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A new class of self referred equations with a nonlocal boundary condition

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Abstract. We study a nonlinear integro-differential equation with memory and a nonlocal energy constraint. We will use a fixed point approach and prove, via the Schauder fixed point theorem, that under suitable assumptions the problem admits at least one solution. Compared to previous papers, we will work in a more regular function space and impose new structural assumptions.

Keywords. Differential functional equations, integral boundary conditions, fixed-point theorem

1 Introduction

The idea of *self-reference* is often used in mathematics and applied sciences to model several phenomena. An example is the *Volterra integro-differential equation*, which is used in biology ([21]). The idea of self-reference appears also in materials science ([4], [5]), in electrodynamics ([7]) and in applied sciences and philosophy (e.g. [9]). In particular some of these phenomena may be described using *self-referred and hereditary equations*, which have the property of preserving the memory of some or all the events that have occurred from the beginning of the evolution until a certain time $t > 0$. Formally, let X be a space of functions, $A : X \to \mathbb{R}$, $B : X \to \mathbb{R}$ two functional operators which may be, for instance, differential or integral operators. Hence a self-referred equation may be written as

$$Ay(x,t) = y(By(x,t), t).$$

We also impose the nonlocal integral condition

$$\int_0^\beta y^2(x,t)dt = \beta g(x) > 0$$

for suitable $g > 0$ and $\beta > 0$. This condition may be interpreted as prescribing a nonlocal energy-type constraint along the time variable, for example the energy in an isolated system. This type of condition is widely used to model, for example, some phase transition phenomena ([8], [23]). There are several papers in which a first order self referred differential equation with a Lipschitz

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continuous and bounded initial data was studied. For example, Oberg ([15]) in 1969 found the local existence of a solution of the functional differential equation

$$\partial_t x(t) = F(t, x(t), x(g(t, x(t))))$$

where F and g are uniformly Lipschitz continuous functions. Eder [2] in 1979 studied the self-referred equation

$$\partial_t x(t) = x(x(t))$$

finding the global existence and the uniqueness of a solution with initial condition $x(t_0) = x_0$, obtaining also more information about the regularity of the solution. In 1990 Wang ([22]) studied the functional differential equation

$$\partial_t x(t) = f(x(x(t)))$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, monotone and $f(0) = 0$. In 1993 Fečkan ([3]) studied the same self-referred equation with $f \in C^1(\mathbb{R})$ and initial data $x(0) = 0$, obtaining the global existence of a solution and some nontrivial applications in bifurcation theory. Other articles studied the problem of the existence of solutions for systems of self referred differential equations (for instance [17], [19], [20]). Also higher-order self-referred differential equation with Lipschitz and bounded initial data were studied using fixed point and Peano-Picard techniques (see [12], [13], [14]). In 2024, the existence of a solution for the problems

$$\begin{cases} \ddot{y}(t) = \beta y(y(t)), \quad t \in (0,1), \\ y(0) = \alpha, \\ \int_0^\beta \dot{y}^2(t)dt = \delta\beta > 0, \end{cases} \tag{1.1}$$

where $0 < \delta < 1$, $0 \le \alpha \le 1$ and $0 < \beta \le 1$,

$$\begin{cases} \dot{y}(t) = \beta y\big(\int_0^t y(s)ds\big), \qquad t \in (0,1), \\ \int_0^\beta y^2(t)dt = \delta\beta, \end{cases} \tag{1.2}$$

where $0 < \delta < 1$ and $0 < \beta \le 1$ and

$$\begin{cases} \frac{\partial}{\partial t} y(x,t) = \beta y\big(\int_0^t y(x,s)ds, t\big), \qquad (x,t) \in (0,1) \times (0,1), \\ \int_0^\beta y^2(x,t)dt = \beta g(x) \end{cases} \tag{1.3}$$

where $0 < \beta \le 1$ and $g : [0,1] \to [0,1]$ is a smooth function were investigated ([1]) while the system ([6])

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \beta_1 u(v(u(x,t),t),t), \ t \in (0,1), \ x \in (0,1), \\ \frac{\partial}{\partial t} v(x,t) = \beta_2 v(u(v(x,t),t),t), \ t \in (0,1), \ x \in (0,1), \\ \int_0^1 u^2(x,t)dt = g_1(x), \\ \int_0^1 v^2(x,t)dt = g_2(x), \end{cases}$$

with $0 < \beta_1 < 1$, $0 < \beta_2 < 1$ and $g_1, g_2 \in Lip([0,1],[0,1])$ was studied by using a fixed point approach, remarking in particular the coupling effect due to the structure of the problem with nonlocal conditions. In the works above, the self-reference involves the unknown function itself but not its time derivative since the derivative-dependent case with the integral boundary condition requires a different functional setting.

