\| g - \sum_{n=1}^{2^{k-2}} \sum_{m=0}^{J-1} c_{n,m}\, \varphi_{n,m}(t) \right\|_2 .$$

Moreover, the approximation error satisfies the asymptotic behavior

$$E_{2^k-1,J}(g) = O\left(\frac{1}{J!\, 2^{\,J(k+1)}} \right),$$

which establishes the convergence rate of the corresponding second-kind Chebyshev wavelet expansion in $L^2[0,1]$.

Proof. Since a function g is J times differentiable, by Taylor's expansion, we have

$$g(b+h) = g_{J-1} = g(b) + \frac{h}{1!}g'(b) + \cdots + \frac{h^{J-1}}{(J-1)!}g^{(J-1)}(b) + \frac{h^J}{J!}g^{(J)}(b+\zeta h),$$

where $0 < \zeta < 1$, and

$$g_J = g(b) + \frac{h}{1!}g'(b) + \cdots + \frac{h^{J-1}}{(J-1)!}g^{(J-1)}(b).$$

Now, we write

$$g_{J+1} - g_J = \frac{h^J}{J!}g^{(J)}(b+\zeta h), \qquad 0 < \zeta < 1. \tag{3.2}$$

Using (3.2) and dividing the interval $[0,1]$ into subintervals $\left[\frac{l-1}{2^k}, \frac{l}{2^k}\right]$, we get

$$\begin{aligned}
\|g - S_{2^k-2,J-1}\|_2^2 &= \int_0^1 \left| g(t) - \sum_{l=1}^{2^k-2} \sum_{m=0}^{J-1} c_{lm}\varphi_{lm}(t) \right|^2 dt \\
&= \sum_{l=0}^{2^k-1} \int_{\frac{l-1}{2^k}}^{\frac{l}{2^k}} \left| g(t) - \sum_{l=1}^{2^k-2} \sum_{m=0}^{J-1} c_{lm}\varphi_{lm}(t) \right|^2 dt \\
&\leq \sum_{l=1}^{2^k} \int_{\frac{l-1}{2^k}}^{\frac{l}{2^k}} \left(\frac{1}{J!} \left(\frac{1}{2^{k-1}} \right)^J \sup_{x\in[0,1]} |g^{(J)}(t)| \right)^2 dt \\
&= \int_0^1 \left(\frac{1}{J!} \right)^2 \left(\frac{1}{2^{J(k-1)}} \right)^2 \sup_{x\in[0,1]} |g^{(J)}(t)|^2 \, dt.
\end{aligned}$$

Now,

$$\|f - S_{2^k-2,J-1}\|_2^2 \leq \left(\frac{1}{J!} \right)^2 \left(\frac{1}{2^{J(k-1)}} \right)^2 \sup_{x\in[0,1]} |g^{(J)}(t)|^2.$$

Hence,

$$\|f - S_{2^k-2,J-1}\|_2 \leq \left(\frac{1}{J!} \right) \left(\frac{1}{2^{J(k-1)}} \right) \sup_{x\in[0,1]} |g^{(J)}(t)|.$$

Thus,

$$\begin{aligned}
E_{2^k-1}(f) = \|f - S_{2^k-2,J-1}\|_2 &\leq \left(\frac{1}{J!} \right) \left(\frac{1}{2^{J(k-1)}} \right) \sup_{x\in[0,1]} |g^{(J)}(t)| \\
&= O\left(\frac{1}{J! 2^{J(k-1)}} \right).
\end{aligned}$$

4 Operational matrix of integration (OMI) and product operation matrix

In this section, we present the construction of the operational matrix of integration, as well as the matrix associated with the product operation for second-kind Chebyshev wavelet.

4.1 Operational matrix of integration of second-order Chebyshev wavelet for $k = 3$ and $M = 4$.

Here, we introduce the structure of OMI for second-kind Chebyshev wavelet, in particular for $k = 3$ and $M = 4$. The following is an analysis of sixteen basis functions defined on the interval [0,1):

$$\left.\begin{aligned}
\varphi_{1,0}(t) &= \tfrac{4}{\sqrt{\pi}}, \\
\varphi_{1,1}(t) &= \tfrac{4}{\sqrt{\pi}}(16t - 2), \\
\varphi_{1,2}(t) &= \tfrac{4}{\sqrt{\pi}}(256t^2 - 64t + 3), \\
\varphi_{1,3}(t) &= \tfrac{4}{\sqrt{\pi}}(4096t^3 - 1536t^2 + 160t - 4),
\end{aligned}\right\} 0 \leq t < \frac{1}{4}; \tag{4.1a}$$

$$\left.\begin{aligned} \varphi_{2,0}(t) &= \tfrac{4}{\sqrt{\pi}}, \\ \varphi_{2,1}(t) &= \tfrac{4}{\sqrt{\pi}}(16t-6), \\ \varphi_{2,2}(t) &= \tfrac{4}{\sqrt{\pi}}(256t^2-192t+35), \\ \varphi_{2,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(4096t^3-4608t^2+1696t-204), \end{aligned}\right\} \frac{1}{4} \leq t < \frac{1}{2}; \tag{4.1b}$$

