TAMKANG JOURNAL OF MATHEMATICS
Volume vol, Number i, xxx-xxx, 20xx
DOI:10.5556/j.tkjm.57.2026.5975

Golden Poisson Geometry: from algebraic structures to Lie algebras and Poisson manifolds

E. Hinamari Mang-massou,G.F. Wankap Nono, Djagwa Dehainsala and E. Kilanta

Abstract. We introduce a unified framework for **Golden Poisson Geometry**, built around an operator φ satisfying the Golden identity $\varphi^2 = \varphi + I$. By developing the notions of Golden algebras, Golden Poisson algebras, and Golden Lie algebras, we describe how φ governs new compatibility patterns between algebraic operations, Lie brackets, and Poisson tensors. Linear Golden Poisson structures naturally arise on the duals of Golden Lie algebras, revealing Poisson manifolds endowed with intrinsic Fibonacci-type symmetries. Conceptual parallels with the Poisson–Nijenhuis theory of Kosmann-Schwarzbach (1990) show how Golden operators extend classical recursion phenomena. This framework opens new directions in deformation theory and in the geometric study of polynomial identities.

Our main results include the construction of an infinite Golden-Fibonacci hierarchy of compatible Poisson tensors . We establish Golden recurrence relations for hamiltonian vector fields and prove functorial properties showing that the Golden Poisson morphisms transport the entire hierarchical structure. We completely characterize the compatibility conditions between Golden structures and linear Poisson brackets, analyze the associated symplectic foliations, and develop the theory of Golden Poisson morphisms.

Keywords. Golden algebra, Golden Poisson algebras, ideal, subalgebra

1 Introduction

The differential geometry of Golden structure on manifolds was first introduced by M. Crasmareau and C. Hretcanu in [5]. The concepts of a Golden-Riemannian structures and a Golden-Riemannian manifold were subsequently developed in [5]-[13] using a corresponding almost product structure, with various properties of these manifolds being extensively studied. In [7], A. Gezer, N. Cengiz and A.Salimov investigated the integrability conditions for Golden-Riemannian structures.

The concept of an algebra over a field $\mathbb{K}$ is one of the most classical notions in mathematics. Given a field $\mathbb{K}$, an algebra over $\mathbb{K}$ is a vector space $\mathcal{A}$ endowed with a bilinear multiplication

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Received date: December 8, 2025; Published online: May 27, 2026.
2010 *Mathematics Subject Classification.* 58F15, 58F17, 53C35.
Corresponding author: Emmanuel Hinamari Mang-massou.

$\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. This structure provides the natural setting for studying associative and Lie algebras, as well as more elaborate algebraic systems.

Among these, Poisson algebras occupy a distinguished place. Introduced probably in the first time by A. M. Vinogradov and J. S. Krasil'shchik in 1975 under the name "canonical algebra" and J. Braconnier in his short note " algèbres de Poisson (Comptes rendus Ac.Sci) in 1975 (See [18]). It was natural algebraic interpretation of the notion of Poisson structure and Poisson brackets in XIX century in the framework of classical mechanics. A Poisson algebra $(\mathcal{A}, \{\cdot,\cdot\}, \cdot)$ is defined as a commutative associative algebra $(\mathcal{A}, \cdot)$ equipped with a Lie bracket $\{.,.\}$ satisfying the Leibniz rule:

$$\{ab, c\} = a\{b, c\} + \{a, c\}\, b \text{ for any } a, b, c \in \mathcal{A}.$$

Lie algebras and Poisson geometry play a central role, providing the algebraic and geometric backbone for many areas such as differential geometry, mathematical physics and integrable systems.

It is also important to recall that the modern theory of compatibility between Poisson structures and tensorial endomorphisms has a strong foundation. In particular, the seminal paper of Yvette Kosmann-Schwarzbach (1990) on Poisson-Nijenhuis structures (see reference [16]),established how Nijenhuis operators control the compatibility between Poisson brackets, Lie algebroids, and integrable hierarchies. Their framework shows that polynomial identities satisfied by an operator strongly influence the geometry. In this sense, the Golden operator φ, characterized by the relation $\varphi^2 = \varphi + I$, provides a natural analogue of Nijenhuis operators, though governed by a distinct Fibonacci-type identity that produces new deformation patterns and compatibility behaviors.

The purpose of this article is to develop a coherent and systematic theory of Golden Poisson Geometry, linking Golden algebraic structures, Lie algebras, and Poisson manifolds. We introduce Golden algebras, associative algebras equipped with an operator φ satisfying appropriate compatibility conditions. This leads naturally to the notions of Golden Poisson algebras and Golden Poisson modules, where the operator φ interacts in a structured way with Poisson brackets and module operations.

We then develop the theory of Golden Lie algebras $(\mathfrak{g}, [\cdot,\cdot], \varphi)$, where φ is compatible with the Lie bracket. We establish fundamental properties of these algebras, including their integrability conditions, morphisms, and subalgebra structures. Particular attention is devoted to solvable and nilpotent Golden Lie algebras, by examining their structural properties and the possible interactions with the compatibility conditions under consideration. These structures constitute the algebraic basis for constructing linear Golden Poisson structures on the dual of a Golden Lie algebra, yielding new families of Poisson manifolds endowed with intrinsic Golden symmetry. Throughout the article, we analyze in detail how the operator φ interacts with Poisson tensors, derivations, and Lie brackets. We determine when it behaves as a Poisson morphism, a Lie morphism, or a recursion-like operator, and we highlight conceptual parallels with the Poisson–Nijenhuis structures of [16].

Building on this algebraic foundation, we define Linear Golden Poisson algebras $(\mathfrak{g}, \pi_\varphi, \varphi)$ on the dual space $\mathfrak{g}^*$, where the classical linear Poisson bivector π is twisted by the dual map φ^* of the Golden endomorphism. We explore the associated anchor map $\pi_\varphi^\sharp$, verify the Jacobi identity, and establish the notion of a linear Golden Poisson morphism, which provides a direct correspondence between the Golden Lie algebra and the Poisson algebra of function on $\mathfrak{g}^*$.

Our approach reveals several remarkable structures, we construct an infinite Golden-Fibonacci hierarchy of compatible Poisson tensors $\pi_n = F_n\pi_\varphi + F_{n-1}\pi_{id}$ where F_n denotes the Fibonacci sequence, we establish the Golden recurrence of Hamiltonian vector fields

$$X^x_{\varphi^n} = F_n\, X^x_{\varphi} + F_{n-1}\, X^x_{id}$$

and show that the associated recursion operator $\mathcal{R}$ satisfies the Golden identity $\mathcal{R}^2 = \mathcal{R} + \mathrm{id}$, we also demonstrate the functoriality of this construction, providing that Golden Poisson morphisms transport the entire hierarchy of Poisson structures, Hamiltonain vector fields, and Casimir functions in a structure-preserving manner, and we build a Golden tower of Casimir functions $C_n := C \circ (\varphi^*)^n$ satisfying the Fibonacci recurrence $C_{n+2} = C_{n+1} + C_n$. These results establish deep interconnection between algebraic and geometric structures, generalizing classical Lie-Poisson theory and opening new avenues for applications in integrable systems, generalized geometry, and mathematical physics. The emergence of Fibonacci sequences from the simple Golden relation $\varphi^2 = \varphi + \mathrm{Id}_{\mathfrak{g}}$ to generate complete hierarchies of compatible structures represents a particularly powerful aspect of this theory . This article is organized as follows: Section 2 covers the preliminary concepts, while Sections 3 and 4 present our main results on Lie a algebras and linear Golden Poisson structures, including the hierarchical constructions and their functorial properties.

2 Preliminaries

2.1 $\mathbb{K}$-algebra and Poisson algebra

Definition 1. ([1]-[3]) Let $\mathbb{K}$ be a field. An algebra over $\mathbb{K}$(or a $\mathbb{K}$-algebra) is a $\mathbb{K}$-vector space $\mathcal{A}$ with a bilinear map $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Remark 1. An algebra $\mathcal{A}$ is unitary if there exists an element 1 such that $1a = a1$ for all $a \in \mathcal{A}$. Such an element is called the identity in $\mathcal{A}$. We say that the algebra is associative if for all $a, b, c \in \mathcal{A}$, we have $a(bc) = (ab)c$. An algebra $\mathcal{A}$ is commutative if $ab = ba$ for all $a, b \in \mathcal{A}$. We say that the algebra is finite dimensional if the underlying vector space is finite dimensional.

Definition 2. ([1]) Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{K}$-algebras.
A map $f : \mathcal{A} \to \mathcal{B}$ is called an algebra homomorphism if it is $\mathbb{K}$-linear and satisfies the following properties:

1. $f(a \cdot_A b) = f(a) \cdot_B f(b)$, for all $a,\ b \in A$;
2. $f(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ when the algebras are unital.

If $\mathcal{B} = \mathcal{A}$, such a homomorphism is referred to as endomorphism on $\mathcal{A}$. If, in addition, f is bijective, then f is an endomorphism of $\mathbb{K}$-algebras.

Proposition 2.1. *([1]) Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. Then*

1. *$1_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is an algebra endomorphism.*
2. *If $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ are homomorphisms of $\mathbb{K}$-algebras, then their composition $g \circ f : \mathcal{A} \to \mathcal{C}$ is also a homomorphism of $\mathbb{K}$-algebras.*

Proposition 2.2. *([1]) Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of $\mathbb{K}$-algebras. Then*

1. *$Im(f) = \{f(a)/a \in \mathcal{A}\}$ is a subalgebra of $\mathcal{B}$.*
2. *If $\mathcal{A}$ is commutative, $Im(f)$ is also commutative.*

3. *$\ker(f) = \{a \in \mathcal{A} / f(a) = 0\}$ is an ideal of $\mathcal{A}$.*
4. *f is injective if and only if $\ker(f) = \{0\}$.*

We thus arrive at the first isomorphism theorem for algebras:

Theorem 2.1. *([1]) Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of $\mathbb{K}$-algebras. There exists a unique homomorphism $\overline{f} : \mathcal{A}/\ker(f) \to Im(f)$ making the diagram commute*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d, "\pi"] & \mathcal{B} \\
\mathcal{A}/\ker(f) \arrow[r, "\overline{f}"] & Im(f) \arrow[u, "i"]
\end{tikzcd}$$

i.e, $f = i \circ \overline{f} \circ \pi$, where π is the canonical projection and i is the canonical injection. Moreover, $\overline{f}$ is an isomorphism.

Definition 3. ([3]-[4]-[19]) A Poisson algebra is a triple $(\mathcal{A}, \cdot, \{\cdot, \cdot\})$ consisting of an associative algebra $(\mathcal{A}, \cdot)$ over a field $\mathbb{K}$, together with a bilinear map

$$\{\cdot, \cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A},$$

called the Poisson bracket, such that for all $a, b, c \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{K}$, the following properties hold:

1. $\{a, b\} = -\{b, a\}$
2. $\{\lambda a, \mu b\} = \lambda\mu \{b, a\}$
3. $\{ab, c\} = a\{b, c\} + \{a, c\} b$
4. $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0.$

Example 1. Every Lie algebra is a Poisson algebra w.r.t the null associative product: $a \cdot b = 0$, and every associative algebra is a Poisson algebra w.r.t the null Poisson bracket: $\{a, b\} = 0$, such an algebra is called null Poisson algebra.

Definition 4. ([19])Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra.

1. A vector subspaces $\mathcal{B} \subset \mathcal{A}$ is called a Poisson subalgebra if $\mathcal{B} \cdot \mathcal{B} \subseteq \mathcal{B}$ and $\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B}$.
2. A vector subspace $I \subset \mathcal{A}$ is called a Poisson ideal if $I \cdot \mathcal{A} \subset I$ and $\{I, \mathcal{A}\} \subset I$.

Definition 5. ([19])Let $(A, \cdot, \{\cdot, \cdot\})$ be a Poisson algebra.

1. A Poisson ideal $P \subsetneq \mathcal{A}$ is called a Poisson prime ideal, if for all Poisson ideals $I,\ J$ of $\mathcal{A}$,the inclusion $IJ \subset P$ implies that $I \subset P$ and $J \subset P$.
2. The Poisson spectrum of $\mathcal{A}$, denoted by $PSpec(\mathcal{A})$, is the set of all Poisson prime ideals of $\mathcal{A}$.

Definition 6. ([4]-[19])Let $(\mathcal{A}, \cdot_{\mathcal{A}}, \{\cdot, \cdot\}_{A})$ and $(\mathcal{B}, \cdot_{\mathcal{B}}, \{\cdot, \cdot\}_{\mathcal{B}})$ be two Poisson algebras.

A linear map $f : \mathcal{A} \to \mathcal{B}$ is called a morphism of Poisson algebras linear map for all $a, b \in \mathcal{A}$, the following conditions hold:

1. $f(a \cdot_A b) = f(a) \cdot_B f(b)$;

2. $f(\{a,b\}_{\mathcal{A}}) = \{f(a), f(b)\}_{\mathcal{B}}$.

Example 2. Let $(\mathcal{A}, \cdot_{\mathcal{A}}, \{\cdot,\cdot\}_A)$ and $(\mathcal{B}, \cdot_{\mathcal{B}}, \{\cdot,\cdot\}_{\mathcal{B}})$ be two Poisson algebras. Then their tensor product $\mathcal{A} \otimes \mathcal{B}$ carries a natural structure of Poisson algebra defined by

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\}_{\mathcal{A}} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}_{\mathcal{B}},$$

for all $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$.