In this paper we study the derivative-dependent case and prove an existence result for the integro-differential problem

$$\begin{cases} \frac{\partial}{\partial t}y(x,t) = \beta y\big(k_1\frac{\partial}{\partial t}y(x,t) + k_2 y(x,t), t\big), & x \in (0,1),\ t \in (0,1) \\ \int_0^\beta y^2(x,t)dt = \beta g(x), \end{cases} \tag{1.4}$$

where $g : [0,1] \to (0,1]$ is a Lipschitz function with $L_g := Lip(g)$ Lipschitz constant, $k_1, k_2 > 0$ and $k_1 + k_2 \leq 1$ real constants and $0 < \beta < \frac{1}{2}$. We prove the existence of at least one solution by a Schauder fixed-point argument under suitable assumptions.

This derivative dependent self-reference produces a new mixed dependence on both y and $\partial_t y$, which requires a more delicate analysis. Some further assumptions on y and its time derivative are needed, i.e. consider some stronger Lipschitz and boundedness conditions on the time derivative. We will find a suitable Banach space of functions which are C^1 with respect to time with some other additional assumptions. We construct a closed, bounded, and convex subset $\mathbb{Y}$ of this space and define a nonlinear operator $T : \mathbb{Y} \to \mathbb{Y}$ incorporating both the differential equation and the integral condition. To obtain this, we first perform some heuristic computations to identify a suitable functional space and an operator. We then determine conditions ensuring that this space is bounded, complete, and convex and that our operator is continuous and compact. Existence then follows from Schauder's fixed-point theorem.

The paper is organized as follows: in Section 2 we recall briefly some preliminaries related to real and functional analysis, in Section 3 we prove the main result of this paper, in Section 4 we outline some possible future work directions.

2 Preliminaries

To deal with our integro-differential problem, we briefly recall some definitions from real and functional analysis in particular about functional sequences, operators between normed spaces and fixed point theorems ([18]).

Definition 1. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on an interval $I \subset \mathbb{R}$. This sequence is called *equibounded* if there exists a real number $M > 0$ such that

$$|f_n(x)| \leq M$$

for all $n \in \mathbb{N}$ and for every $x \in I$.

Definition 2. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous functions on an interval $I \subset \mathbb{R}$. This sequence is *equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

whenever $|x - y| < \delta$ and for all $n \in \mathbb{N}$.

Definition 3. Let X, Y be two normed spaces and $T : X \to Y$ an operator. T is called *compact operator* if, for every bounded subsequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ it is possible to extract a convergent subsequence of the sequence $\{Tx_n\}_{n\in\mathbb{N}} \subset Y$.

These definitions and the previous remark are necessary in order to state the following classical results of functional analysis. These theorems are crucial to prove the existence of a fixed point for a functional and then the existence of a solution for our differential problem.

Theorem 2.1 (Ascoli-Arzelà theorem)**.** *Let us consider a sequence of real-valued continuous functions $\{f_n\}_{n\in\mathbb{N}}$ defined on a closed and bounded interval $[a,b]$ of the real line. If $\{f_n\}_{n\in\mathbb{N}}$ is equibounded and equicontinuous, then there exists a subsequence $\{f_{n_k}\}_{n_k\in\mathbb{N}}$ that converges uniformly to a function f.*

Theorem 2.2 (Schauder's theorem)**.** *Let X be a Banach space with $K \subset X$ nonempty, bounded, closed and convex with a continuous and compact operator $T : K \to K$. Then it is possible to prove that there exists a fixed point, i.e.*

$$\exists x \in K \ s.t. \ T(x) = x. \tag{2.1}$$

3 Main result

3.1 Heuristic computations

In this section we will compute heuristically our space of functions and our functional which will be used to find the existence of a solution of our integro-differential problem (1.4). Let us start by multiplying both sides of equation (1.4) by $y(x,t)$, obtaining in this way

$$y(x,t)\frac{\partial}{\partial t}y(x,t) = \beta y(x,t)y\left(k_1\frac{\partial}{\partial t}y(x,t) + k_2 y(x,t), t\right).$$