$$\left.\begin{aligned} \varphi_{3,0}(t) &= \tfrac{4}{\sqrt{\pi}}, \\ \varphi_{3,1}(t) &= \tfrac{4}{\sqrt{\pi}}(16t-10), \\ \varphi_{3,2}(t) &= \tfrac{4}{\sqrt{\pi}}(256t^2-320t+99), \\ \varphi_{3,3}(t) &= 2\sqrt{\tfrac{2}{\pi}}(4096t^3-7680t^2+4768t-980), \end{aligned}\right\} \frac{1}{2} \leq t < \frac{3}{4}; \tag{4.1c}$$

$$\left.\begin{aligned} \varphi_{4,0}(t) &= \tfrac{4}{\sqrt{\pi}}, \\ \varphi_{4,1}(t) &= \tfrac{4}{\sqrt{\pi}}(16t-14), \\ \varphi_{4,2}(t) &= \tfrac{4}{\sqrt{\pi}}(256t^2-448t+195), \\ \varphi_{4,3}(t) &= \tfrac{4}{\sqrt{\pi}}(4096t^3-10752t^2+9376t-2716), \end{aligned}\right\} \frac{3}{4} \leq t < 1. \tag{4.1d}$$

Let

$$\varphi_{16}(t) = [\varphi_{1,0}(t)\ \varphi_{1,1}(t)\ \varphi_{1,2}(t)\ \varphi_{1,3}(t)\ ...\varphi_{4,0}(t)\ \varphi_{4,1}(t)\ \varphi_{4,2}(t)\ \varphi_{4,3}]^T. \tag{4.2}$$

By integrating the first basis function between 0 and t, we get

$$\int_0^t \varphi_{1,0}(t')dt' = \begin{cases} \frac{4}{\sqrt{\pi}}t & , 0 \leq t < \frac{1}{4} \\ \frac{1}{\sqrt{\pi}} & , \frac{1}{4} \leq t < \frac{1}{2} \\ \frac{1}{\sqrt{\pi}} & , \frac{1}{2} \leq t < \frac{3}{4} \\ \frac{1}{\sqrt{\pi}} & , \frac{3}{4} \leq t < 1. \end{cases} \tag{4.3}$$

Expanding the L.H.S. of (4.3) in the form of the basis function, we have

$$\begin{aligned} \int_0^t \varphi_{1,0}(t')\,dt' = a_{1,0}\varphi_{1,0} + a_{1,1}\varphi_{1,1} + a_{1,2}\varphi_{1,2} + a_{1,3}\varphi_{1,3} + a_{2,0}\varphi_{2,0} + a_{2,1}\varphi_{2,1} + a_{2,2}\varphi_{2,2} \\ + a_{2,3}\varphi_{2,3} + a_{3,0}\varphi_{3,0} + a_{3,1}\varphi_{3,1} + a_{3,2}\varphi_{3,2} + a_{3,3}\varphi_{3,3} \\ + a_{4,0}\varphi_{4,0} + a_{4,1}\varphi_{4,1} + a_{4,2}\varphi_{4,2} + a_{4,3}\varphi_{4,3}, \end{aligned} \tag{4.4}$$

where the first coefficient is

$$\begin{aligned} a_{1,0} &= \left\langle \int_0^t \varphi_{1,0}(t')dt', \varphi_{1,0}(t) \right\rangle_{w_n} \\ &= \frac{1}{8}. \end{aligned}$$

Other coefficients of (4.4) can be calculated in the same manner, which are as follows: $a_{1,1} = \frac{1}{16}, a_{1,2} = a_{1,3} = 0, a_{2,0} = \frac{1}{4}, a_{2,1} = a_{2,2} = a_{2,3} = 0, a_{3,0} = \frac{1}{4}, a_{3,1} = a_{3,2} = a_{3,3} = 0$, $a_{4,0} = \frac{1}{4}, a_{4,1} = a_{4,2} = a_{4,3} = 0$.

Thus, (4.4) is defined as

$$\int_0^t \varphi_{1,0}(t')dt' = \frac{1}{8}\varphi_{1,0}(t) + \frac{1}{16}\varphi_{1,1}(t) + \frac{1}{4}\varphi_{2,0}(t) + \frac{1}{4}\varphi_{3,0}(t) + \frac{1}{4}\varphi_{2,0}(t)$$

$$= \left[\frac{1}{8}\ \frac{1}{16}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\varphi_{16}(t). \tag{4.5}$$

By integrating the second basis function from 0 to t, we get

$$\int_0^t \varphi_{1,1}(t')dt' = \begin{cases} \frac{4}{\sqrt{\pi}}(8t^2 - 2t), & 0 \leq t < \frac{1}{4} \\ 0, & \frac{1}{4} \leq t < \frac{1}{2} \\ 0, & \frac{1}{2} \leq t < \frac{3}{4} \\ 0, & \frac{3}{4} \leq t < 1. \end{cases} \tag{4.6}$$