Moreover, the canonical maps $\mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{B}, \quad a \longmapsto a \otimes 1$ and $\mathcal{B} \longrightarrow \mathcal{A} \otimes \mathcal{B}, \quad b \longmapsto 1 \otimes b$ are morphism of Poisson algebras, and they satisfy $\{a \otimes 1,\ 1 \otimes b\} = 0, \quad$ for all$(a,b) \in \mathcal{A} \times \mathcal{B}$.

Proposition 2.3. *([4])*
Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of Poisson algebras. Then

1. $\ker(f)$ *is an ideal of* $\mathcal{A}$.
2. $Im(f)$ *is a subalgebra of* $\mathcal{B}$
3. *The quotient* $\mathcal{A}/\ker(f)$ *is isomorphic, as a Poisson algebra, to* $f(\mathcal{A}) = Im(f)$.

Proposition 2.4. *([2]) Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of Poisson algebras. If f is bijective, then its inverse map $f^{-1} : \mathcal{B} \to \mathcal{A}$ is also a morphism of Poisson algebras. In this situation, f is called an isomorphism of Poisson algebras.*

Proof. Let $u, v \in \mathcal{B}$. Since f is bijective, there exist unique elements $a, b \in \mathcal{A}$ such that $u = f(a)$ and $v = f(b)$.

We first verify that f^{-1} preserves the associative product:

$$f^{-1}(u \cdot_{\mathcal{B}} v) = f^{-1}(f(a) \cdot_{\mathcal{B}} f(b)) = f^{-1}(f(a \cdot_{\mathcal{A}} b)) = a \cdot_{\mathcal{A}} b = f^{-1}(u) \cdot_{\mathcal{A}} f^{-1}(v).$$

Next, we check compatibility with the Poisson bracket:

$$\begin{aligned} f^{-1}(\{u,v\}_{\mathcal{B}}) &= f^{-1}(\{f(a), f(b)\}_{\mathcal{B}}) \\ &= f^{-1}(f(\{a,b\}_{\mathcal{A}})) \\ &= \{a,b\}_{\mathcal{A}} \\ &= \{f^{-1}(u), f^{-1}(v)\}_{\mathcal{A}}. \end{aligned}$$

Thus, f^{-1} preserves both the associative product and the Poisson bracket, which proves that it is a Poisson algebra morphism. ☐

Definition 7. ([2]) Let $\mathcal{A}$ be a Poisson algebra and E a left $\mathcal{A}$-module. A *Poisson module structure* on E is given by a bilinear map

$$\{\cdot,\cdot\}_E : \mathcal{A} \times E \to E$$

satisfying, for all $a, b \in \mathcal{A}$ and $e \in E$:

1. $\{\{a,b\}_{\mathcal{A}}, e\}_E = \{a, \{b,e\}_E\}_E - \{b, \{a,e\}_E\}_E$;
2. $\{ab, e\}_E = a \cdot \{b,e\}_E + b \cdot \{a,e\}_E$;
3. $\{a, b \cdot e\}_E = \{a,b\}_{\mathcal{A}} \cdot e + b \cdot \{a,e\}_E$.

Example 3. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of Poisson algebras. Then $\mathcal{B}$ becomes naturally a Poisson $\mathcal{A}$-module by defining

$$a \cdot b := f(a) \cdot_{\mathcal{B}} b, \qquad \{a, b\} := \{f(a), b\}_{\mathcal{B}}.$$

Proposition 2.5. *([2]-([3])) Let E and F be two Poisson modules over a Poisson algebra $\mathcal{A}$. Then their tensor product $E \otimes F$ admits a natural Poisson module structure defined by*

$$a \cdot (e \otimes f) = (a \cdot e) \otimes f + e \otimes (a \cdot f),$$
$$\{a, e \otimes f\} = \{a, e\} \otimes f + e \otimes \{a, f\}.$$

Let $\mathcal{A}$ be a Poisson algebra and E a Poisson $\mathcal{A}$-module. One can define a Poisson algebra structure on the direct sum $\mathcal{A} \oplus E$ by setting

$$(a + e) \cdot (a_1 + e_1) = aa_1 + a \cdot e_1 + a_1 \cdot e,$$
$$\{a + e, a_1 + e_1\} = \{a, a_1\}_{\mathcal{A}} + \{a, e_1\}_E - \{a_1, e\}_E.$$

Proposition 2.6. *([2]-([3])) The above construction endows $\mathcal{A} \oplus E$ with a Poisson algebra structure if and only if E is a Poisson $\mathcal{A}$-module. Moreover, E is a square-zero ideal (i.e., $E^2 = 0$), and the canonical projection $\mathcal{A} \oplus E \to \mathcal{A}$ is a morphism of algebras.*

2.2 Lie algebras

Definition 8. ([14]-[15]) A *Lie algebra* over a field $\mathbb{K}$ is a vector space $\mathfrak{g}$ equipped with a bilinear map

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that, for all $u, v, w \in \mathfrak{g}$:

1. (Skew-symmetry) $[u, v] = -[v, u]$;
2. (Jacobi identity) $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$.

The Lie algebra $\mathfrak{g}$ is called *abelian* if $[u, v] = 0$ for all $u, v \in \mathfrak{g}$.

Definition 9. ([14]-[15]) Let $\mathfrak{g}$ be a Lie algebra.

- A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a linear subspace stable under the bracket, i.e. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.
- An ideal $\mathfrak{i} \subset \mathfrak{g}$ is a subspace satisfying $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$. In particular, every ideal is a Lie subalgebra.

Definition 10. ([14]-[15]) A Lie algebra homomorphism between Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ over a field $\mathbb{K}$, is a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ commutes with the bracket operations

$$\varphi([x, y]_{\mathfrak{g}}) = [\varphi(x), \varphi(y)]_{\mathfrak{g}'}, \text{ for all } x, y \in \mathfrak{g}.$$

An isomorphism is a bijective Lie algebra homomorphism, establishing structural equivalence between $\mathfrak{g}$ and $\mathfrak{g}'$.

Definition 11. ([14]-[15]) A Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$ is solvable if its derived series stabilizes at the zero. This series is defined inductively:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \text{ for all } k \geq 1.$$

Solvability requires that $\mathfrak{g}^{(n)} = \{0\}$ for some finite $n \in \mathbb{N}$.

Example 4. The Lie algebra $\mathfrak{t}(2,\mathbb{K})$ of 2×2 upper-triangular matrices provides a fundamental example of a solvable algebra under the commutator bracket $[X,Y] = XY - YX$. the derived series satisfies:

- $\mathfrak{t}(2,\mathbb{K})^{(1)} = \mathfrak{n}(2,\mathbb{K})$, the space of strict upper-triangular matrices;
- $\mathfrak{t}(2,\mathbb{K})^{(2)} = \{0\}$.
  Thus $\mathfrak{t}(2,\mathbb{K})$ is solvable of derived length 2.

Proposition 2.7. *([14]-[15])The class of solvable Lie algebras exhibits the following stability properties*

1. *Every subalgebra of a solvable algebra is solvable.*
2. *If $\mathfrak{i}$ is an ideal in a solvable algebra $\mathfrak{g}$, the quotient algebra $\mathfrak{g}/\mathfrak{i}$ inherits solvability. These closure properties make solvability a categorical property in Lie theory.*
3. *Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be a Lie algebra homomorphism. If $\mathfrak{g}$ is solvable, then the image $\varphi(\mathfrak{g}) \subseteq \mathfrak{h}$ is also solvable.*

Theorem 2.2. *([14])(Lie-Kolchin Theorem) Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field of characteristic zero. Then any finite-dimensional representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, there exits a basis of V in which all operators $\rho(X)$, for $X \in \mathfrak{g}$, are simultaneously represented upper-triangular matrices.*

Definition 12. ([14]-[15]) A Lie algebra $\mathfrak{g}$ is called nilpotent if its lower central series terminates at the zero subalgebra after a finitely many steps. This series is defined recursively by:

$$\mathcal{C}^0\mathfrak{g} = \mathfrak{g}, \quad \mathcal{C}^k\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{k-1}\mathfrak{g}], \text{ for all } k \geq 1.$$

The algebra $\mathfrak{g}$ is nilpotent if there exists $n \in \mathbb{N}$ such that $\mathcal{C}^n\mathfrak{g} = \{0\}$.

Example 5. The three-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$ is defined by the basis $\{x, y, z\}$ with non-trivial bracket $[x, y] = z$, and all other brackets vanishing. Its lower central series is:

$$\mathcal{C}^1\mathfrak{h}_3 = [\mathfrak{h}_3, \mathfrak{h}_3] = \mathbb{K}z, \quad \mathcal{C}^2\mathfrak{h}_3 = [\mathfrak{h}_3, \mathcal{C}^1\mathfrak{h}_3] = \{0\}.$$

Therefore, $\mathfrak{h}_3$ is nilpotent.

Remark 2. Every nilpotent Lie algebra is automatically solvable. This follows from the fact that the derived series $\mathfrak{g}^{(k)}$ is always contained within the lower central series $\mathcal{C}^k\mathfrak{g}$. The converse does not hold in general, as demonstrated by the algebra of upper-triangular matrices.

Proposition 2.8. *([14]-[15]) The case of nilpotent Lie algebras exhibits the following stability properties:*

1. *Any subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ of a nilpotent Lie algebra $\mathfrak{g}$ is nilpotent.*
2. *If $\mathfrak{i} \lhd \mathfrak{g}$ is an ideal of a nilpotent algebra $\mathfrak{g}$, then the quotient algebra $\mathfrak{g}/\mathfrak{i}$ is nilpotent.*
3. *The image of a nilpotent algebra under a Lie algebra homomorphism is nilpotent.*

Theorem 2.3. *([14])(Engel's theorem) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic zero. Then $\mathfrak{g}$ is nilpotent if and only if every element $x \in \mathfrak{g}$, the adjoint operator $ad_x : \mathfrak{g} \longrightarrow \mathfrak{g}$, defined by $ad_x(y) = [x, y]$, is nilpotent (i.e, $(ad_x)^n = 0$ for some n).*

Proposition 2.9. *([14]-[15]) If $\mathfrak{g}$ is nilpotent Lie algebra and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional representation, then there exists a basis of V in which all element of $\rho(\mathfrak{g})$ is represented by strictly upper-triangular matrix.*

Proposition 2.10. *([14]-[15]) Every nonzero nilpotent Lie algebra $\mathfrak{g}$ has non-trivial center:*

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g}/[x, y] = 0 \ \forall y \in \mathfrak{g}\} \neq \{0\}.$$

2.3 Linear Poisson structures

Linear Poisson structures establish a natural connection between Lie algebras and symplectic geometry, emerging canonically on dual spaces of Lie algebras and facilitating important applications in Hamiltonian mechanics

Definition 13. ([19]) Consider a finite-dimensional Lie algebra $\mathfrak{g}$ with dual space $\mathfrak{g}^*$. A Poisson structure on $\mathfrak{g}^*$ is called linear if the Poisson bracket operation preserves the subspace of linear functions.

For any Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ with basis $\{e_i\}$ and structure constants determined by $[e_i, e_j] = c_{ij}^k e_k$, the corresponding linear Poisson bivector field on $\mathfrak{g}^*$ is expressed as

$$\pi = \frac{1}{2} c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \tag{2.1}$$

where the coordinates $\{x_i\}$ on $\mathfrak{g}^*$ are dual to the basis $\{e_i\}$. This bivector field satisfies the Poisson condition $[\pi, \pi]_{SN} = 0$ and generates the Lie-Poisson bracket

$$\{x_i, x_j\} = c_{ij}^k x_k. \tag{2.2}$$

For elements $x,\ y \in \mathfrak{g}$, the associated linear functionals $l_x,\ l_y \in C^\infty(\mathfrak{g}^*)$ are defined by:

$$l_x(\alpha) = \langle \alpha, x\rangle, \quad l_y(\alpha) = \langle \alpha, y\rangle \quad \text{for any } \alpha \in \mathfrak{g}^*, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ represents the dual pairing. In this case, the Poisson structure respects the Lie algebra operation through the identity:

$$\{l_x, l_y\} = l_{[x,y]}, \tag{2.4}$$

where $[\cdot,\cdot]$ is a Lie bracket on $\mathfrak{g}$.

Theorem 2.4. *([19]) For any finite-dimensional vector space V, there is a natural one-to-one correspondence between linear Poisson structures on V^* and Lie algebra structures on V.*

2.4 Golden structures on manifolds

Let M be a smooth manifold.

Definition 14. ([5]) A golden structure on a smooth manifold M is defined as $(1,1)$-tensor field $\varphi \in \Gamma(End(TM))$ that satisfies the fundament equation

$$\varphi^2 = \varphi + \mathrm{id}_{TM}. \tag{2.5}$$

This algebraic relation reflects the characteristic property of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$, which satisfies $\phi^2 = \phi + 1$.

Definition 15. ([5]) A Golden manifold refers to a pair (M, φ) consisting of a smooth manifold M equipped with a Golden structure φ.