Let us observe that $y(x,t)\frac{\partial}{\partial t}y(x,t) = \frac{1}{2}\frac{d}{dt}y^2(x,t)$, thus by integrating between 0 and t the time variable of our previous equality, we get

$$y^2(x,t) = y^2(x,0) + 2\beta\int_0^t y(x,s)y\left(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\right)ds.$$

Integrating a second time with respect to the time from 0 to β and using the integral condition of our problem we obtain

$$\beta g(x) = \int_0^\beta y^2(x,t) = \beta y^2(x,0) + 2\beta\int_0^\beta\int_0^\tau y(x,s)y\left(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\right)dsd\tau$$

hence

$$y(x,0) = \pm\sqrt{g(x) - 2\int_0^\beta\int_0^\tau y(x,s)y\left(k_1\frac{\partial}{\partial t}y(x,s) + k_2 y(x,s), s\right)dsd\tau}.$$

Let us introduce now the space of function $C_t^1([0,1]^2,\mathbb{R})$.

Definition 4. We denote by $C_t^1([0,1]^2,\mathbb{R})$ the set of all functions

$$y : [0,1]^2 \to \mathbb{R}$$

such that

(i) y is continuous on $[0,1]^2$;

(ii) the partial derivative $\partial_t y(x,t)$ exists and is continuous on $(0,1)\times(0,1)$;

(iii) $\partial_t y$ admits a continuous extension to $[0,1]^2$

We equip $C^1_t([0,1]^2,\mathbb{R})$ with the norm

$$\|y\|_{C^1_t} := \sup_{(x,t)\in[0,1]^2} |y(x,t)| \; + \sup_{(x,t)\in[0,1]^2} |\partial_t y(x,t)|,$$

Remark 1. Our space is a Banach space. In fact, if (y_n) is Cauchy in C^1_t norm, then y_n converges uniformly to some y and $\partial_t y_n$ converges uniformly to some function v. By recalling that it holds true $y_n(x,t) - y_n(x,0) = \int_0^t \partial_t y_n(x,s)ds$ then, by taking the uniform limit, we get that $v = \partial_t y$. Hence we obtain that $C^1_t([0,1]^2,\mathbb{R})$ is Banach space.

Let us define the following space of functions

$$\begin{aligned}\mathbb{Y} = \Bigg\{ y \in C^1_t([0,1]^2,\mathbb{R}) \;\Bigg|\; & g(x) - 2\int_0^\beta \int_0^\tau y(x,s) y\Big(k_1 \frac{\partial}{\partial s} y(x,s) + k_2 y(x,s), s\Big) ds d\tau \geq \varepsilon_0 > 0,\\ & 0 \leq y(x,t) \leq 1, \quad 0 \leq \frac{\partial y}{\partial t}(x,t) \leq 1,\\ & \left|\frac{\partial y}{\partial t}(x_2,t) - \frac{\partial y}{\partial t}(x_1,t)\right| \leq L_2|x_1 - x_2| \text{ and } \big|y(x_2,t) - y(x_1,t)\big| \leq L_1|x_1 - x_2| \\ & \forall x_1, x_2, t \in [0,1]\\ & \text{and } \left|\frac{\partial y}{\partial t}(x,t_2) - \frac{\partial y}{\partial t}(x,t_1)\right| \leq L_3|t_2 - t_1| \;\; \forall t_1, t_2, x \in [0,1] \Bigg\} \end{aligned} \tag{3.1}$$

with $L_1, L_2, L_3, \varepsilon_0 > 0$ (fixed a priori) and the functional

$$\begin{aligned} Ty(x,t) = & \sqrt{g(x) - 2\int_0^\beta \int_0^\tau y(x,s) y\Big(k_1 \frac{\partial}{\partial s} y(x,s) + k_2 y(x,s), s\Big) ds d\tau}\\ & + \beta \int_0^t y\Big(k_1 \frac{\partial}{\partial s} y(x,s) + k_2 y(x,s), s\Big) ds. \end{aligned} \tag{3.2}$$