Expanding the L.H.S. of (4.6) in the form of the basis function, we have

$$\begin{aligned} \int_0^t \varphi_{1,1}(t)dt' &= \frac{-3}{32}\varphi_{1,0} + \frac{1}{32}\varphi_{1,2} \\ &= \left[\frac{-3}{32}\ 0\ \frac{1}{32}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t). \end{aligned} \tag{4.7}$$

Adopting a similar procedure, we obtain

$$\begin{aligned}
\int_0^t \varphi_{1,2}(t')dt' &= \left[\frac{1}{24}\ \frac{-1}{48}\ 0\ \frac{1}{48}\ \frac{1}{12}\ 0\ 0\ 0\ \frac{1}{12}\ 0\ 0\ 0\ \frac{1}{12}\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{1,3}(t')dt' &= \left[\frac{-1}{32}\ 0\ \frac{-1}{64}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \right]\varphi_{16}(t). \\
\int_0^t \varphi_{2,0}(t')dt' &= \left[0\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{16}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{2,1}(t')dt &= \left[0\ 0\ 0\ 0\ \frac{-3}{32}\ 0\ \frac{1}{32}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{2,2}(t')dt' &= \left[0\ 0\ 0\ 0\ \frac{1}{24}\ \frac{-1}{48}\ 0\ \frac{1}{48}\ \frac{1}{12}\ 0\ 0\ 0\ \frac{1}{12}\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{2,3}(t')dt' &= \left[0\ 0\ 0\ 0\ \frac{-1}{32}\ 0\ \frac{-1}{64}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{3,0}(t')dt' &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{16}\ 0\ 0\ \frac{1}{4}\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{3,1}(t')dt' &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-3}{32}\ 0\ \frac{1}{32}\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{3,2}(t')dt' &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{24}\ \frac{-1}{48}\ 0\ \frac{1}{48}\ \frac{1}{12}\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{3,3}(t')dt' &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-1}{32}\ 0\ \frac{-1}{64}\ 0\ 0\ 0\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{4,0}(t')dt' &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{8}\ \frac{1}{16}\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t i\varphi_{4,1}(t')dt' &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-3}{32}\ 0\ \ \frac{1}{32}\ 0\ 0\right]\varphi_{16}(t). \\
\int_0^t \varphi_{4,2}(t')dt &= \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{1}{24}\ \frac{-1}{48}\ 0\ \frac{1}{48}\right]\varphi_{16}(t).
\end{aligned}$$

$$\int_0^t \varphi_{4,3}(t')dt = \Big[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{-1}{32}\ 0\ \frac{-1}{64}\ 0\Big]\,\varphi_{16}(t).$$

Now, we write

$$\int_0^t \varphi_{16}(t')dt' = P\ \varphi(t), \tag{4.8}$$

where $P_{16\times 16}$ is an operational matrix of integration (OMI), given by

$$P_{16\times16} = \left[\begin{array}{cccccccccccccccc}
\frac{1}{8} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
\frac{-3}{32} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{24} & \frac{-1}{48} & 0 & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 0 & 0 & 0 \\
\frac{-1}{32} & 0 & \frac{-1}{64} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-3}{32} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{24} & \frac{-1}{48} & 0 & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 & \frac{1}{12} & 5 & 6 & 6 \\
0 & 0 & 0 & 0 & \frac{-1}{32} & 0 & \frac{-1}{64} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{16} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3}{32} & 0 & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & \frac{-1}{48} & 0 & \frac{1}{48} & \frac{1}{12} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{32} & 0 & \frac{-1}{64} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{8} & \frac{1}{16} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3}{32} & 0 & \frac{1}{32} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{24} & \frac{-1}{48} & 0 & \frac{1}{48} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{32} & 0 & \frac{-1}{64} & 0
\end{array}\right]_{16\times16.} \tag{4.9}$$

This matrix can be written in block form:

$$P_{16\times16} = \frac{1}{8}\begin{bmatrix} M_{4\times4} & N_{4\times4} & N_{4\times4} & N_{4\times4} \\ O_{4\times4} & M_{4\times4} & N_{4\times4} & N_{4\times4} \\ O_{4\times4} & O_{4\times4} & M_{4\times4} & N_{4\times4} \\ O_{4\times4} & O_{4\times4} & O_{4\times4} & M_{4\times4} \end{bmatrix},$$

where

$$M_{4\times4} = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{4} & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} \\ \frac{\sqrt{2}}{8} & 0 & -\frac{1}{4} & 0 \end{bmatrix} \quad \text{and} \quad N_{4\times4} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2\sqrt{2}}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we prove the following lemma:

Lemma 4.1.