Definition 16. ([5]) Given two golden manifolds (M, φ_M) and (N, φ_N), a smooth mapping $\Phi : M \to N$ is called a Golden morphism if its bundle map $d\Phi : TM \to TN$ satisfies the compatibility condition:

$$d\Phi \circ \varphi_M = \varphi_N \circ d\Phi. \tag{2.6}$$

The integrability of a golden structure is characterized by the vanishing of its associated Nijenhuis tensor vanishes defined as:

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \tag{2.7}$$

for all smooth vector fields $X,\ Y \in \mathfrak{X}(M)$, where the bracket denotes the Lie bracket of vector fields.

Proposition 2.11. *([5])*

1. *The eigenvalues of a Golden structure φ are the Golden ratio ϕ and $1-\phi$.*
2. *A Golden-structure φ is an isomorphism on the tangent of the manifold, T_xM, for every $x \in M$.*
3. *It follows that φ is invertible and its is inverse $\widetilde{\varphi} = \varphi^{-1}$ satisfies: $\widetilde{\varphi}^2 = -\widetilde{\varphi} + id$.*

Proposition 2.12. *([5])Let M be a smooth manifold.*

1. *Any Golden-structure φ on M induces two almost product structures on M define as follows*

$$P_- = -\frac{1}{\sqrt{5}}(2\varphi - id) \quad \textit{and} \quad P_+ = \frac{1}{\sqrt{5}}(2\varphi - id_M). \tag{2.8}$$

2. *Conversely, any almost product structure P on M induces two Golden-structures on M define as follows*

$$\varphi_- = \frac{1}{2}(id_M - \sqrt{5}P) \quad \textit{and} \quad \varphi_+ = \frac{1}{2}(id_M + \sqrt{5}P). \tag{2.9}$$

3 Main results

3.1 Golden algebras

We now adopt the definition to algebras.

Definition 17. Let $\mathcal{A}$ be an associative and commutative algebra over a field $\mathbb{K}$. A Golden structure on $\mathcal{A}$ is a linear map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\varphi^2 = \varphi + Id_{\mathcal{A}}$.
In other words, for every element $a \in \mathcal{A}$: $\varphi(\varphi(a)) = \varphi(a) + a$.

Once a Golden structure φ is defined on $\mathcal{A}$, the triple $(\mathcal{A}, \cdot, \varphi)$ is called a Golden algebra.

Example 6. Let $\mathcal{A} = \mathbb{R}^3$ be endowed with its canonical basis. We define a linear endomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ by the matrix

$$\begin{pmatrix} 0 & \frac{1+\sqrt{5}}{4} & \frac{\sqrt{3}+\sqrt{15}}{4} \\ \frac{-1+\sqrt{5}}{4} & \frac{5+3\sqrt{5}}{8} & \frac{\sqrt{3}-\sqrt{15}}{8} \\ \frac{-\sqrt{3}+\sqrt{15}}{4} & \frac{\sqrt{3}-\sqrt{15}}{8} & \frac{7+\sqrt{5}}{8} \end{pmatrix}.$$

A straightforward computation shows that φ satisfies the Golden identity $\varphi^2 = \varphi + id_{\mathbb{R}^3}$. Therefore, the triple $(\mathcal{A}, \cdot, \varphi)$ provides a nontrivial example of Golden algebra.

Example 7. Let $\mathcal{A} = \mathcal{M}_2(\mathbb{R})$ and define $\varphi(X) = PX + XQ$, with $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. We verify the Golden identity $\varphi^2 = \varphi + \mathrm{id}$. Since $P^2 = P + I,\ Q^2 = Q$, we compute $\varphi^2(X) = P(PX + XQ) + (PX + XQ)Q = (P + I)X + 2\,PXQ + XQ$. A direct multiplication shows that $PXQ = \frac{1}{2}X$, hence $\varphi^2(X) = PX + XQ + X = \varphi(X) + X$. Thus, $\varphi^2 = \varphi + \mathrm{id}$. Hence, the triple $(\mathcal{A}, \cdot, \varphi)$ is a Golden algebra.

Proposition 3.1. *Let $(\mathcal{A}, \varphi)$ be a Golden algebra. Then*

1. *The eigenvalues of φ are $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = 1 - \phi = -\frac{1}{\phi}$.*
2. *The space $\mathcal{A}$ decomposes into the direct sum of the corresponding eigenspaces:*

$$\mathcal{A} = \mathcal{A}_\phi \oplus \mathcal{A}_{\overline{\phi}}$$

where $\mathcal{A}_\phi = \{x \in \mathcal{A} | \varphi(x) = \phi x\}$ and $\mathcal{A}_{\overline{\phi}} = \{x \in \mathcal{A} | \varphi(x) = (1 - \phi)(x)\}$.

3. *The linear projections onto eigenspaces are:*

$$P_\phi = \frac{\varphi - \overline{\phi} id}{\phi - \overline{\phi}}, \quad P_{\overline{\phi}} = \frac{\varphi - \phi id}{\overline{\phi} - \phi}, \quad P_\phi^2 = P_\phi, \quad P_{\overline{\phi}}^2 = P_{\overline{\phi}}, \quad P_\phi + P_{\overline{\phi}} = id.$$

Proof. The result follows directly from the definition of a Golden algebra and the standard linear algebra. □

3.2 Ideal of Golden algebra

Definition 18. Let $(\mathcal{A}, \varphi)$ be a Golden algebra. A subset I of $\mathcal{A}$ is called an ideal of $(\mathcal{A}, \varphi)$ if

1. $a \in \mathcal{A}, b \in I \Rightarrow ab \in I$.
2. $\varphi(I) \subset I$, stability under the Golden structure .

Remark 3. An ideal of a Golden algebra is an algebraic ideal that is also stable under the Golden map.

Proposition 3.2. *Let $(\mathcal{A}, \varphi)$ be a Golden algebra and let I, J be φ-stable ideals of $\mathcal{A}$. Then:*

1. *$I \cap J$ is a φ-stable ideal of $\mathcal{A}$.*
2. *$I + J = \{x + y | x \in I, y \in J\}$ is a φ-stable ideal of $\mathcal{A}$.*

Proof.

1. $I \cap J$ is an ideal because the intersection of two ideals is always an ideal. For any $x \in I \cap J$, we have $x \in I$ and $x \in J$, so $\varphi(x) \in I$ and $\varphi(x) \in J$, hence $\varphi(x) \in I \cap J$. Then $\varphi(I \cap J) \subseteq I \cap J$. Hence $I \cap J$ is a φ-stable ideal of $\mathcal{A}$.
2. $I + J$ is an ideal because the sum of two ideals is always an ideal. For any $x + y \in I + J$, with $x \in I, y \in J$, we have $\varphi(x + y) = \varphi(x) + \varphi(y) \in I + J$. Then $\varphi(I + J) \subseteq I + J$. Hence $I + J$ is a φ-stable ideal of $\mathcal{A}$.

□

We obtain the following important result:

Theorem 3.1. *Let $(\mathcal{A}, \varphi)$ be a Golden algebra and I an ideal of $\mathcal{A}$. If $\varphi(I) \subset I$ then*

1. *I is stable under the Golden structure.*
2. *The quotient algebra $\mathcal{A}/I$ inherits a Golden structure $\overline{\varphi}$ satisfying the Golden relation $\overline{\varphi}^2 = \overline{\varphi} + Id$.*

Proof.

1. Consider the linear map $\overline{\varphi} : \mathcal{A}/I \to \mathcal{A}/I$, defined by: $\overline{\varphi}(a + I) = \varphi(a) + I$. Let us show that it is well-defined. If $a + I = b + I$ in A/I , then $a - b \in I$. By φ-stability, $\varphi(a) - \varphi(b) = \varphi(a - b) \in \varphi(I) \subset I$. Hence, $\varphi(a) + I = \varphi(b) + I$. Thus φ induces a well-defined endomorphism $\overline{\varphi}$ on the quotient $\mathcal{A}/I$.
2. Let us now show that $\overline{\varphi}^2 = \overline{\varphi} + Id$. For any $a + I \in \mathcal{A}/I$, we have
$$\begin{array}{rcll} \overline{\varphi}^2(a+I) & = & \overline{\varphi}(\overline{\varphi}(a+I)) & \\ & = & \overline{\varphi}(\varphi(a)+I) & \\ & = & \varphi^2(a)+I & \\ & = & (\varphi(a)+a)+I & \text{Golden relation for } \varphi \\ & = & (\varphi(a)+I)+(a+I) & \\ & = & \overline{\varphi}(a+I)+(a+I) & \text{since } \varphi \text{ is linear} \\ & = & (\overline{\varphi}+Id)(a+I). & \end{array}$$
Hence $\overline{\varphi}^2 = \overline{\varphi} + Id$.

Thus, I is stable under φ and the quotient $\mathcal{A}/I$ naturally inherits a Golden structure. ☐

Definition 19. Let $(\mathcal{A}, \varphi)$ be a Golden algebra and let $I \subset \mathcal{A}$ be a φ-stable ideal. The pair $(\mathcal{A}/I, \overline{\varphi})$ is called a Golden quotient algebra.

Definition 20. Let $(\mathcal{A}, \varphi)$ be a Golden algebra.

A φ-stable ideal $M \subseteq \mathcal{A}$ is called a maximal Golden ideal if $M \neq A$ and there is no φ-stable ideal I such that $M \subset\neq I \subsetneq A$.

Proposition 3.3. *Let $(\mathcal{A}, \varphi)$ be a Golden algebra and let $M \subseteq \mathcal{A}$ be a maximal φ-stable ideal. Then the quotient $\mathcal{A}/M$, endowed with the induced structure $\overline{\varphi}$, is simple as a Golden algebra, i.e., it has no non trivial $\overline{\varphi}$-stable ideals.*

Proof. Let $\pi : \mathcal{A} \to \mathcal{A}/M$ be the canonical projection. The map φ induces a well-defined map $\overline{\varphi}$ on $\mathcal{A}/M$ by

$$\overline{\varphi}(a + M) = \varphi(a) + M$$

since M is φ-stable.

Let $J \subset \mathcal{A}/M$ be a $\overline{\varphi}$-stable ideal. Then $\pi^{-1}(J)$ is a φ-stable ideal of $\mathcal{A}$ containing M. By maximality of M, we must have either

$$\pi^{-1}(J) = M \quad \text{or} \quad \pi^{-1}(J) = A.$$

Applying π, we obtain either $J = \{0\}$ or $J = \mathcal{A}/M$. Hence, $\mathcal{A}/M$has no non-trivial $\overline{\varphi}$-stable ideals and is therefore simple as a Golden algebra. ☐

3.3 Golden subalgebras

Definition 21. Let $(\mathcal{A}, \varphi)$ be a Golden algebra. A subalgebra $\mathcal{B} \subset \mathcal{A}$ is called a Golden subalgebra if it is stable under φ, that is $\varphi(\mathcal{B}) \subseteq \mathcal{B}$.

Proposition 3.4. *Let $(\mathcal{A}, \varphi)$ be a Golden algebra.*

1. *If $I \subseteq \mathcal{A}$ is a φ-stable ideal, then I is Golden subalgebra of A.*
2. *Conversely, a φ-stable subalgebra $\mathcal{B} \subset \mathcal{A}$ is a Golden subalgebra, but it is not necessarily an ideal of $\mathcal{A}$.*

Proof. 1. Since I is an ideal of $\mathcal{A}$, it is in particular a subalgebra of $\mathcal{A}$. Moreover, as I is φ-stable, we have $\varphi(I) \subseteq I$. Hence the pair $(I, \varphi|_I)$ forms a Golden algebra, i.e, a Golden subalgebra of A.

2. Let $\mathcal{B} \subset \mathcal{A}$ be a φ-stable subalgebra. Then $\mathcal{B}$ is closed under the algebra operations and stable under φ, so $(\mathcal{B}, \varphi|_{\mathcal{B}})$ is a Golden algebra. However, in general, $\mathcal{B}$ is not stable under multiplication by arbitrary elements of $\mathcal{A}$, so it need may not be an ideal.

□

Thus, every φ-stable ideal is automatically a Golden subalgebra, whereas a Golden subalgebra need not be an ideal.

3.4 Golden algebra morphisms

Definition 22. Let $(\mathcal{A}, \cdot_{\mathcal{A}}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \cdot_{\mathcal{B}}, \varphi_{\mathcal{B}})$ be Golden algebras. A Golden algebra morphism is a linear map $f : \mathcal{A} \to \mathcal{B}$ satisfying two conditions

1. $f(a \cdot_{\mathcal{A}} b) = f(a) \cdot_{\mathcal{B}} f(b), \forall a, b \in \mathcal{A}$.
2. f is compatible with the Golden structures:$f \circ \varphi_{\mathcal{A}} = \varphi_{\mathcal{B}} \circ f$.

In others terms, f preserves multiplication and f is compatible with the Golden structures.

Remark 4. If f is bijective,it is called an isomorphism of Golden algebras.
If such map exists, we say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as Golden algebras, and write $\mathcal{A} \cong \mathcal{B}$.

Proposition 3.5. *The composition of two Golden algebra morphisms is again a Golden algebra morphism.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be two Golden algebra morphisms.