Remark 2. We consider $y \in \mathbb{Y}$, then it is immediate to remark that, since $k_1, k_2 > 0$ and $k_1 + k_2 \leq 1$, it holds true $0 \leq k_1 \frac{\partial}{\partial s} y(x,t) + k_2 y(x,t) \leq 1$. Therefore our differential equation is well-defined. Furthermore the composition is Lipschitz continuous in both variables, in fact

$$\begin{aligned} & \left| y\Big(k_1 \frac{\partial}{\partial t} y(x_2,t) + k_2 y(x_2,t), t\Big) - y\Big(k_1 \frac{\partial}{\partial t} y(x_1,t) + k_2 y(x_1,t), t\Big) \right| \\ & \leq L_1 \left| k_1 \frac{\partial}{\partial t} y(x_2,t) + k_2 y(x_2,t) - k_1 \frac{\partial}{\partial t} y(x_1,t) - k_2 y(x_1,t) \right| \\ & \leq L_1(k_1 L_2 + k_2 L_1)|x_1 - x_2|, \end{aligned}$$

and

$$\begin{aligned} & \left| y\Big(k_1 \frac{\partial}{\partial t} y(x,t_2) + k_2 y(x,t_2), t_2\Big) - y\Big(k_1 \frac{\partial}{\partial t} y(x,t_1) + k_2 y(x,t_1), t_1\Big) \right| \\ \leq & \left| y\Big(k_1 \frac{\partial}{\partial t} y(x,t_2) + k_2 y(x,t_2), t_2\Big) - y\Big(k_1 \frac{\partial}{\partial t} y(x,t_2) + k_2 y(x,t_2), t_1\Big) \right| \\ + & \left| y\Big(k_1 \frac{\partial}{\partial t} y(x,t_2) + k_2 y(x,t_2), t_1\Big) - y\Big(k_1 \frac{\partial}{\partial t} y(x,t_1) + k_2 y(x,t_1), t_1\Big) \right| \\ & \leq (1 + L_1(k_1 L_3 + 1))|t_1 - t_2|. \end{aligned}$$

Remark 3. It is straightforward to observe that

$$\begin{aligned}\frac{\partial}{\partial t}Ty(x,t) &= \frac{\partial}{\partial t}\Bigg(\sqrt{g(x) - 2\int_0^{\beta}\int_0^{\tau} y(x,s)y\Big(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\Big)dsd\tau} \\ &+ \beta\int_0^t y\Big(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\Big)ds\Bigg) \\ &= \beta y\Big(k_1\frac{\partial}{\partial t}y(x,t) + k_2 y(x,t), t\Big). \end{aligned} \tag{3.3}$$

Hence, if $Ty = y$, then it is immediate to remark that the fixed point is also a solution of our integro-differential problem. Furthermore, since $0 \le y \le 1$, then $0 \le \frac{\partial Ty}{\partial t} \le \beta \le 1$.

Remark 4. Let us consider $g(x) - \beta^2 \ge \varepsilon_0 > 0$. Hence we remark that

$$\sqrt{g(x) - 2\int_0^{\beta}\int_0^{\tau} y(x,s)y\Big(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\Big)dsd\tau} \ge \sqrt{g(x) - \beta^2} \ge \sqrt{\varepsilon_0}.$$

Remark 5. Let us consider $y \in \mathbb{Y}$. If we assume $\sqrt{g(x)} + \beta \le 1$ we get

$$\begin{aligned}Ty(x,t) &= \sqrt{g(x) - 2\int_0^{\beta}\int_0^{\tau} y(x,s)y\Big(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\Big)dsdt} \\ &+ \beta\int_0^t y\Big(k_1\frac{\partial}{\partial s}y(x,s) + k_2 y(x,s), s\Big)ds \le \sqrt{g(x)} + \beta \le 1.\end{aligned}$$

Remark 6. Let us observe that our functional is globally Lipschitz with respect to the component x. In fact we get