$$A_1 = \int_0^t \varphi_1(t)\,dt' = M\varphi_1(t) + N\varphi_2(t) + N\varphi_3(t) + \cdots + N\,\varphi_{2^k-1}(t), \tag{4.10}$$

where

$$M = P_{11}, \qquad N = P_{1j}, \quad j = 2, 3, \ldots, 2^{k-1}.$$

The matrices M and N are given as follows:

$$M = \frac{1}{2^k}\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{4} & 0 & \cdots & 0 \\ -\frac{\sqrt{2}}{3} & -\frac{1}{2} & 0 & \frac{1}{6} & \cdots & 0 \\ \frac{\sqrt{2}}{8} & 0 & -\frac{1}{4} & 0 & \cdots & 0 \\ -\frac{\sqrt{2}}{15} & 0 & 0 & -\frac{1}{6} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\sqrt{2}}{(M-3)(M-1)} & 0 & 0 & 0 & \cdots & \frac{1}{2M-2} \\ \frac{\sqrt{2}}{(M-2)M} & 0 & 0 & 0 & \cdots & -\frac{1}{2(M-2)} \end{bmatrix} \tag{4.11}$$

and

$$N = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ -\dfrac{2\sqrt{2}}{3} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ -\dfrac{2\sqrt{2}}{15} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -\dfrac{2\sqrt{2}}{(M-3)(M-1)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{4.12}$$

Similar to (4.1a)–(4.1d), we can obtain

$$\int_0^t \varphi_{nm}(t')\,dt', \qquad m = 0, 1, \ldots, M-1.$$

In (2.7), $\varphi_i(t)$; $i = 2, 3 \ldots n$ is obtained by translating the vector $\varphi_1(t)$ to the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$.

Theorem 4.1.

$$A_1 = \int_0^t \varphi_1(t')\,dt' = M\varphi_1(t) + N\varphi_2(t) + N\varphi_3(t) + \cdots + N\varphi_{2^k-1}(t), \tag{4.13}$$

$$A_2 = \int_0^t \varphi_2(t')\,dt' = O\varphi_1(t) + M\varphi_2(t) + N\varphi_3(t) + \cdots + N\varphi_{2^k-1}(t), \tag{4.14}$$

$$\vdots$$

$$A_{2^k-1} = \int_0^t \varphi_{2^k-1}(t')\,dt' = O\varphi_1(t) + O\varphi_2(t) + O\varphi_3(t) + \cdots + M\varphi_{2^k-1}(t), \tag{4.15}$$

where O represents the zero matrix, and the matrices M and N are defined in Eqs. (4.11) and (4.12) respectively.

Proof. From $(4.1a) - (4.1d)$, we observe that the term A_2 has the same form as A_1 except that the second-kind Chebyshev wavelet functions are shifted to the next subinterval. Because of this shift, the corresponding coefficient blocks P_{2j} in the first position must be zero matrices. For the remaining blocks $j = 2, 3, \ldots, 2^{k-1}$, each P_{2j} appears one position below its location in the previous row; specifically, $P_{22} = M,\ P_{2j} = N,\ j = 1, \ldots, 2^{k-1}$. Repeating this shifting pattern for subsequent rows eventually produces the final row, which reveals the complete structural pattern of the following operational matrix of integration (OMI).

$$P = \begin{bmatrix} M & N & N & \ldots & \ldots & N \\ O & M & N & \ldots & \ldots & N \\ O & O & M & \ldots & \ldots & N \\ \vdots & & & \ldots & \ddots & N \\ O & O & O & \ldots & \ldots & M \end{bmatrix}.$$

□

4.2 Product operation matrix (POM)

The following relation is used to represent the product of two vectors generated by the second-kind Chebyshev wavelet functions:

$$F^T\varphi(t)\varphi^T(t) = \varphi^T(t)\tilde{F}, \tag{4.16}$$

where $\varphi(t)\varphi^T(t)$ is a product operation matrix of order 16×16, given by

$$\varphi\varphi^T(t) = \begin{bmatrix} \varphi_{10}\varphi_{10} & \varphi_{10}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{10}\varphi_{42} & \varphi_{10}\varphi_{43} \\ \varphi_{11}\varphi_{10} & \varphi_{11}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{11}\varphi_{12} & \varphi_{11}\varphi_{43} \\ \varphi_{12}\varphi_{10} & \varphi_{12}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{12}\varphi_{12} & \varphi_{12}\varphi_{43} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_{42}\varphi_{10} & \varphi_{42}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{42}\varphi_{12} & \varphi_{42}\varphi_{43} \\ \varphi_{43}\varphi_{10} & \varphi_{43}\varphi_{11} & \cdots & \cdots & \cdots & \varphi_{43}\varphi_{12} & \varphi_{43}\varphi_{43} \end{bmatrix}_{16\times 16} \tag{4.17}$$

and

$$\tilde{F} = \begin{bmatrix} H_1 & 0 & 0 & 0 \\ 0 & H_2 & 0 & 0 \\ 0 & 0 & H_3 & 0 \\ 0 & 0 & 0 & H_4 \end{bmatrix}_{16\times 16,} \tag{4.18}$$