1. Since f and g preserve multiplication, we have
$$\begin{aligned} (g \circ f)(a \cdot_{\mathcal{A}} b) &= g(f(a \cdot_{\mathcal{A}} b)) \\ &= g(f(a) \cdot_{\mathcal{B}} f(b)) \\ &= g(f(a)) \cdot_{\mathcal{B}} g(f(b)). \end{aligned}$$
2. Since f and g commute with the Golden structures, we have
$$\begin{aligned} (g \circ f) \circ \varphi_{\mathcal{A}} &= g \circ (f \circ \varphi_{\mathcal{A}}) \\ &= g \circ (\varphi_{\mathcal{B}} \circ f) \\ &= (g \circ \varphi_{\mathcal{B}}) \circ f \\ &= (\varphi_{\mathcal{C}} \circ g) \circ f \\ &= \varphi_{\mathcal{C}} \circ (g \circ f). \end{aligned}$$
Thus, $g \circ f$ is a Golden algebra morphism.

□

Proposition 3.6. *The inverse of a Golden algebra isomorphism is again a Golden algebra morphism.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden algebra isomorphism. Since f is bijective, it has an inverse $f^{-1} : \mathcal{B} \to \mathcal{A}$. By Proposition 2.4, the image $f^{-1} : \mathcal{B} \to \mathcal{A}$ preserves the multiplication. Let us show that it is compatible with the Golden structure.
For all $u = f(a) \in \mathcal{B}$, we have:
$f^{-1} \circ \varphi_{\mathcal{B}}(u) = f^{-1}(\varphi_{\mathcal{B}}(f(a))) = f^{-1}(f(\varphi_{\mathcal{A}}(a))) = \varphi_{\mathcal{A}}(a) = \varphi_{\mathcal{A}}(f^{-1}(u)) = \varphi_{\mathcal{A}} \circ f^{-1}(u)$. Thus f^{-1} is compatible with the Golden structures and therefore f^{-1} is a Golden algebra morphism. □

Proposition 3.7. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden algebra morphism.*

1. *The kernel* $\ker(f) = \{a \in \mathcal{A} | f(a) = 0\}$ *of f is a Golden ideal of $\mathcal{A}$.*
2. *The image $Im(f) = \{f(x) \in \mathcal{B} | x \in \mathcal{A}\}$ of f is a Golden subalgebra of $\mathcal{B}$.*

Proof.

1. Since f is linear, $\ker(f)$ is a subspace of $\mathcal{A}$. For all $a \in \mathcal{A}, x \in \ker(f)$: $f(a \cdot_{\mathcal{A}} x) = f(a) \cdot_{\mathcal{B}} f(x) = f(a) \cdot_{\mathcal{B}} 0 = 0$,hence $a \cdot_{\mathcal{A}} x \in \ker(f)$. For all $x \in \ker(f)$ $f(\varphi_{\mathcal{A}}(x)) = \varphi_{\mathcal{B}}(f(x)) = \varphi_{\mathcal{B}}(0) = 0$, then $\varphi_{\mathcal{A}}(x) \in \ker(f)$,we conclude that $\varphi_{\mathcal{A}}(\ker(f)) \subset \ker(f)$. Thus, $\ker(f)$ is a Golden ideal of $\mathcal{A}$.
2. Since f is linear, $Im(f)$ is a subspace of $\mathcal{B}$. For $x, y \in Im(f)$: $f(x) \cdot_y f(b) = f(x \cdot_A y) \in Im(f)$ and $\varphi_{\mathcal{B}}(f(x)) = f(\varphi_{\mathcal{A}}(x)) \in Im(f)$.
Hence, $Im(f)$ is a Golden subalgebra of $\mathcal{B}$.

□

We get the first isomorphism theorem for Golden algebras:

Theorem 3.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden algebra morphism. Then*

1. *The quotient $\mathcal{A}/\ker(f)$ is a Golden algebra.*
2. *There exists a natural isomorphism of Golden algebras:$\mathcal{A}/\ker(f) \cong Im(f)$.*

Proof. By the classical first isomorphism theorem for algebras $\mathcal{A}/\ker(f) \cong Im(f)$(see Theorem 2.1) as algebras. Since f is a Golden algebra morphism, the Golden structure φ is compatible with f, i.e, $f \circ \varphi_{\mathcal{A}} = \varphi_{\mathcal{B}} \circ f$. This compatibility descends naturally to the quotient and is preserved in the image (see Proposition 3.7). Therefore, $\mathcal{A}/\ker(f)$ and Imf inherit the Golden structure, making them Golden algebras, and the isomorphism is a Golden algebra morphism. □

Corollary 3.3. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden algebra morphism. Then the canonical projection $\pi_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}/\ker(f)$ is a Golden algebra morphism.*

3.5 Golden Poisson algebras

3.5.1 Golden Poisson structures

We now introduce a structure which combines both Golden algebras and Poisson algebras.

Definition 23. A Golden Poisson algebra is a quadruple $(\mathcal{A}, \cdot, \{\cdot, \cdot\}, \varphi)$, where

1. $(\mathcal{A}, \cdot, \{\cdot, \cdot\})$ is a Poisson algebra;
2. $(\mathcal{A}, \cdot, \varphi)$ is a Golden algebra;
3. the Golden structure φ is compatible with both structures:
   $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b), \varphi(\{a, b\}) = \{\varphi(a), \varphi(b)\}$, for any $a, b \in \mathcal{A}$.

Proposition 3.8. *Let $(\mathcal{A}, \cdot, \{\cdot, \cdot\}, \varphi)$ be a Golden Poisson algebra. Then*

1. *$Im(\varphi)$ is a Poisson subalgebra of $\mathcal{A}$.*
2. *$\ker \varphi$ is a Poisson ideal f $\mathcal{A}$.*

Proof.
1. Let $u = \varphi(a), v = \varphi(b) \in Im(\varphi)$. Then $u \cdot v = \varphi(a) \cdot \varphi(b) = \varphi(a \cdot b) \in Im(\varphi)$. In addition, we have: $\{u, v\} = \{\varphi(a), \varphi(b)\} = \varphi(\{a, b\}) \in Im(\varphi)$.
   Hence $Im(\varphi)$ is a Poisson subalgebra of $\mathcal{A}$.
2. • If $a \in Ker\varphi$ and $b \in \mathcal{A}$, $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) = 0 \cdot \varphi(b) = 0$, hence $a \cdot b \in \ker \varphi$.
   • If $a \in \ker \varphi$ and $b \in \mathcal{A}$, we have $\varphi(\{a, b\}) = \{\varphi(a), \varphi(b)\} = \{0, \varphi(b)\} = 0$. Thus $\ker \varphi$ is a Poisson ideal of $\mathcal{A}$.

□

Theorem 3.4. *Let $(\mathcal{A}, \cdot, \{., .\}, \varphi)$ be a Golden Poisson algebra. Then the induced map $\widetilde{\varphi}$: $\mathcal{A}/\ker \varphi \to Im\varphi$, defined by $\widetilde{\varphi}([a]) = \varphi(a)$ is bijective.*

Proof. The argument for well-definedness, injectivity, surjectivity, and preservation of the associative product are similar to those in Theorem 3.2. We only prove that $\widetilde{\varphi}$ is a Poisson morphism. Let $[a], [b] \in \mathcal{A}/Ker\varphi$ with representatives $a, b \in \mathcal{A}$.
Then $\widetilde{\varphi}(\{[a], [b]\}) = \widetilde{\varphi}([\{a, b\}]) = \varphi(\{a, b\})$. But $\varphi(\{a, b\}) = \{\varphi(a), \varphi(b)\}$.
Then $\widetilde{\varphi}(\{[a], [b]\}) = \widetilde{\varphi}([\{a, b\}]) = \{\widetilde{\varphi}([a]), \widetilde{\varphi}([b])\}$. □

3.5.2 Golden Poisson algebra morphisms

Let $(\mathcal{A}, \cdot_{\mathcal{A}}, \{\cdot, \cdot\}_{\mathcal{A}}, \varphi_{\mathcal{A}})$ and $(\mathcal{B}, \cdot_{\mathcal{B}}, \{\cdot, \cdot\}_{\mathcal{B}}, \varphi_{\mathcal{B}})$ be two Golden Poisson algebras.

Definition 24. A Golden Poisson algebra morphism is linear map $f : \mathcal{A} \to \mathcal{B}$ satisfying the following properties:

1. $f(a \cdot_A b) = f(a) \cdot_{\mathcal{B}} f(b)$
2. $f(\{a, b\}_{\mathcal{A}}) = \{f(a), f(b)\}_{\mathcal{B}}$
3. $f \circ \varphi_{\mathcal{A}} = \varphi_{\mathcal{B}} \circ f$ (Golden structure compatibility),
   for any $a, b \in \mathcal{A}$.

Remark 5. f preserves the three structures simultaneously: associative multiplication, Poisson bracket, and Golden map.

Proposition 3.9. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden Poisson algebra morphism. If f is bijective, then its inverse $f^{-1} : \mathcal{B} \to \mathcal{A}$ is also a Golden Poisson algebra morphism.*

Proof. By Proposition 2.4, the inverse f^{-1} of a bijective Poisson algebra morphism is itself a Poisson algebra morphism. By Proposition 3.6, f^{-1} is compatible with the Golden structure. Then combining these results, f^{-1} preserves the multiplication, the Poisson bracket and the Golden structure. Hence $f^{-1} : \mathcal{B} \to \mathcal{A}$ is also a Golden Poisson algebra morphism. □

The following theorem shows that morphisms of Golden Poisson algebras respect the iterative action of the Golden structure and fully preserve the compatibility between multiplication, the Poisson bracket and Golden map.

Theorem 3.5. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden Poisson algebra morphism. Then, for all $a, b \in A$ and all $n \in \mathbb{N}$, we have*

$$f(\varphi^n_{\mathcal{A}}(a \cdot_{\mathcal{A}} b)) = \varphi^n_{\mathcal{B}}(f(a) \cdot_{\mathcal{B}} f(b)) \text{ and } f(\varphi^n_{\mathcal{A}} \{a, b\}_{\mathcal{A}}) = \varphi^n_{\mathcal{B}}(\{f(a), f(b)\}_{\mathcal{B}}).$$

Proof. We proceed by induction on n.
For $n = 1$, by the definition of a morphism,
$f(\varphi_{\mathcal{A}}(a \cdot_{\mathcal{A}} b)) = \varphi_{\mathcal{B}}(f(a) \cdot_{\mathcal{B}} f(b))$ and $f(\varphi_{\mathcal{A}} \{a, b\}_{\mathcal{A}}) = \varphi_{\mathcal{B}}(\{f(a), f(b)\}_{\mathcal{B}})$.
Assume that the property holds for $k \geq 1$,i.e,
$f(\varphi^k_{\mathcal{A}}(a \cdot_{\mathcal{A}} b)) = \varphi^k_{\mathcal{B}}(f(a) \cdot_{\mathcal{B}} f(b))$ and $f(\varphi^k_{\mathcal{A}} \{a, b\}_{\mathcal{A}}) = \varphi^k_{\mathcal{B}}(\{f(a), f(b)\}_{\mathcal{B}})$.
Let us show that the property is true for $k + 1$, we have:

$$\begin{aligned} f(\varphi^{k+1}_{\mathcal{A}}(a \cdot_{\mathcal{A}} b)) &= f(\varphi_{\mathcal{A}}(\varphi^k_{\mathcal{A}}(a \cdot_{\mathcal{A}} b))) \\ &= \varphi_{\mathcal{B}}(f(\varphi^k_{\mathcal{A}}(a \cdot_{\mathcal{A}} b))) \quad \text{by the compatibility of } f \text{ with } \varphi \\ &= \varphi_{\mathcal{B}}(\varphi^k_{\mathcal{B}}(f(a) \cdot_{\mathcal{B}} f(b))) \quad \text{by the induction hypothesis} \\ &= \varphi^{k+1}_{\mathcal{B}}(f(a) \cdot_{\mathcal{B}} f(b)). \end{aligned}$$

Similarly, for the Poisson bracket: $f(\varphi^{k+1}_{\mathcal{A}} \{a, b\}_{\mathcal{A}}) = \varphi^{k+1}_{\mathcal{B}}(\{f(a), f(b)\}_{\mathcal{B}})$.
Thus, the property holds for $k + 1$. By the principe of induction, the property holds for all $n \in \mathbb{N}$.
Thus,

$$f(\varphi^n_{\mathcal{A}}(a \cdot_{\mathcal{A}} b)) = \varphi^n_{\mathcal{B}}(f(a) \cdot_{\mathcal{B}} f(b)) \text{ and } f(\varphi^n_{\mathcal{A}} \{a, b\}_{\mathcal{A}}) = \varphi^n_{\mathcal{B}}(\{f(a), f(b)\}_{\mathcal{B}}).$$

□

Proposition 3.10. *Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be two Golden Poisson algebra morphisms. Then the composition $g \circ f : \mathcal{A} \to \mathcal{C}$ is also a morphism of Golden Poisson algebras.*

Proof. Since the proof is similar to the one for Golden algebra morphisms, we only check the preservation of the Poisson bracket. For all $a, b \in \mathcal{A}$:

$$\begin{aligned} (g \circ f)(\{a, b\}_{\mathcal{A}}) &= g(f \{a, b\}_{\mathcal{A}}) \\ &= g(\{f(a), f(b)\}_{\mathcal{B}}) \\ &= \{g(f(a)), g(f(b))\}_{\mathcal{C}} \\ &= \{(g \circ f)(a), (g \circ f)(b)\}_{\mathcal{C}}\,. \end{aligned}$$

Hence, $g \circ f$ preserves the Poisson bracket, and by analogy with the Golden algebra case, it also preserves multiplication and the Golden map. Therefore, $g \circ f$ is a Golden Poisson algebra morphism. □

3.5.3 Golden Poisson ideals and Golden Poisson subalgebra

Definition 25. Let $(\mathcal{A}, \cdot, \{\cdot, \cdot\}, \varphi)$ be a Golden Poisson algebra. A subspace $I \subseteq \mathcal{A}$ is called a Golden Poisson ideal or GP-ideal if:

1. I is a two-side ideal of the associative algebra $(\mathcal{A}, \cdot)$;
2. I is Poisson ideal;
3. I is φ-stable i.e $\varphi(I) \subseteq I$.