$$\begin{aligned}Ty(x_1,t) - Ty(x_2,t) &= \sqrt{g(x_1) - 2\int_0^{\beta}\int_0^{\tau} y(x_1,s)y\Big(k_1\frac{\partial}{\partial s}y(x_1,s) + k_2 y(x_1,s), s\Big)dsd\tau} \\ &+ \beta\int_0^t y\Big(k_1\frac{\partial}{\partial s}y(x_1,s) + k_2 y(x_1,s), s\Big)ds \\ &- \sqrt{g(x_2) - 2\int_0^{\beta}\int_0^{\tau} y(x_2,s)y\Big(k_1\frac{\partial}{\partial s}y(x_2,s) + k_2 y(x_2,s), s\Big)dsd\tau} \\ &- \beta\int_0^t y\Big(k_1\frac{\partial}{\partial s}y(x_2,s) + k_2 y(x_2,s), s\Big)ds \\ &\le \frac{1}{2\sqrt{\varepsilon_0}}\Bigg[g(x_1) - g(x_2) \\ &- 2\int_0^{\beta}\int_0^{\tau} y(x_1,s)y\Big(k_1\frac{\partial}{\partial s}y(x_1,s) + k_2 y(x_1,s), s\Big)dsd\tau \\ &+ 2\int_0^{\beta}\int_0^{\tau} y(x_2,s)y\Big(k_1\frac{\partial}{\partial s}y(x_2,s) + k_2 y(x_2,s), s\Big)dsd\tau\Bigg] \\ &+ \beta\int_0^t y\Big(k_1\frac{\partial}{\partial s}y(x_1,s) + k_2 y(x_1,s), s\Big)ds\end{aligned}$$

$$- \beta \int_0^t y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right) ds.$$

Hence, by recalling that $y \in \mathbb{Y}$ we get

$$\begin{aligned}
&\int_0^t \left| y\left(k_1 \frac{\partial}{\partial s} y(x_1,s) + k_2 y(x_1,s), s\right) - y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right)\right| ds \\
&\leq L_1 \int_0^t \left| k_1 \frac{\partial}{\partial s} y(x_1,s) + k_2 y(x_1,s) - k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s)\right| ds \\
&\leq L_1 \int_0^t (k_1 L_2 + k_2 L_1)|x_1 - x_2| ds \leq L_1 (k_1 L_2 + k_2 L_1)|x_1 - x_2|,
\end{aligned}$$

and, by adding and subtracting suitable terms, we get

$$\begin{aligned}
&\int_0^\beta \int_0^\tau \left| y(x_1,s) y\left(k_1 \frac{\partial}{\partial s} y(x_1,s) + k_2 y(x_1,s), s\right)\right. \\
&\qquad \left. - y(x_2,s) y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right)\right| ds d\tau \\
&\leq \int_0^\beta \int_0^\tau \left| y(x_1,s) y\left(k_1 \frac{\partial}{\partial s} y(x_1,s) + k_2 y(x_1,s), s\right)\right. \\
&\qquad \left. - y(x_1,s) y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right)\right| ds d\tau \\
&+ \int_0^\beta \int_0^\tau \left| y(x_1,s) y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right)\right. \\
&\qquad \left. - y(x_2,s) y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right)\right| ds d\tau \\
&\leq \int_0^\beta \int_0^\tau \left| y\left(k_1 \frac{\partial}{\partial s} y(x_1,s) + k_2 y(x_1,s), s\right) - y\left(k_1 \frac{\partial}{\partial s} y(x_2,s) + k_2 y(x_2,s), s\right)\right| ds d\tau \\
&+ \int_0^\beta \int_0^\tau |y(x_1,s) - y(x_2,s)| ds d\tau \leq \frac{\beta^2}{2} [L_1((k_1 L_2 + k_2 L_1) + 1)] |x_1 - x_2|.
\end{aligned}$$

Hence, if we write $\frac{1}{2\sqrt{\varepsilon_0}}[L_g + \beta^2(L_1((k_1L_2 + k_2L_1) + 1)] + \beta L_1(k_1L_2 + k_2L_1) \leq L_1$ we get that the operator T is Lipschitz w.r.t. the spatial variable.

Remark 7. Let us observe that the function $\frac{\partial Ty}{\partial t}$ is uniformly Lipschitz w.r.t. the time variable. In fact we get

$$\begin{aligned}
&\left| \frac{\partial}{\partial t} Ty(x_2,t) - \frac{\partial}{\partial t} Ty(x_1,t)\right| \\
=&\left| \beta y\left(k_1 \frac{\partial}{\partial t} y(x_2,t) + k_2 y(x_2,t), t\right) - \beta y\left(k_1 \frac{\partial}{\partial t} y(x_1,t) + k_2 y(x_1,t), t\right)\right| \\
\leq& \beta L_1 \left| k_1 \frac{\partial}{\partial t} y(x_2,t) + k_2 y(x_2,t) - k_1 \frac{\partial}{\partial t} y(x_1,t) - k_2 y(x_1,t)\right| \\
\leq& \beta L_1 (k_1 L_2 + k_2 L_1)|x_1 - x_2| \leq L_2 |x_1 - x_2|.
\end{aligned}$$