where $H_i, i = 1, 2, 3, 4$ are 4×4 matrices.
Here, by computation, we get

$$\tilde{F}=\begin{bmatrix}
\frac{1}{8\sqrt{2}} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{16\sqrt{2}} & \frac{1}{8\sqrt{2}} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{32} & \frac{1}{8\sqrt{2}} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{1}{32} & \frac{1}{8\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{3}{8\sqrt{2}} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{3}{8\sqrt{2}} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{3}{8\sqrt{2}} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{3}{8\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{8\sqrt{2}} & \frac{1}{16\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{5}{8\sqrt{2}} & \frac{1}{32} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{5}{8\sqrt{2}} & \frac{1}{32} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{5}{8\sqrt{2}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{8\sqrt{2}} & \frac{1}{16\sqrt{2}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{16\sqrt{2}} & \frac{7}{8\sqrt{2}} & \frac{1}{32} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{7}{8\sqrt{2}} & \frac{1}{32}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{32} & \frac{7}{8\sqrt{2}}
\end{bmatrix}_{16\times 16}. \tag{4.19}$$

5 Solution of linear differential equations

In this section, we obtain numerical solution to four very famous linear differential equations and compare their results with the exact solutions.

Example 1. First-order linear differential equation

$$y'(t)+2y(t)=t; \tag{5.1a}$$

$$y(0)=0. \tag{5.1b}$$

Exact solution of (5.1a) with (5.1b) is $y(t)=\frac{t}{2}-\frac{1}{4}+\frac{1}{4}e^{-2t}$.
Here, we solve the above initial value problem using second-kind Chebyshev wavelet for $k=3$ and $M=4$.
Consider

$$y(t)=C^T\varphi(t). \tag{5.2}$$

In (5.2),

$$C=[c_{1,0}\ c_{1,1}\ c_{1,2}\ c_{1,3}\ c_{2,0}\ c_{2,1}\ c_{2,2}\ c_{2,3}\ c_{3,0}\ c_{3,1}\ c_{3,2}\ c_{3,3}\ c_{4,0}\ c_{4,1}\ c_{4,2}\ c_{4,3}] \tag{5.3}$$

and

$$\varphi(t)=[\varphi_{1,0}\ \varphi_{1,1}\ \varphi_{1,2}\ \varphi_{1,3}\ \varphi_{2,0}\ \varphi_{2,1}\ \varphi_{2,2}\ \varphi_{2,3}\ \varphi_{3,0}\ \varphi_{3,1}\ \varphi_{3,2}\ \varphi_{3,3}\ \varphi_{4,0}\ \varphi_{4,1}\ \varphi_{4,2}\ \varphi_{4,3}]. \tag{5.4}$$

The components of $\varphi(t)$ are provided in (4.1a) to (4.1d). Now, we also express

$$t=\left[\frac{\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\ \frac{3\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\ \frac{5\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\ \frac{7\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\right]=E^T\varphi(t). \tag{5.5}$$

By integrating (5.1a) from 0 to t and applying (5.1b) and (5.5) to the resulting expression, we obtain

$$C^T\varphi(t)+2C^T\int_0^t\varphi(t')dt'=\int_0^t E^T\varphi(t)dt. \tag{5.6}$$

Using (4.8), we have

$$C^T + 2C^T P = E^T P$$

i.e.

$$(I + 2P^T)C = P^T E. \tag{5.7}$$

Now, (5.7) can be written as

$$DC = F, \tag{5.8}$$

where $D = (I + 2P^T)$ and $F = P^T E$.

Here, (5.8) is a set of 16 algebraic equations with 16 unknowns. We solve (5.8) for C and obtain the following matrix:

$$C = \begin{bmatrix} 0.00386631324475085 \\ 0.0030067287043509 \\ 0.000676654377478186 \\ -2.81939323949244 \times 10^{-05} \\ 0.0250433582291843 \\ 0.00727215250648188 \\ 0.000410412059529829 \\ -1.71005024804095 \times 10^{-05} \\ 0.0596818086046451 \\ 0.00985926555217641 \\ 0.000248927760188688 \\ -1.0371990007862 \times 10^{-05} \\ 0.102485022697866 \\ 0.0114284305922034 \\ 0.00015098247810638 \\ -6.29093658776585 \times 10^{-06} \end{bmatrix}_{16\times 1.} \tag{5.9}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 1.

In Table 1, a comparison is shown between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 1: Estimated and exact solutions for $y(t)$ in Example 1.

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.1	0.004728	0.004683	0.000045
0.2	0.017539	0.017580	0.000041
0.3	0.037230	0.037203	0.000027
0.4	0.062303	0.062332	0.000029
0.5	0.091872	0.091970	0.000098
0.6	0.125316	0.125299	0.000017
0.7	0.161635	0.161649	0.000014
0.8	0.200484	0.200474	0.000010
0.9	0.241314	0.241325	0.000011