We denote by $Id^{GP}(\mathcal{A})$ for the set all GP-ideals of $\mathcal{A}$.

Definition 26. Let $(\mathcal{A}, \cdot, \{\cdot,\cdot\}, \varphi)$ be a Golden Poisson algebra and let $I \subseteq \mathcal{A}$ be GP-ideal.

1. A maximal GP-ideal is a proper GP-ideal which maximal with respect to inclusion in $Id^{GP}(\mathcal{A})$.
2. A prime GP-ideal $P \subset \mathcal{A}$, with $P \neq \mathcal{A}$, is a GP-ideal such that for all GP-ideals $I, J \subset \mathcal{A}$,
$$IJ \subseteq P \implies I \subseteq P \text{ or } J \subseteq P.$$

Definition 27. Let $(\mathcal{A}, \cdot, \{\cdot,\cdot\}, \varphi)$ be a Golden Poisson algebra. A subspace $\mathcal{S} \subseteq \mathcal{A}$ is a Golden Poisson subalgebra if:

1. for all $x, y \in \mathcal{S}, x \cdot y \in \mathcal{S}$;
2. for all $x, y \in \mathcal{S}$, $\{x, y\} \in \mathcal{S}$;
3. $\mathcal{S}$ is φ-stable under the Golden map i.e for all $x \in \mathcal{S}, \varphi(x) \in \mathcal{S}$.

We get the isomorphism theorem for Golden Poisson algebras:

Theorem 3.6. *Let $f : \mathcal{A} \to \mathcal{B}$ be a Golden Poisson algebra morphism. Then:*

1. *$Kerf$ is a GP-ideal of $\mathcal{A}$;*
2. *$f(\mathcal{A})$ is a Golden Poison subalgebra of $\mathcal{B}$.*
3. *The quotient $\mathcal{A}/\ker(f)$ is isomorphic, as a Golden Poisson algebra, to $f(\mathcal{A})$.*

Proof.

1. It is standard that the kernel of an associative algebra morphism is an ideal, and that the kernel of a Poisson algebra morphism is a Poisson ideal, hence $Kerf$ is both ideal and a Poisson ideal(see Proposition 2.3). It remains to check invariance under the Golden map φ. For any $a \in Kerf$ we have $f(\varphi_{\mathcal{A}}(a)) = \varphi_{\mathcal{B}}(f(a)) = \varphi_{\mathcal{B}}(0) = 0$, so $\varphi_{\mathcal{A}}(\ker(f)) \subset Ker(f)$.
2. It is classical that the image of an algebra morphism is a subalgebra and the image of a Poisson algebra morphism is a Poisson subalgebra (see Proposition2.3). Thus we only need to check the stability under the Golden map. Let $u = f(x) \in f(\mathcal{A})$. Then $\varphi_{\mathcal{B}}(u) = \varphi_{\mathcal{B}}(f(x)) = f(\varphi_{\mathcal{A}}(x)) \in f(\mathcal{A})$.
Hence $f(\mathcal{A})$ is a Golden Poison subalgebra of $\mathcal{B}$.
3. Define $\widetilde{f} : \mathcal{A}/\ker(f) \to Im(f)$ by $\widetilde{f}(x + \ker(f)) = f(x)$.
Combine the algebraic isomorphism result (Proposition 2.1), the Poisson isomorphism theorem (Theorem2.3) an the Golden isomorphism result (Theorem 3.2): each structure descends to the quotient and is preserved by $\widetilde{f}$, hence $\widetilde{f}$ is an isomorphism of Golden Poisson algebras.

□

Remark 6. Let $(\mathcal{A}, \cdot, \{\cdot,\cdot\}, \varphi)$ be a Golden Poisson algebra.

1. The intersection of any family of Golden Poisson ideals is a Golden Poisson ideal.
2. The sum of any family of Golden Poisson ideals is a Golden Poisson ideal.

3.6 Golden Poisson modules

Definition 28. Let $(\mathcal{A}, \cdot, \{.,.\}, \varphi)$ be a Golden Poisson algebra. A Golden Poisson module (E, φ_E) over $\mathcal{A}$ is a Poisson module $(E, \{.,.\}_E)$ over $\mathcal{A}$ endowed with a linear endomorphism $\varphi_E : E \to E$ satisfying:

1. $\varphi_E(a \cdot e) = \varphi(a) \cdot \varphi_E(e)$;
2. $\{\varphi(a), \varphi_E(e)\}_E = \varphi_E(\{a, e\}_E)$, for any $a \in \mathcal{A}$ and $e \in E$.

Definition 29. Let (E, φ_E) be a Poisson module over a Poisson algebra $(\mathcal{A}, \cdot, \{.,.\}, \varphi)$.
We say that (E, φ_E) is strong if, in addition to the classical Poisson module conditions and the compatibility with φ, it satisfies the mixed Leibniz identity:

$$\{a, b \cdot e\} = \{a, b\} \cdot \varphi_E(e) + \varphi(b) \cdot \{a, e\}, \varphi_E(\{a, b \cdot e\}) = \{\varphi(a), \varphi_E(b \cdot e)\},$$

for all $a, b \in \mathcal{A}$ and $e \in E$.

Lemma 3.1. *Let (E, φ_E) and (F, φ_F) be Poisson modules over a Poisson algebra $(\mathcal{A}, \cdot, \{.,.\}, \varphi)$. Then the direct sum $E \oplus F$ is a Poisson module with*
$a \cdot (e, f) = (a \cdot e, a \cdot f), \{a, (e, f)\} = (\{a, e\}, \{a, f\})$.

Theorem 3.7. *Let (E, φ_E) and (F, φ_F) be Golden Poisson modules over a Golden Poisson algebra $(\mathcal{A}, \cdot, \{.,.\}, \varphi)$. Then:*

1. *their direct sum $E \oplus F$ with $\varphi_{E \oplus F}(e, f) = (\varphi_E(e), \varphi_F(f))$ is a strong Golden Poisson module.*
2. *their tensor product $E \otimes F$ with $\{a, e \otimes f\} = \{a, e\} \otimes \varphi_F(f) + \varphi_E(e) \otimes \{a, f\}$ and $\varphi_{E \otimes F}(e \otimes f) = \varphi_E(e) \otimes \varphi_F(f)$ is a strong Golden Poisson module.*

Proof. 1. The Lie-module and Leibniz conditions are classical and inherited from E and F. We only check compatibility of the Golden structure.

- Compatibility of the Golden structure with the Poisson bracket.

For any $a \in \mathcal{A}, (e, f) \in E \oplus F$, we compute:

$$\begin{aligned} \varphi_{E \oplus F}(\{a, (e, f)\} &= \varphi_{E \oplus F}(\{a, e\}, \{a, f\}) \\ &= (\varphi_E(\{a, e\}), \varphi_F(\{a, f\})) \\ &= (\{\varphi(a), \varphi_E(e)\}, \{\varphi(a), \varphi_F(f)\}) \\ &= \{\varphi(a), (\varphi_E(e), \varphi_F(f))\} \\ &= \{\varphi(a), \varphi_{E \oplus F}((e, f))\}. \end{aligned}$$

Thus the Golden compatibility with the Poisson bracket holds.

- Compatibility of the Golden structure with multiplication. For any $a \in \mathcal{A}, (e, f) \in E \oplus F$, we have:

$$\begin{aligned} \varphi_{E \oplus F}(a \cdot (e, f)) &= \varphi_{E \oplus F}(a \cdot e, a \cdot f)) \\ &= (\varphi_E(a \cdot e), \varphi_F(a \cdot f)) \\ &= (\varphi(a) \cdot \varphi_E(e), \varphi(a) \cdot \varphi_F(f)) \\ &= \varphi(a) \cdot (\varphi_E(e), \varphi_F(f)) \\ &= \varphi(a) \cdot \varphi_{E \oplus F}(e, f). \end{aligned}$$

- Mixed Leibniz identity. For all $a, b \in \mathcal{A}, (e, f) \in E \oplus F$, we have:

$$\begin{aligned} \{a, b \cdot (e, f)\} &= (\{a, b \cdot e\}, \{a, b \cdot f\}) \\ &= (\{a, b\} \cdot \varphi_E(e) + \varphi(b) \cdot \{a, e\}, \{a, b\} \cdot \varphi_F(f) + \varphi(b) \cdot \{a, f\}) \\ &= \{a, b\} \cdot \varphi_{E \oplus F}(e, f) + \varphi(b) \cdot \{a, (e, f)\}; \end{aligned}$$

$$\begin{aligned}
\varphi_{E\oplus F}(\{a, b\cdot(e,f)\} &= \varphi_{E\oplus F}(\{a, b\cdot e\}, \{a, b\cdot f\}) \\
&= (\varphi_E(\{a, b\cdot e\}), \varphi_F(\{a, b\cdot f\})) \\
&= (\{\varphi(a), \varphi_E(b\cdot e)\}, \{\varphi(a), \varphi_F(b\cdot f)\}) \\
&= (\{\varphi(a), \varphi(b)\cdot\varphi_E(e)\}, \{\varphi(a), \varphi(b)\cdot\varphi_F(f)\}) \\
&= \{\varphi(a), (\varphi(b)\cdot\varphi_E(e), \varphi(b)\cdot\varphi_F(f))\} \\
&= \{\varphi(a), \varphi_{E\oplus F}(b\cdot(e,f))\}.
\end{aligned}$$

Hence, $E \oplus F$ is a strong Golden Poisson module.

2. The Lie-module and Leibniz conditions are classical and inherited from E and F. We only check compatibility of the Golden structure.

   • Compatibility of the Golden structure with the Poisson bracket.

   For any $a \in \mathcal{A}, (e,f) \in E \oplus F$, we compute:

$$\begin{aligned}
\varphi_{E\otimes F}(\{a, (e\otimes f)\}) &= \varphi_{E\otimes F}(\{a, e\}\otimes f + e\otimes\{a, f\})) \\
&= \varphi_E(\{a, e\})\otimes\varphi_F(f) + \varphi_E(e)\otimes\varphi_F(\{a, f\})) \\
&= \{\varphi(a), \varphi_E(e)\}\otimes\varphi_F(f) + \varphi_E(e)\otimes\{\varphi(a), \varphi_F(f)\} \\
&= \{\varphi(a), \varphi_E(e)\otimes\varphi_F(f)\} \\
&= \{\varphi(a), \varphi_{E\otimes F}(e\otimes f)\}.
\end{aligned}$$

   • Compatibility of the Golden structure with multiplication. For any $a \in \mathcal{A}, (e,f) \in E\oplus F$, we have:

$$\begin{aligned}
\varphi_{E\otimes F}(a\cdot(e\otimes f)) &= \varphi_{E\oplus F}((a\cdot e)\otimes f) \\
&= \varphi_E(a\cdot e)\otimes\varphi_F(f)) \\
&= (\varphi(a)\cdot\varphi_E(e))\otimes\varphi_F(f)) \\
&= \varphi(a)\cdot(\varphi_E(e)\otimes\varphi_F(f)) \\
&= \varphi(a)\cdot\varphi_{E\otimes F}(e\otimes f).
\end{aligned}$$

   • Mixed Leibniz identity. For all $a, b \in \mathcal{A}, (e,f) \in E \oplus F$, we have:

$$\begin{aligned}
\{a, b\cdot(e\otimes f)\} &= \{a, (b\cdot e)\otimes\varphi_F(f) + \varphi_E(e)\otimes(b\cdot f)\} \\
&= (\{a, b\}\cdot\varphi_E(e) + \varphi(b)\cdot\{a, e\})\otimes\varphi_F^2(f) + \varphi_E^2(e)\otimes(\{a, b\}\cdot\varphi_F(f) + \varphi(b)\cdot\{a, e\}) \\
&= \{a, b\}\cdot\varphi_{E\otimes F}(e\otimes f) + \varphi(b)\cdot\{a, e\otimes f\};
\end{aligned}$$

$$\begin{aligned}
\varphi_{E\otimes F}(\{a, b\cdot(e\otimes f)\}) &= \varphi_{E\otimes F}(\{a, b\cdot e\}\otimes f + (b\cdot e)\otimes\{a, f\})) \\
&= \varphi_E(\{a, b\cdot e\}\otimes\varphi_F(f) + \varphi_E(b\cdot e)\otimes\varphi_F(\{a, f\})) \\
&= \{\varphi(a), \varphi_E(b\cdot e)\}\otimes\varphi_F(f) + \varphi_E(b\cdot e)\otimes\{\varphi(a), \varphi_F(f)\} \\
&= \{\varphi(a), \varphi_E(b\cdot e)\otimes\varphi_F(f)\} \\
&= \{\varphi(a), \varphi_{E\otimes F}(b\cdot(e\otimes f))\}.
\end{aligned}$$

   Hence, $E \otimes F$ is a strong Golden Poisson module.