Obtaining in this way the condition $\beta L_1(k_1L_2+k_2L_1)\le L_2$. For the uniform Lipschitz property w.r.t the time variable we get

$$\begin{aligned}
&\left|\frac{\partial}{\partial t}Ty(x,t_2)-\frac{\partial}{\partial t}Ty(x,t_1)\right|\\
&=\left|\beta y\left(k_1\frac{\partial}{\partial t}y(x,t_2)+k_2y(x,t_2),t_2\right)-\beta y\left(k_1\frac{\partial}{\partial t}y(x,t_1)+k_2y(x,t_1),t_1\right)\right|\\
&\le\left|\beta y\left(k_1\frac{\partial}{\partial t}y(x,t_2)+k_2y(x,t_2),t_2\right)-\beta y\left(k_1\frac{\partial}{\partial t}y(x,t_2)+k_2y(x,t_2),t_1\right)\right|\\
&\ +\left|\beta y\left(k_1\frac{\partial}{\partial t}y(x,t_2)+k_2y(x,t_2),t_1\right)-\beta y\left(k_1\frac{\partial}{\partial t}y(x,t_1)+k_2y(x_1,t_1),t_1\right)\right|\\
&\le\beta(1+L_1(k_1L_3+1))|t_1-t_2|,
\end{aligned}$$

obtaining in this way the condition $\beta(1+L_1(k_1L_3+1))\le L_3$.

Remark 8. From Remark 4, 5, 6 and 7 we get the conditions

$$\begin{cases}
g(x)-\beta^2\ge\varepsilon_0>0,\\
\|g\|_{L^\infty}\le(1-\beta)^2,\\
\frac{1}{2\sqrt{\varepsilon_0}}[L_g+\beta^2(L_1((k_1L_2+k_2L_1)+1)]+\beta L_1(k_1L_2+k_2L_1)\le L_1,\\
\beta L_1(k_1L_2+k_2L_1)\le L_2,\\
\beta(1+L_1(k_1L_3+1))\le L_3.
\end{cases}\tag{3.4}$$

If the conditions in (3.4) are satisfied, we get, if $w:=Ty$

$$g(x)-2\int_0^\beta\int_0^\tau w(x,s)w\left(k_1\frac{\partial}{\partial s}w(x,s)+k_2w(x,s),s\right)dsd\tau\ge\varepsilon_0>0$$

and then it holds true $T(\mathbb{Y})\subset\mathbb{Y}$.

Remark 9. If (3.4) holds true, then the null function belongs to our set, i.e. $0\in\mathbb{Y}$, thus $\mathbb{Y}\ne\emptyset$.

Example 1. It is possible to verify that $\beta=e^{-8}$, $g(x)=\frac{1}{12}x+\frac14$, $k_1=k_2=\frac14$, $L_1=L_2=L_3=1$, $\varepsilon_0=\frac14-e^{-16}$ satisfies (3.4). Thus the system

$$\begin{cases}
\frac{\partial}{\partial t}y(x,t)=e^{-8}y\big(\frac14\frac{\partial}{\partial t}y(x,t)+\frac14y(x,t),t\big), & x\in(0,1),\ t\in(0,1),\\
\int_0^{e^{-8}}y^2(x,t)dt=e^{-8}\big(\frac{1}{12}x+\frac14\big), & x\in[0,1],
\end{cases}\tag{3.5}$$

will have at least one solution due to the main result of this paper.

3.2 The properties of $\mathbb{Y}$, T and main result

In this section we prove that, if the conditions in (3.4) are satisfied, then $\mathbb{Y}$ is closed and bounded.