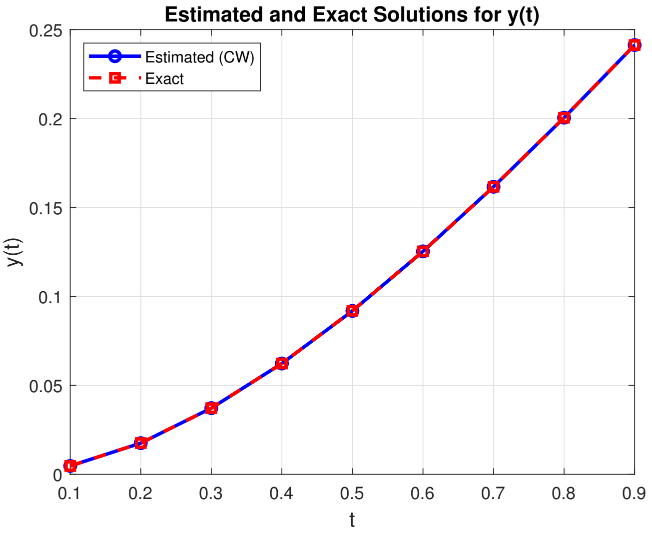

Figure 1: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 1.

Example 2. Consider the Lane–Emden-type differential equation

$$y''(t) + \frac{2}{t}y'(t) + y(t) = 0; \tag{5.10a}$$

$$y(0) = 1, y'(0) = 0. \tag{5.10b}$$

The exact solution of (5.10a) is $y(t) = \frac{sint}{t}$.
We solve this example using second-kind Chebyshev wavelet for $k = 3$ and $M = 4$. Initially, the unknown function $y(t)$ is considered as

$$y''(t) = C^T\varphi(t), \tag{5.11}$$

where C and $\varphi(t)$ are given by (5.3) and (5.4) respectively, and the elements of $\varphi(t)$ are given in (4.1a) to (4.1d).
Integrating (5.11) from 0 to t and utilizing (4.8) and (5.10b) in this, we obtain

$$\begin{aligned} y'(t) &\simeq C^TP\varphi(t) + y'(0) \\ &= C^TP\varphi(t) + A^T\varphi(t) \\ y'(t) &= C^TP\varphi(t) && (5.12) \\ y(t) &\simeq C^TP^2\varphi(t) + ty'(0) + y(0) \\ y(t) &= C^TP^2\varphi(t) + B^T\varphi(t), && (5.13) \end{aligned}$$

where

$$B^T = \left[\frac{\sqrt{\pi}}{4}\ 0\ 0\ 0\ \frac{\sqrt{\pi}}{4}\ 0\ \ 0\ 0\ \frac{\sqrt{\pi}}{4}\ 0\ 0\ 0\ \frac{\sqrt{\pi}}{4}\ 0\ 0\ 0\right] \tag{5.14}$$

and P is the 16×16 second-kind Chebyshev wavelet OMI given by (4.9).
Also, we express t as $F^T\varphi(t)$ in the following manner:

$$\begin{aligned} t &= \left[\frac{\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\ \frac{3\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\ \frac{5\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\ \frac{7\sqrt{\pi}}{32}\ \frac{\sqrt{\pi}}{64}\ 0\ 0\right] \\ &= F^T\varphi(t). \end{aligned} \tag{5.15}$$

Substituting (5.11) to (5.15) in (5.10a), we get

$$F^T\varphi(t)\varphi^T(t)C + 2\varphi^T(t)P^TC + F^T\varphi(t)\varphi^T(t)P^{2T}C + F^T\varphi(t)\varphi^T(t)B = 0. \tag{5.16}$$

Now using (4.16) in (5.16), we get

$$(\tilde{F} + 2P^T + \tilde{F}P^{2T})C + \tilde{F}B = 0. \tag{5.17}$$

Here, (5.17) gives a set of 16 algebraic equations with 16 unknowns. We solve (5.17) for C and obtain the following matrix:

$$C = \begin{bmatrix} -0.115160183318089 \\ 0.000439877246537369 \\ 7.73762602078076 \times 10^{-05} \\ 1.54352216389588 \times 10^{-05} \\ -0.11145789279826 \\ 0.00135662523789703 \\ 5.36586159121016 \times 10^{-05} \\ 1.00260066494751 \times 10^{-05} \\ -0.104228465136533 \\ 0.0022195207514643 \\ 4.1603514277296 \times 10^{-05} \\ 6.09028432049204 \times 10^{-06} \\ -0.0937438993194888 \\ 0.00299006768580203 \\ 2.89827432071026 \times 10^{-05} \\ 3.88800841635918 \times 10^{-06} \end{bmatrix}_{16\times 1.} \tag{5.18}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 2.
In Table 2, a comparison is shown between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 2: Estimated and exact solutions for $y(t)$ in Example 2

t	**Estimated value of** $y(t)$ **using CW**	**Exact value of** $y(t)$	**Absolute error**
0.1	0.998694	0.998334	0.000360
0.2	0.994786	0.993346	0.001440
0.3	0.988289	0.985067	0.003222
0.4	0.979247	0.973540	0.005707
0.5	0.957709	0.958850	0.001141
0.6	0.943699	0.941070	0.002629
0.7	0.927336	0.920310	0.007026
0.8	0.898672	0.896690	0.001982
0.9	0.877823	0.870360	0.007463