□

Definition 30. Let (E, φ_E) and (F, φ_F) be Golden Poisson modules over a Golden Poisson algebra $(\mathcal{A}, \cdot, \{.,.\}, \varphi)$. A morphism of Golden Poisson modules is a morphism of Poisson modules $f : E \to F$ such that $f \circ \varphi_E = \varphi_F \circ f$.

Proposition 3.11. *Let $f : (E, \varphi_E) \to (F, \varphi_F)$ and $g : (F, \varphi_F) \to (H, \varphi_H)$ be Golden Poisson modules morphisms. Then the composition $g \circ f : E \to H$ is a morphism of Golden Poisson modules.*

Proof. We assume classically that the composition $g \circ f$ is a morphism of Poisson modules. It remains to check the Golden compatibility. Since f and g Golden morphisms, we have $f \circ \varphi_E = \varphi_F \circ f$ and $g \circ \varphi_F = \varphi_H \circ g$. For any $m \in M$, one has:

$$\begin{aligned}(g\circ f)(\varphi_E(m)) &= g(f(\varphi_E(m)))\\ &= g(\varphi_F(f(m)))\\ &= \varphi_H(g(f(m)))\\ &= \varphi_H((g\circ f)(m)).\end{aligned}$$

Therefore, $g\circ f$ is indeed a morphism of Golden Poisson modules. □

4 Golden Lie algebras

4.1 Golden Lie structures

Golden Lie algebras extend the concept of golden structures to the algebraic setting of Lie algebraic, providing a framework for studying deformations and integrability conditions within Lie theory.

Definition 31. Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie algebra. A Golden structure on $\mathfrak{g}$ is a linear endomorphism $\varphi:\mathfrak{g}\to\mathfrak{g}$ satisfying the Golden identity:

$$\varphi^2=\varphi+\mathrm{id}_{\mathfrak{g}}.$$

Remark 7. The characteristic equation $\varphi^2=\varphi+\mathrm{id}_{\mathfrak{g}}$ induces a natural decomposition of the Lie algebra $\mathfrak{g}$. Let $\phi=\frac{1+\sqrt{5}}{2}$ and $\overline{\phi}=\frac{1-\sqrt{5}}{2}$ denote the two roots of the equation $\lambda^2=\lambda+1$. The Lie algebra splits into complementary subspaces

$$\mathfrak{g}=\mathfrak{g}_{\phi}\oplus\mathfrak{g}_{\overline{\phi}},$$

where $\mathfrak{g}_{\phi}$ and $\mathfrak{g}_{\overline{\phi}}$ correspond to the generalized eigenspaces associated with ϕ and $\overline{\phi}$, respectively.

The Golden structure φ is said to be integrable if its associated Golden Nijenhuis tensor vanishes identically:

$$N_{\varphi}(x,y)=\varphi^2[x,y]+[\varphi x,\varphi y]-\varphi[\varphi x,y]-\varphi[x,\varphi y]=0,\text{for all } x,\ y\in\mathfrak{g}.$$

When this condition is satisfied, φ is called an integrable Golden structure on $\mathfrak{g}$.

Definition 32. A Golden Lie algebra is a triple $(\mathfrak{g},[\cdot,\cdot]_{\varphi},\varphi)$, where:

- $(\mathfrak{g},[\cdot,\cdot])$ is a Lie algebra,
- φ is a Golden structure on $\mathfrak{g}$,
- The deformed bracket $[\cdot,\cdot]_{\varphi}$ is defined by:

$$[x,y]_{\varphi}=[\varphi x,y]+[x,\varphi y]-\varphi([x,y]),\quad \forall x,y\in\mathfrak{g}. \tag{4.1}$$

Example 8 (Abelian case)**.** Consider the abelian Lie algebra $\mathfrak{g}=\mathbb{R}^2$ with the trivial bracket $[x,y]=0$, and let $\{e_1,e_2\}$ be the standard basis of $\mathbb{R}^2$. Define $\varphi:\mathbb{R}^2\to\mathbb{R}^2$ by

$$\varphi(e_1)=\phi e_1,\qquad \varphi(e_2)=\overline{\phi}e_2,$$

where $\phi=\frac{1+\sqrt{5}}{2}$ and $\overline{\phi}=\frac{1-\sqrt{5}}{2}$. Then $\varphi^2=\varphi+\mathrm{id}$, and the deformed bracket satisfies $[x,y]_{\varphi}=0$. Thus, $(\mathbb{R}^2,\varphi)$ forms a golden Lie algebra.

Example 9 (Golden deformation of $\mathfrak{aff}(\mathbb{R})$)**.** Consider the Lie algebra $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ of the affine group on $\mathbb{R}$, with basis $\{e_1, e_2\}$ of $\mathbb{R}^2$ and bracket $[e_1, e_2] = e_2$. Define the golden structure φ by

$$\varphi(e_1) = \phi e_1 \text{ and } \qquad \varphi(e_2) = e_1 + \overline{\phi} e_2.$$

A direct verification shows that $\varphi^2 = \varphi + \mathrm{id}$. The deformed bracket evaluates as:

$$[e_1, e_2]_\varphi = [\varphi(e_1), e_2] + [e_1, \varphi(e_2)] - \varphi([e_1, e_2]) = \phi e_2 + \overline{\phi} e_2 - (e_1 + \overline{\phi} e_2) = \phi e_2 - e_1.$$

This nontrivial deformation yields a golden Lie algebra structure on $\mathfrak{aff}(\mathbb{R})$.

Example 10 (Golden structure on $\mathfrak{so}(3)$)**.** Let $\mathfrak{g} = \mathfrak{so}(3)$ with the standard basis $\{E_1, E_2, E_3\}$ satisfying

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

Define $\varphi : \mathfrak{so}(3) \to \mathfrak{so}(3)$ by

$$\varphi(E_1) = \phi E_1, \qquad \varphi(E_2) = E_3, \qquad \varphi(E_3) = E_2 + E_3.$$

This map satisfies $\varphi^2 = \varphi + \mathrm{id}$. The induced deformed bracket is given by:

$$[E_1, E_2]_\varphi = \phi E_3, \quad [E_1, E_3]_\varphi = \overline{\phi} E_1, \quad [E_2, E_3]_\varphi = -\phi E_2 + E_3,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $\overline{\phi} = \frac{1-\sqrt{5}}{2}$. This bracket defines a new Lie algebra structure on $\mathfrak{so}(3)$, making $(\mathfrak{so}(3), [\cdot,\cdot]_\varphi, \varphi)$ a golden Lie algebra.

In the context of a Golden Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$, where $\varphi : \mathfrak{g} \to \mathfrak{g}$ is a linear operator and $[\cdot,\cdot]_\varphi$ denotes a deformed Lie bracket, the modified Nijenhuis tensor is defined as:

$$T_\varphi(x, y) = [\varphi x, \varphi y] - \varphi([x, y]_\varphi), \text{ for all } x,\ y \in \mathfrak{g}. \tag{4.2}$$

Definition 33. The operator φ is called Nijenhuis if its associated modified tensor vanishes identically i.e., $T_\varphi(x, y) = 0$ for all $x,\ y \in \mathfrak{g}$.

This characterization is purely algebraic and independent of any geometric notion of integrability.

Proposition 4.1. *Let* $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ *be a Lie Golden algebra.*

If φ *is Nijenhuis, then the deformed bracket* $[\cdot,\cdot]_\varphi$ *endows* $\mathfrak{g}$ *with a Lie algebra structure.*

Proof. • The bilinearity and antisymmetry of $[\cdot,\cdot]_\varphi$ follow directly from the definition and the basic properties of the original Lie bracket and the operator φ.

• Now, we focus on the Jacobi identity. Let us denote the Jacobiator for the deformed bracket as :

$$J_\varphi(x, y, z) = [x, [y, z]_\varphi]_\varphi + [y, [z, x]_\varphi]_\varphi + [z, [x, y]_\varphi]_\varphi. \tag{4.3}$$

But

$$\begin{aligned}
[x, [y, z]_\varphi]_\varphi &= [\varphi(x), [y, z]_\varphi] + [x, \varphi([y, z]_\varphi)] - \varphi([x, [y, z]_\varphi]) \\
&= [\varphi(x), [\varphi(y), z]] + [\varphi(x), [y, \varphi(z)]] - [\varphi(x), \varphi([y, z])] + [x, [\varphi(y), \varphi(z)]] - \\
&\quad \varphi([x, [\varphi(y), z]]) - \varphi([x, [y, \varphi(z)]]) + \varphi([x, \varphi([y, z])]).
\end{aligned}$$

Hence,

$$\begin{aligned}
J_\varphi(x,y,z) &= [\varphi(x),[\varphi(y),z]] + [\varphi(y),[z,\varphi(y)]] + [x,[\varphi(y),\varphi(z)]] \\
&\quad +[\varphi(x),[y,\varphi(z)]] + [\varphi(z),[\varphi(x),y]] + [y,[\varphi(z),\varphi(x)]] \\
&\quad +[\varphi(y),[\varphi(z),x]] + [\varphi(z),[x,\varphi(y)]] + [x,[\varphi(y),\varphi(z)]] \\
&\quad -\varphi([x,[\varphi(y),z]]) - \varphi([z,[x,\varphi(y)]]) - \varphi([y,[\varphi(z),x]]) - \varphi([x,[y,\varphi(z)]]) \\
&\quad -\varphi([z,[\varphi(x),y]]) - \varphi([y,[z,\varphi(x)]]) \\
&\quad -[\varphi(x),\varphi([y,z])] - [\varphi(y),\varphi([z,x])] - [\varphi(z),\varphi([x,y])] \\
&\quad +\varphi([\varphi(x),\varphi([y,z])]) + \varphi([\varphi(y),\varphi([z,x])]) + \varphi([\varphi(z),\varphi([x,y])]) \\
&= \varphi(-[x,[\varphi(y),z]] - [z,[x,\varphi(y)]]) + \varphi(-[y,[\varphi(z),x]] - [x,[y,\varphi(z)]]) \\
&\quad +\varphi(-[z,[\varphi(x),y]] - [y,[z,\varphi(x)]]) \\
&\quad -[\varphi(x),\varphi([y,z])] - [\varphi(y),\varphi([z,x])] - [\varphi(z),\varphi([x,y])] \\
&\quad +\varphi([\varphi(x),\varphi([y,z])]) + \varphi([\varphi(y),\varphi([z,x])]) + \varphi([\varphi(z),\varphi([x,y])]) \\
&\quad \text{(by applying the Jacobi identity to the original bracket.)} \\
&= \varphi([\varphi(y),[z,x]]) + \varphi([\varphi(z),[x,y]]) + \varphi([\varphi(y),[y,z]]) \\
&\quad -[\varphi(x),\varphi([y,z])] - [\varphi(y),\varphi([z,x])] - [\varphi(z),\varphi([x,y])] \\
&\quad +\varphi([x,\varphi([y,z])]) + \varphi([y,\varphi([z,x])]) + \varphi([z,\varphi([x,y])]) \\
&\quad \text{(by applying the Jacobi identity .)} \\
&= \varphi([\varphi(y),\varphi([z,x])]) + \varphi([y,\varphi([z,x])]) - [\varphi(y),\varphi([z,x])] \\
&\quad +\varphi([\varphi(z),\varphi([x,y])]) + \varphi([z,\varphi([x,y])]) - [\varphi(z),\varphi([x,y])] \\
&\quad +\varphi([\varphi(x),\varphi([y,z])]) + \varphi([x,\varphi([y,z])]) - [\varphi(x),\varphi([y,z])] \\
&\quad \text{since } \varphi([x,y]_\varphi) = [\varphi(x),\varphi(y)] \\
&= 0
\end{aligned}$$

Thus, the deformed bracket defines a Lie algebra structure on $\mathfrak{g}$ when φ is Nijenhuis. Hence, $(\mathfrak{g},[\cdot,\cdot]_\varphi)$ is a Lie algebra. ☐

Remark 8. The deformed bracket also satisfies the Jacobi identity if the image of φ is central in $\mathfrak{g}$, or if φ is a scalar multiple of the identity.

Theorem 4.1. *Let $(\mathfrak{g},[\cdot,\cdot]_\varphi,\varphi)$ be a Golden Lie algebra.*
If φ is Nijenhuis, then for each eigenvalue $\lambda \in \{\phi,\overline{\phi}\}$ (where $\phi = \frac{1+\sqrt{5}}{2}$, $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ are the roots of $\lambda^2 = \lambda + 1$), the eigenspace $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid \varphi x = \lambda x\}$ is a Lie subalgebra of $\mathfrak{g}$.