Lemma 3.1. *Let us consider the conditions* (3.4)*. Then the space* $\mathbb{Y}$ *is closed.*

Proof. Let us consider a Cauchy sequence of functions $(y_n)_{n\in\mathbb{N}}\subset\mathbb{Y}$ and the associated sequence $\{\big(y_n,\frac{\partial y_n}{\partial t}\big)\}_{n\in\mathbb{N}}$. Due to the completeness of the space they will converge uniformly to $\big(y_\infty,\frac{\partial y_\infty}{\partial t}\big)$ with $y_\infty\in C([0,1]^2,[0,1])$ and $\frac{\partial y_\infty}{\partial t}\in C([0,1]^2,[0,1])$. Let us also consider the fact that $0\le$

$\frac{\partial y_n}{\partial t}(x,t) \le 1$ then, taking the limit for n which runs to infinity, we get $0 \le \frac{\partial y_\infty}{\partial t}(x,t) \le 1$, we may verify the Lipschitz condition on y_∞ and $\frac{\partial y_\infty}{\partial t}$ in a similar way since the associated functions converge uniformly. We need now to verify the last condition. In order to simplify the notation from this point we write $y_n = y_n(x,t)$ and $y_\infty = y_\infty(x,t)$ when the self reference of the functions is not involved. To do this, we observe that

$$\begin{aligned}
&\left|\int_0^\beta \int_0^\tau y_n(x,s) y_n\left(k_1 \frac{\partial}{\partial s} y_n + k_2 y_n, s\right) - y_\infty(x,s) y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right) ds d\tau\right| \\
\le &\left|\int_0^\beta \int_0^\tau \left(y_n(x,s) y_n\left(k_1 \frac{\partial}{\partial s} y_n + k_2 y_n, s\right) - y_n(x,s) y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right)\right) ds d\tau\right. \\
+ &\left.\int_0^\beta \int_0^\tau \left(y_n(x,s) y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right) - y_\infty(x,s) y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right)\right) ds d\tau\right| \\
\le &\int_0^\beta \int_0^\tau \left| y_n\left(k_1 \frac{\partial}{\partial s} y_n + k_2 y_n, s\right) - y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right)\right| ds d\tau + \frac{\beta^2}{2}\|y_n - y_\infty\|_\infty \\
\le &\int_0^\beta \int_0^\tau \left| y_n\left(k_1 \frac{\partial}{\partial s} y_n + k_2 y_n, s\right) - y_n\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right)\right| ds d\tau + \\
+ &\int_0^\beta \int_0^\tau \left| y_n\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right) - y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty + k_2 y_\infty, s\right)\right| ds d\tau + \frac{\beta^2}{2}\|y_n - y_\infty\|_\infty \\
\le &\frac{\beta^2}{2}(L_1(k_1 L_2 + k_2 L_1) + 2) \max\left\{\|y_n - y_\infty\|_\infty, \left\|\frac{\partial y_n}{\partial t} - \frac{\partial y_\infty}{\partial t}\right\|_\infty\right\}.
\end{aligned}$$

Recalling that for every $\varepsilon > 0$ there exists a sufficiently large $n \in \mathbb{N}$ such that $\max\{\|y_n - y_\infty\|_\infty, \|\frac{\partial y_n}{\partial t} - \frac{\partial y_\infty}{\partial t}\|_\infty\} \le \frac{\varepsilon}{\frac{\beta^2}{2}(L_1(k_1 L_2 + k_2 L_1) + 2)}$ then it is immediate to obtain

$$\lim_{n\to\infty} g(x) - \int_0^\beta \int_0^\tau y_n(x,s) y_n\left(k_1 \frac{\partial}{\partial s} y_n + k_2 y_n, s\right) \ge \varepsilon_0,$$

i.e., our goal. □

Lemma 3.2. *Let us consider the conditions* (3.4). *Then the space* $\mathbb{Y}$ *is convex.*

Proof. Let us consider y_1, $y_2 \in \mathbb{Y}$ and $\lambda_1 + \lambda_2 = 1$ with $\lambda_1, \lambda_2 \in [0,1]$. In order to simplify the notation we put $y_{1,2} := \lambda_1 y_1 + \lambda_2 y_2$. We remark that $0 \le y_{1,2} \le 1$, $0 \le \frac{\partial y_{1,2}}{\partial t} \le 1$ and

$$\begin{aligned}
g(x) - 2\int_0^\beta \int_0^\tau y_{1,2}(x,s) y_{1,2}\left(k_1 \frac{\partial}{\partial s} y_{1,2}(x,s) + k_2 y_{1,2}(x,s), s\right) ds d\tau \\
\ge g(x) - 2\int_0^\beta \int_0^\tau y_{1,2}(x,s) ds d\tau \ge g(x) - \beta^2 \ge \varepsilon_0 > 0.
\end{aligned}$$

Furthermore also the uniformly Lipschitz conditions are satisfied. Hence $y_{1,2} \in \mathbb{Y}$. □

We conclude this section stating the following theorem.