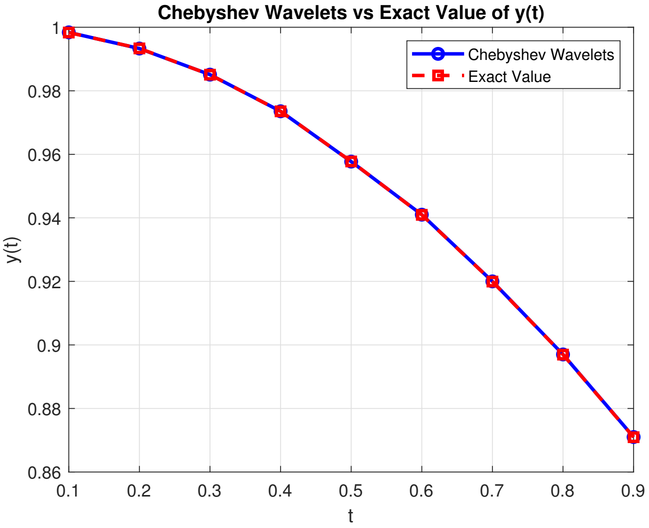

Figure 2: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 2.

Example 3. Consider the following third-order differential equation

$$y'''(t) - y''(t) = 0; \tag{5.19a}$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2. \tag{5.19b}$$

The exact solution of (5.19a) is $y(t) = 2e^t - t - 2$.
We solve this example using of second-kind Chebyshev wavelet for $k = 3$ and $M = 4$. Initially, the unknown function $y(t)$ is treated as

$$y'''(t) = C^T \varphi(t), \tag{5.20}$$

where C and $\varphi(t)$ are given by (5.3) and (5.4) respectively, and the elements of $\varphi(t)$ are given in (4.1a) to (4.1d).

Integrating (5.20) over the interval 0 to t, and employing the identities given in (4.8) and (5.19b) in this, we obtain

$$\begin{aligned} y''(t) &\simeq C^T P\varphi(t) + y''(0) \\ &= C^T P\varphi(t) + 2B^T\varphi(t), \end{aligned} \tag{5.21}$$

Now, integrating (5.21) over the interval 0 to t, we have

$$\begin{aligned} y'(t) &\simeq C^T P^2\varphi(t) + 2B^T P\varphi(t) + y'(0) \\ y'(t) &= C^T P^2\varphi(t) + 2B^T P\varphi(t) + B^T\varphi(t), \end{aligned} \tag{5.22}$$

Now, integrating (5.22) over the interval, we have

$$\begin{aligned} y(t) &\simeq C^T P^3\varphi(t) + 2B^T P^2\varphi(t) + B^T P\varphi(t) + y(0) \\ y(t) &= C^T P^3\varphi(t) + 2B^T P^2\varphi(t) + B^T P\varphi(t). \end{aligned} \tag{5.23}$$

Using (5.20) and (5.21) in (5.19a), we get

$$\varphi^T(t)C - \varphi^T(t)P^T C - 2\varphi^T(t)B = 0. \tag{5.24}$$

i.e.

$$(I - P^T)C - 2B = 0. \tag{5.25}$$

Here, (5.25) gives a set of 16 algebraic equations with 16 unknowns. We solve (5.25) for C and obtain the following matrix:

$$C = \begin{bmatrix} 1.0061893089767 \\ 0.0628459298066136 \\ 0.00196329621263746 \\ 4.0902004429947 \times 10^{-05} \\ 1.29197260526241 \\ 0.0806957686173046 \\ 0.00252092215660958 \\ 5.25192115960329 \times 10^{-05} \\ 1.65892560958149 \\ 0.103615414598453 \\ 0.00323692801869558 \\ 6.74360003894913 \times 10^{-05} \\ 2.13010257873567 \\ 0.133044821635259 \\ 0.00415629771460619 \\ 8.65895357209623 \times 10^{-05} \end{bmatrix}_{16\times 1.} \tag{5.26}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 3. In Table 3, a comparison is shown between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 3: Estimated and exact solutions for $y(t)$ in Example 3.

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.1	0.11027	0.11034	0.00007
0.2	0.24286	0.24280	0.00006
0.3	0.39964	0.39971	0.00007
0.4	0.58373	0.58364	0.00009
0.5	0.79803	0.79744	0.00059
0.6	1.04412	1.04423	0.00011
0.7	1.32761	1.32750	0.00011
0.8	1.65095	1.65108	0.00013
0.9	2.01935	2.01920	0.00015

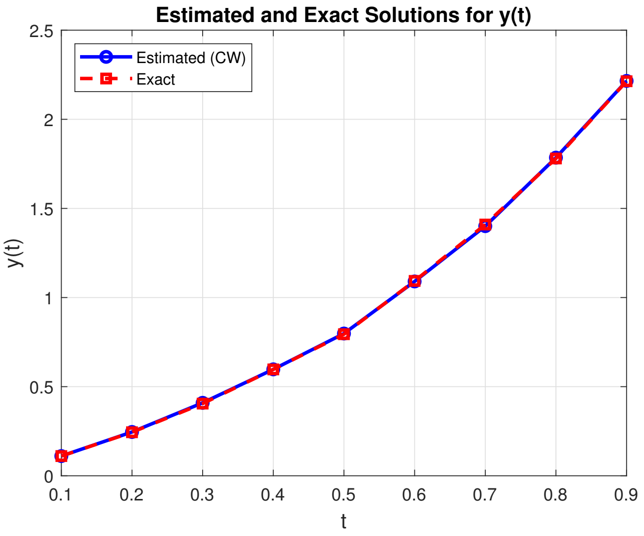

Figure 3: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 3.