Proof. Let $x,y \in \mathfrak{g}_\lambda$. We need to show that $[x,y] \in \mathfrak{g}_\lambda$, i.e, $\varphi([x,y]) = \lambda[x,y]$. One has:

$$\begin{aligned}
T_\varphi(x,y) &= [\varphi x,\varphi y] - \varphi([x,y]_\varphi) \\
&= [\varphi x,\varphi y] - \varphi([\varphi x,y] + [x,\varphi y] - \varphi([x,y])) \\
&= [\lambda x,\lambda y] - \varphi([\lambda x,y]) - \varphi([x,\lambda y]) + \varphi^2([x,y]) \\
&= \lambda^2[x,y] - \lambda\varphi([x,y]) - \lambda\varphi([x,y]) + \varphi([x,y]) + [x,y] \\
&= \varphi[x,y] + [x,y] + \lambda^2[x,y] - \lambda\varphi[x,y] - \varphi\lambda[x,y] \\
&= (-2\lambda+1)\varphi([x,y]) + (\lambda^2+1)[x,y].
\end{aligned}$$

Hence, $T_\varphi(x,y) = 0 \Leftrightarrow \varphi([x,y]) = \frac{\lambda^2+1}{2\lambda-1}[x,y] = \lambda[x,y] \Leftrightarrow [x,y] \in \mathfrak{g}_\lambda$. Therefore, $\mathfrak{g}_\lambda$ is a Lie subalgebra of $\mathfrak{g}$. ☐

Corollary 4.2. *Let $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ be a Golden Lie algebra with integrable φ. If $\mathfrak{g}$ is realized as vector fields on a manifold M, then for any eigenspace $\mathfrak{g}_\lambda$, the vector fields in $\mathfrak{g}_\lambda$ generate a foliation on M.*

Proof. Since $\mathfrak{g}_\lambda$ is a Lie subalgebra, the distribution spanned by $\mathfrak{g}_\lambda$ is involutive. The result follows directly from Frobenius's theorem. □

4.2 Solvable and Nilpotent Golden Lie algebra

Definition 34. A Golden Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ is solvable if:

- $\mathfrak{g}$ is solvable with respect to the original bracket $[\cdot,\cdot]$, and
- φ preserves the derived series, i.e., $\varphi(g^{(k)}) \subseteq \mathfrak{g}^{(k)}, \forall k \geq 0$, where $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$.

In the following, the Solvability of a Golden Lie algebra is considered with respect to classical Lie bracket $[\cdot,\cdot]$. The modified bracket $[\cdot,\cdot]_\varphi$ is not used in defining the derived series or solvability. It only appears when defining the Golden structure on the Lie algebra $\mathfrak{g}$ or on the abelian quotient $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$.

Theorem 4.3. *Let $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ be a solvable Golden Lie algebra. If φ is Nijenhuis, then for each k, the derived $\mathfrak{g}^{(k)}$ is an ideal of $\mathfrak{g}$ and is φ-stable.*

Proof.

1. It is well-known that each $\mathfrak{g}^{(k)}$ is an ideal of $\mathfrak{g}$(see [15], Lie algebras, Theorm 2.2).
2. Let us show that $\mathfrak{g}^{(k)}$ is φ-stable. We proceed by induction on k.
   For $k = 0$, we have, $\mathfrak{g}^{(0)} = \mathfrak{g}$ is trivially φ-stable by hypothesis,
   since $\varphi(g) \subseteq \mathfrak{g}$. Assume that $\mathfrak{g}^{(k)}$ is φ-stable, i.e, $\varphi(g^{(k)}) \subseteq \mathfrak{g}^{(k)}$.
   Let us show that $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$ is φ-stable. Let $y = [a, b] \in \mathfrak{g}^{(k+1)}$, where $a, b \in \mathfrak{g}^{(k)}$.
   Using the strong Nijenhuis property and the inductive, one has:
   $\varphi(y) = \varphi([a, b]) = [\varphi(a), \varphi(b)] \in [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}] = \mathfrak{g}^{(k+1)}$. Thus $\varphi(g^{(k+1)}) \subseteq \mathfrak{g}^{(k+1)}$.
   Hence $\mathfrak{g}^{(k+1)}$ is φ-stable.

□

Proposition 4.2. *Let $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ be a solvable Golden Lie algebra. Then φ induces a well-defined endomorphism $\overline{\varphi}_k$ on the quotient $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ for each k, and φ_k satisfies the Golden relation $\overline{\varphi}_k^2 = \overline{\varphi}_k + id$.*

Proof. Since $\varphi(\mathfrak{g}^{(k)}) \subset \mathfrak{g}^{(k)}$ and $\varphi(\mathfrak{g}^{(k+1)}) \subset \mathfrak{g}^{(k+1)}$, the linear map
$\overline{\varphi}_k : \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)} \to \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ defined by:

$$\overline{\varphi}_k(x + \mathfrak{g}^{(k+1)}) = \varphi(x) + \mathfrak{g}^{(k+1)}$$

is well-defined. For any $x + \mathfrak{g}^{(k+1)} \in \mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$, one has:

$$\begin{aligned}
\overline{\varphi}_k^2(x + \mathfrak{g}^{(k+1)}) &= \overline{\varphi_k}(\overline{\varphi_k}(x + \mathfrak{g}^{(k+1)})) \\
&= \overline{\varphi_k}(\varphi(x) + \mathfrak{g}^{(k+1)}) \\
&= \varphi^2(x) + \mathfrak{g}^{(k+1)} \\
&= (\varphi(x) + x) + \mathfrak{g}^{(k+1)}(\text{ Golden relation for } \varphi) \\
&= (\varphi(x) + \mathfrak{g}^{(k+1)}) + (x + \mathfrak{g}^{(k+1)}) \\
&= \overline{\varphi}_k(x + \mathfrak{g}^{(k+1)}) + (x + \mathfrak{g}^{(k+1)})(\text{ since } \varphi \text{ is linear}) \\
&= (\overline{\varphi}_k + id)(x + \mathfrak{g}^{(k+1)}).
\end{aligned}$$

Hence, $\overline{\varphi}_k^2 = \overline{\varphi}_k + id.$ □

Proposition 4.3. *Let* $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ *be a solvable Golden Lie algebra. Then the quotient* $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ *is abelian.*

Proof. By definition, $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$. For any $x,\ y \in \mathfrak{g}^{(k)}$, one has:

$$[x + \mathfrak{g}^{(k+1)}, y + \mathfrak{g}^{(k+1)}] = [x, y] + \mathfrak{g}^{(k+1)} = 0 + \mathfrak{g}^{(k+1)} = \overline{0}.$$

Hence, $\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}$ is abelian. □

Definition 35. The pair $(\mathfrak{g}^{(k)}/\mathfrak{g}^{(k+1)}, \overline{\varphi_k})$, where $\overline{\varphi_k}$ is the induced endomorphism, is called an Abelian Golden algebra quotient. $\overline{\varphi_k}$ is an induced Golden structure.

Definition 36. A Golden Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ is nilpotent if:

- $\mathfrak{g}$ is nilpotent with respect to the original bracket $[\cdot,\cdot]$, and
- φ preserves the lower central series, i.e., $\varphi(\mathcal{C}^k g) \subseteq \mathcal{C}^k\mathfrak{g}, \forall k \geq 0$, where $\mathcal{C}^0\mathfrak{g} = \mathfrak{g}$ and $\mathcal{C}^{k+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^k\mathfrak{g}]$.

Proposition 4.4. *If* $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ *is nilpotent then:*

1. $\mathfrak{g}_k$ *is* φ*-stable ideal.*
2. *The quotient* $\mathfrak{g}_k/\mathfrak{g}_{k+1}$ *is abelian.*
3. *The quotient carries a well-defined Golden structure induced by* φ*.*

Corollary 4.4. *If* $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ *is nilpotent, then it is also solvable.*

4.3 Golden Lie algebra morphism and Golden Lie subalgebra

Let $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ and $(\mathfrak{g}', [\cdot,\cdot]'_{\varphi'}, \varphi')$ be two Golden Lie algebras.

Definition 37. A morphism of Golden Lie algebras is a linear map $\eta : \mathfrak{g} \to \mathfrak{g}'$ satisfying the two conditions:

- Preservation of the Lie bracket:
  $\eta([x, y]) = [\eta(x), \eta(y)]', \forall x, y \in \mathfrak{g};$
- Compatibility with the Golden structures: $\eta \circ \varphi = \varphi' \circ \eta$.

Moreover, If η is bijective, it is called an isomorphism of Golden Lie algebras.

Proposition 4.5.
1. *The composition of Golden Lie algebra morphisms is also a Golden Lie algebra morphism.*
2. *If* $\eta : \mathfrak{g} \to \mathfrak{g}'$ *is an isomorphism of Golden Lie algebras, then* η^{-1} *is also a Golden Lie algebra morphisms.*

Definition 38. Let $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ be a Golden Lie algebra. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a Golden Lie subalgebra if it satisfies:

1. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ i.e., $[x, y] \in \mathfrak{h}, \forall x, y \in \mathfrak{h};$

2. $\varphi(\mathfrak{h}) \subset \mathfrak{h}$ i.e., $\varphi(x) \in \mathfrak{h}, \forall x \in \mathfrak{h}$.

Example 11. The eigenspace $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} | \varphi x = \lambda x\}$, $\lambda \in \{\phi, \overline{\phi}\}\}$ (where ϕ and $\overline{\phi}$ are the eigenvalues satisfying $\lambda^2 = \lambda + 1$), is a Golden Lie subalgebra of $\mathfrak{g}$.

Proposition 4.6. *Let $\eta : \mathfrak{g} \to \mathfrak{g}'$ be a morphism of Golden Lie algebras. Then:*

1. *The kernel $Ker(\eta) = \{x \in \mathfrak{g} | \eta(x) = 0\}$ is a Golden Lie subalgebra.*
2. *Then image $Im(\eta) = \{\eta(x) | x \in \mathfrak{g}\} \subseteq \mathfrak{g}'$ is a Golden Lie subalgebra.*

4.4 Linear Golden Poisson algebras

4.4.1 Linear Golden Poisson structures

Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite-dimensional real or complex Lie algebra, let φ be a Golden structure on $\mathfrak{g}$ and let π be the linear Poisson structure on the dual space $\mathfrak{g}^*$. On $\mathfrak{g}^*$, we consider linear functions

$$\begin{cases} \ell_x(\mu) := \mu(x) \\ \{\ell_x, \ell_y\}_\varphi(\mu) := \mu\big([x,y]_\varphi\big), \qquad \text{for all} x,\ y \in \mathfrak{g},\ \mu \in \mathfrak{g}^*. \end{cases}$$

Since the bracket $[\cdot,\cdot]_\varphi$ is bilinear and antisymmetric on $\mathfrak{g}$, there exists a unique bivector field $\pi_\varphi \in \Gamma\big(\wedge^2 T\mathfrak{g}^*\big)$such that $\{f,g\}_\varphi(\mu) = \pi_\varphi(\mu)\big(df(\mu), dg(\mu)\big)$, for any $f, g \in C^\infty(\mathfrak{g}^*)$. Intrinsically, for all $x, y \in \mathfrak{g}$, we have

$$\pi_\varphi(\mu)(d\ell_x, d\ell_y) = \{\ell_x, \ell_y\}_\varphi(\mu) = \mu\big([x,y]_\varphi\big).$$

Since the differentials $d\ell_x$ generate the space of constant 1-forms on $\mathfrak{g}^*$, this intrinsic formula completely determines π_φ.

Remark 9. (Coordinate expression)

Let $(e_1, \ldots, e_n)$ be a basis of $\mathfrak{g}$ and $(x_1, \ldots, x_n)$ the dual basis of $\mathfrak{g}^*$, so that

$$x_i(\mu) = \mu(e_i), \qquad \mu \in \mathfrak{g}^*.$$

Write the structure constants of the deformed bracket $[\cdot,\cdot]_\varphi$ as

$$[e_i, e_j]_\varphi = \sum_{k=1}^{n} c^k_{ij,\varphi}\, e_k, \qquad c^k_{ij,\varphi} = -c^k_{ji,\varphi}.$$

Then, for $\mu \in \mathfrak{g}^*$ with coordinates $(x_1(\mu), \ldots, x_n(\mu))$, the bivector π_φ has the coordinate expression

$$\pi_\varphi(\mu) = \frac{1}{2} \sum_{i,j,k=1}^{n} c^k_{ij,\varphi}\, x_k(\mu)\, \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

This shows that π_φ is linear in the coordinates on $\mathfrak{g}^*$.

One has the following result

Proposition 4.7. *Let $(\mathfrak{g}, [\cdot,\cdot]_\varphi, \varphi)$ be a Golden Lie algebra and φ a Golden-Nijenhuis operator. Consider the linear bivector π_φ defined by $\{\ell_x, \ell_y\}_\varphi(\mu) = \mu([x,y]_\varphi)$.*

Then π_φ is a linear Poisson structure, i.e. $[\pi_\varphi, \pi_\varphi]_S = 0$.