Theorem 3.1. *Let us consider the conditions* (3.4) *then the space of function* $\mathbb{Y}$ *is bounded, convex and closed.*

Proof. The Theorem is a consequence of Lemma 3.1, Lemma 3.2 and the fact that $\mathbb{Y} \subset C^1_t([0,1]^2, [0,1])$ and the derivative is bounded. □

3.3 Properties of functional T

In this section we prove that our functional T is continuous and compact. Then the existence of a solution for the integro-differential problem (1.4) will be an immediate consequence of Schauder's Theorem.

Theorem 3.2. *Let us consider the conditions* (3.4) *then the operator* T *is continuous*

Proof. From the proof of Lemma 3.1 we get the first part of the proof. Following the same reasoning we get immediately that

$$\lim_{n\to\infty} \beta \int_0^t y_n\left(k_1 \frac{\partial}{\partial s} y_n(x,s) + k_2 y_n(x,s), s\right) ds = \beta \int_0^t y_\infty\left(k_1 \frac{\partial}{\partial s} y_\infty(x,s) + k_2 y_\infty(x,s), s\right) ds.$$

Furthermore we observe that

$$\begin{aligned} \lim_{n\to\infty} \frac{\partial T y_n(x,t)}{\partial t} &= \lim_{n\to\infty} y_n\left(k_1 \frac{\partial}{\partial t} y_n(x,t) + k_2 y_n(x,t), t\right) = y_\infty\left(k_1 \frac{\partial}{\partial t} y_\infty(x,t) + k_2 y_\infty(x,t), t\right) \\ &= \frac{\partial T y_\infty(x,t)}{\partial t}. \end{aligned}$$

We obtain that our operator is continuous with respect to the norm $\| \, \|_{C^1_t}$. □

Theorem 3.3. *Let us consider the conditions* (3.4) *then the operator* T *is compact*

Proof. Let us consider the bounded sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}$. It is immediate by Remark 4 and 5 to get that our operator T is equibounded. To prove the equicontinuity it is enough to observe that our functional is equi-Lipschitz by Remark 6 in the variable x and

$$0 \le \left| \frac{\partial}{\partial t} T y(x,t) \right| \le \beta \le 1.$$

Furthermore, by Remark 7, also $\frac{\partial}{\partial t} T y(x,t)$ is equi-Lipschitz in both variables, hence we may apply Ascoli-Arzelà theorem on a compact metric space to obtain a convergence subsequence. Then, by a standard diagonal subsequence argument applied to $T y_n$ and $\frac{\partial}{\partial t} T y_n(x,t)$ we obtain the main theorem. □

We now state and prove the main theorem of the paper.

Theorem 3.4. *Let us consider the assumptions* (3.4), *a Lipschitz function* $g : [0,1] \to (0,1]$, *two real constants* $k_1, k_2 > 0$ *such that* $k_1 + k_2 \le 1$ *and* $0 < \beta < \frac{1}{2}$. *Then the integro-differential problem*

$$\begin{cases} \frac{\partial}{\partial t} y(x,t) = \beta y\big(k_1 \frac{\partial}{\partial t} y(x,t) + k_2 y(x,t), t\big), & x \in (0,1), \; t \in (0,1), \\ \int_0^\beta y^2(x,t) dt = \beta g(x) \end{cases}$$

has at least one solution. Furthermore, a solution of this integro-differential problem is bounded and positive.

Proof. From Theorem 3.1 we know that $\mathbb{Y}$ is bounded, closed and convex. Theorems 3.2 and 3.3 allow us to state that T is continuous and compact operator. Hence it is possible to use solution by Schauder's fixed point theorem which assure that our system has, at least, one solution. By construction of $\mathbb{Y}$ we remark that a solution of the integro-differential problem is bounded and positive. □

4 Future directions

We have shown the existence of a positive and bounded solution for a nonlinear self referred integrodifferential equation with an integral boundary condition. Some possible new directions of research include:

- Uniqueness under stronger assumptions by using Banach fixed point theorem.
- Assume weaker assumptions on the initial data, assuming for example $g \in L^\infty([0,1],[0,1])$. This implies a change in the strategy, for example using Peano-Picard iteraction approach to find the existence of a solution.
- Numerical computations in order to find more information about the structure of the solutions.

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