Example 4. Consider the following singular initial value problem

$$y'''(t) - \frac{2}{t}y''(t) + y'(t) = 0; \tag{5.27a}$$

$$y(0) = 0, y'(0) = 1, y''(0) = 2. \tag{5.27b}$$

The exact solution of (5.27a) is $y(t) = te^t$.
Here, we solve the example using of second-kind Chebyshev wavelet for $k = 3$ and $M = 4$. Substituting (5.20) to (5.23) in (5.27a), we get

$$\begin{aligned} F^T\varphi(t)\varphi^T(t)C - 2\varphi^T(t)P^{3T}C - 2\varphi^T(t)P^{2T}B - \varphi^T(t)P^TB - F^T\varphi(t)\varphi^T(t)P^{2T}C \\ -2F^T\varphi(t)\varphi^T(t)P^TB - F^T\varphi(t)\varphi^T(t)B = 0. \end{aligned} \tag{5.28}$$

Now using (4.16) in (5.28), we get

$$(\tilde{F} - 2P^{3T} - \tilde{F}P^{2T})C = (2P^{T} + 4P^{2T} + \tilde{F} + 2\tilde{F}P^{T})B. \tag{5.29}$$

Here, (5.29) gives a set of 16 algebraic equations with 16 unknowns. We solve (5.29) for C and obtain the following matrix:

$$C = \begin{bmatrix} 2.00289979296444 \\ 0.154077408193644 \\ 0.0258351388478971 \\ -0.00582554304797807 \\ 2.77940002797266 \\ 0.230577061110471 \\ 0.0161263732032598 \\ -0.000261875727673331 \\ 3.88134900955495 \\ 0.323795981702866 \\ 0.0182981827669178 \\ 0.000164687419050493 \\ 5.41725561665207 \\ 0.449453693130607 \\ 0.0227681395174711 \\ 0.000386835094826904 \end{bmatrix}_{16\times 1.} \tag{5.30}$$

Now, substituting the calculated value of C, we can determine $y(t)$ in Example 4. In Table 4, a comparison is shown between the solution estimated by the proposed method and the exact solution at different points in the interval [0, 1).

Table 4: Estimated and exact solutions for $y(t)$ in Example 4.

t	**Estimated value of $y(t)$ using CW**	**Exact value of $y(t)$**	**Absolute error**
0.1	0.11054	0.11052	0.00002
0.2	0.24567	0.24428	0.00139
0.3	0.40924	0.40496	0.00428
0.4	0.59708	0.59673	0.00035
0.5	0.79803	0.79436	0.00075
0.6	1.09036	1.09327	0.00291
0.7	1.40064	1.40963	0.00899
0.8	1.78502	1.78043	0.00459
0.9	2.21538	2.21364	0.00174

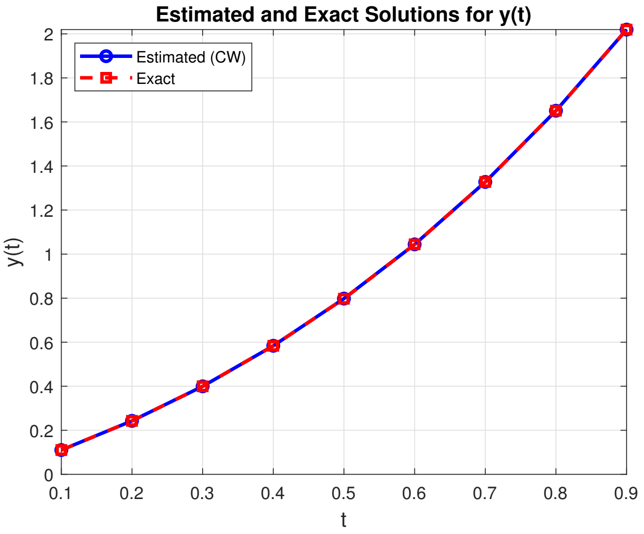

Figure 4: Graphical illustration of estimated and exact solutions for $y(t)$ in Example 4.

6 Conclusion

In this work, the convergence behavior of the proposed wavelet is examined. Also, an efficient and accurate method for solving linear differential equations has been developed. Second-kind Chebyshev wavelet operational matrix for integration and a matrix for product operations have been constructed. We considered first order linear, Lane–Emden type, third order linear and third-order singular differential equations. These differential equations have been solved using the operational matrix of second-kind Chebyshev wavelet for $k = 3$ and $M = 4$, and these solution are compared with their exact solutions.

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