Proof. Since φ is Golden-Nijenhuis, the deformed bracket $[x, y]_\varphi$ satisfies the Jacobi identity:

$$[x, [y, z]_\varphi]_\varphi + [y, [z, x]_\varphi]_\varphi + [z, [x, y]_\varphi]_\varphi = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

Let $x, y, z \in \mathfrak{g}$, and denote by ℓ_x, ℓ_y, ℓ_z the corresponding linear functions on $\mathfrak{g}^*$. By definition of the deformed Poisson bracket, $\{\ell_x, \ell_y\}_\varphi(\mu) = \ell_{[x,y]_\varphi}(\mu) = \mu([x, y]_\varphi)$. Consider the cyclic sum:

$$J_\varphi(\ell_x, \ell_y, \ell_z)(\mu) := \{\ell_x, \{\ell_y, \ell_z\}_\varphi\}_\varphi(\mu) + \{\ell_y, \{\ell_z, \ell_x\}_\varphi\}_\varphi(\mu) + \{\ell_z, \{\ell_x, \ell_y\}_\varphi\}_\varphi(\mu).$$

Since $\{\ell_y, \ell_z\}_\varphi = \ell_{[y,z]_\varphi}$, we obtain

$$\{\ell_x, \{\ell_y, \ell_z\}_\varphi\}_\varphi(\mu) = \{\ell_x, \ell_{[y,z]_\varphi}\}_\varphi(\mu) = \mu\big([x, [y, z]_\varphi]_\varphi\big).$$

Cyclically summing gives

$$J_\varphi(\ell_x, \ell_y, \ell_z)(\mu) = \mu\left([x, [y, z]_\varphi]_\varphi + [y, [z, x]_\varphi]_\varphi + [z, [x, y]_\varphi]_\varphi\right).$$

The expression in parentheses is exactly the Jacobi identity for $[\cdot, \cdot]_\varphi$, which vanishes. Hence

$$J_\varphi(\ell_x, \ell_y, \ell_z)(\mu) = 0, \quad \forall \ell_x, \ell_y, \ell_z \in C^\infty(\mathfrak{g}),\ \forall \mu \in \mathfrak{g}^*.$$

The bracket $\{\cdot, \cdot\}_\varphi$ is bilinear, antisymmetric, and satisfies the Leibniz rule in each argument. The linear functions ℓ_x generate the algebra of polynomial functions on $\mathfrak{g}^*$, and we have just checked the Jacobi identity on these generators. By the Leibniz rule, the Jacobi identity extends to all polynomial functions, and by density and standard regularity arguments, to the whole $C^\infty(\mathfrak{g}^*)$.

Therefore, $\{\cdot, \cdot\}_\varphi$ is a Poisson bracket i.e. its satisfies the Jacobi identity on $C^\infty(\mathfrak{g}^*)$. It follows that $[\pi_\varphi, \pi_\varphi]_S = 0$.

Thus π_φ is a Poisson bivector. □

Proposition 4.8. *If $(\mathfrak{g}, [\cdot, \cdot]_\varphi, \varphi)$ is a Lie Golden algebra, then the dual map*

$$\varphi^* : \mathfrak{g}^* \to \mathfrak{g}^*, \quad \varphi^*(\mu) = \mu \circ \varphi,$$

is a Poisson map from $(\mathfrak{g}^*, \pi_\varphi)$ to $(\mathfrak{g}^*, \pi)$ i.e.,

$$\{f \circ \varphi^*, g \circ \varphi^*\} = \{f, g\}_\varphi \circ \varphi^*, \quad \forall f, g \in C^\infty(\mathfrak{g}^*),$$

where π is the standard linear Poisson structure on $\mathfrak{g}^*$ associated with $[\cdot, \cdot]$, and π_φ is the deformed linear Poisson structure on $\mathfrak{g}^*$ associated with $[\cdot, \cdot]_\varphi$.

Proof. Since φ is Nijenhuis, the vanishing torsion condition

$$[\varphi(x), \varphi(y)] = \varphi([x, y]_\varphi), \quad \forall x, y \in \mathfrak{g},$$

implies that

$$\varphi : (\mathfrak{g}, [\cdot, \cdot]_\varphi) \longrightarrow (\mathfrak{g}, [\cdot, \cdot])$$

is a Lie algebra homomorphism. We now pass to the dual level.

On $\mathfrak{g}^*$, the standard Poisson bracket $\{\cdot, \cdot\}$ associated with $[\cdot, \cdot]$ is given by

$$\{\ell_x, \ell_y\} = \ell_{[x,y]}, \quad \forall x, y \in \mathfrak{g},$$

and the deformed bracket $\{\cdot,\cdot\}_\varphi$ is given by

$$\{\ell_x,\ell_y\}_\varphi=\ell_{[x,y]_\varphi},\quad \forall x,y\in\mathfrak{g}.$$

For $x\in\mathfrak{g}$ and $\mu\in\mathfrak{g}^*$,

$$(\ell_x\circ\varphi^*)(\mu)=\ell_x(\varphi^*(\mu))=\varphi^*(\mu)(x)=\mu(\varphi(x))=\ell_{\varphi(x)}(\mu).$$

Thus,

$$\ell_x\circ\varphi^*=\ell_{\varphi(x)}.$$

Take $x,y\in\mathfrak{g}$, one has

$$\begin{array}{rcl}\{\ell_x\circ\varphi^*,\ell_y\circ\varphi^*\}(\mu) & = & \{\ell_{\varphi(x)},\ell_{\varphi(y)}\}(\mu)\\ & = & \ell_{[\varphi(x),\varphi(y)]}(\mu)\\ & = & \mu([\varphi(x),\varphi(y)])\\ & = & \mu(\varphi([x,y]_\varphi))\text{ by the Golden-Nijenhuis condition}\\ & = & \varphi^*(\mu)([x,y]_\varphi)\\ & = & \ell_{[x,y]_\varphi}(\varphi^*(\mu))\\ & = & \{\ell_x,\ell_y\}_\varphi\circ\varphi^*(\mu)\end{array}$$

Since the linear functions ℓ_x generate the polynomial functions on $\mathfrak{g}^*$, and both Poisson brackets $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_\varphi$ satisfy the Leibniz rule.
We conclude that $\varphi^*:(\mathfrak{g}^*,\pi_\varphi)\longrightarrow(\mathfrak{g}^*,\pi)$ is a Poisson morphism □

Definition 39. A linear Golden Poisson algebra is a quadruple $\big(\mathfrak{g},[\cdot,\cdot],\varphi,\pi_\varphi\big)$ such that:

- $(\mathfrak{g},[\cdot,\cdot],\varphi)$ is a finite-dimensional real Lie algebra;
- π_φ is the linear Poisson bivector on $\mathfrak{g}^*$ associated with $[\cdot,\cdot]_\varphi$, defined by

  $$\{f,g\}_\varphi(\mu)=\pi_\varphi(\mu)(df(\mu),dg(\mu)),$$

  for any $f,g\in C^\infty(\mathfrak{g}^*)$.

Corollary 4.5. *Let $(\mathfrak{g},\pi_\varphi,\varphi)$ be a linear Golden Poisson algebra. Then $(\mathfrak{g}^*,\pi_\varphi)$ is a linear Poisson manifold.*

4.4.2 Linear Golden Poisson morphisms

Let $(\mathfrak{g},\pi_\varphi,\varphi)$ and $(\mathfrak{g}',\pi'_{\varphi'},\varphi')$ be two linear Golden Poisson algebras.

Definition 40. A linear Golden Poisson morphism is a linear map $\zeta:\mathfrak{g}\to\mathfrak{g}'$ satisfying:

1. $\pi(\zeta^*\alpha,\zeta^*\beta)=\pi'(\alpha,\beta)$, for all $\alpha,\beta\in(\mathfrak{g}')^*$;
2. $\zeta\circ\varphi=\varphi'\circ\zeta$;
3. $\zeta([x,y]_\varphi)=[\zeta(x),\zeta(y)]_{\varphi'}$, for all $x,y\in\mathfrak{g}$.

Proposition 4.9. *1. If $\zeta:\mathfrak{g}\to\mathfrak{g}'$ is bijective, then its inverse $\zeta^{-1}:\mathfrak{g}'\to\mathfrak{g}$ is also a linear Golden Poisson morphism.*

2. *If $\zeta : \mathfrak{g} \to \mathfrak{g}'$ and $\eta : \mathfrak{g}' \to \mathfrak{g}''$ are linear Golden Poisson morphisms, then their composition $\eta \circ \zeta : \mathfrak{g} \to \mathfrak{g}''$ is also a linear Golden Poisson morphism.*

Proposition 4.10. *If ζ is a linear Golden Poisson algebra morphism, then $\operatorname{Ker}\zeta$ and $\operatorname{Im}\zeta$ are φ-subspaces:*

1. $\varphi(\ker(\zeta)) \subseteq \ker(\zeta)$;
2. $\varphi'(Im(\zeta)) \subseteq Im(\zeta)$.

Proof. 1. Let $x \in \ker(\zeta)$. Then $\zeta(\varphi(x)) = (\zeta \circ \varphi)(x) = (\varphi' \circ \zeta)(x) = \varphi'(0) = 0$, so $\varphi(x) \in \ker(\zeta)$.

2. Let $y' \in Im(\zeta)$. Then there exists $x \in \mathfrak{g}$ such that $y' = \zeta(x)$. Therefore, $\varphi'(y') = \varphi'(\zeta(x)) = (\varphi' \circ \zeta)(x) = (\zeta \circ \varphi)(x) \in Im(\zeta)$.

☐

Proposition 4.11. *Let ζ be a linear Golden Poisson morphism. If ζ preserves the Lie brackets induced by the Poisson bivectors, then the kernel of ζ is an ideal of the Lie algebra $\mathfrak{g}$.*

Proof. Let $x \in \operatorname{Ker}\zeta$ and $y \in \mathfrak{g}$. Since ζ preserves the brackets, we have:

$$\zeta([x,y]_\varphi) = [\zeta(x), \zeta(y)]_{\varphi'} = [0, \zeta(y)]_{\varphi'} = 0.$$

Therefore, $[x,y]_\varphi \in \ker(\zeta)$, which shows that $\ker(\zeta)$ is an ideal of $\mathfrak{g}$. ☐

4.4.3 Linear Hamiltonian vector fields

Let $(\mathfrak{g}, \pi_\varphi, \varphi)$ be a linear Golden Poisson algebra. For each $x \in \mathfrak{g}$, recall that the linear function on $\mathfrak{g}^*$ is defined by:

$$\ell_x(\alpha) = \langle \alpha, x \rangle, \quad \alpha \in \mathfrak{g}^*.$$

The Hamiltonian vector field associated with ℓ_x and the Golden structure φ is:

$$X_\varphi^x = X_\varphi^{\ell_x} = \pi_\varphi^\sharp \circ \varphi^*(d\ell_x), \quad x \in \mathfrak{g}.$$

Definition 41. The vector field X_φ^x is called the linear Golden Hamiltonian vector field associated to $x \in \mathfrak{g}$.

Proposition 4.12. *Let $(\mathfrak{g}, \pi_\varphi, \varphi)$ be a linear Golden Poisson algebra and $x, y \in \mathfrak{g}$. Then the linear Hamiltonian vector field X_ϕ^x satisfies:*

1. $X_\varphi^{ax+by} = aX_\varphi^x + bX_\varphi^y$ *for all* $a, b \in \mathbb{R}$
2. $[X_\varphi^x, X_\varphi^y] = X_\varphi^{[x,y]_\varphi}$ *i.e., the mapping* $x \mapsto X_\varphi^x$ *is a Lie algebra homomorphism from* $(\mathfrak{g}, [\cdot,\cdot]_\varphi)$ *to the Lie algebra of vector fields on* $\mathfrak{g}^*$.
3. $X_\varphi^x(\ell_y) = \ell_{[x,y]_\varphi}$
4. *Golden Iteration:* $X_{\varphi^2}^x = X_\varphi^x + X_{id}^x$
5. *Fibonacci Relation:* $X_{\varphi^{n+2}}^x = X_{\varphi^{n+1}}^x + X_{\varphi^n}^x$ *for all* $n \geq 0$.

Proof. 1. Let $x, y \in \mathfrak{g}$. Then:

$$\begin{aligned}
X_{\varphi}^{ax+by} &= X_{\phi}^{\ell_{ax+by}} \quad \text{(by definition)}\\
&= \pi^{\sharp} \circ \varphi^{*}(d\ell_{ax+by}) \quad \text{(definition of Hamiltonian vector field)}\\
&= \pi^{\sharp} \circ \varphi^{*}(a\, d\ell_{x} + b\, d\ell_{y}) \quad \text{(linearity of differential)}\\
&= a\pi^{\sharp} \circ \varphi^{*}(d\ell_{x}) + b\pi^{\sharp} \circ \varphi^{*}(d\ell_{y})\\
&= aX_{\phi}^{x} + bX_{\varphi}^{y}.
\end{aligned}$$

2. For linear functions:

$$\begin{aligned}
[X_{\varphi}^{x}, X_{\varphi}^{y}] &= [X_{\varphi}^{\ell_x}, X_{\varphi}^{\ell_y}]\\
&= X_{\varphi}^{\{\ell_x,\ell_y\}_{\varphi}} \quad \text{(bracket of Hamiltonian vector fields)}\\
&= X_{\varphi}^{\ell_{[x,y]}} \quad \text{(Poisson bracket for linear functions)}\\
&= X_{\varphi}^{[x,y]}.
\end{aligned}$$

3. By definition of the Hamiltonian vector field and Poisson bracket:

$$\begin{aligned}
X_{\varphi}^{x}(\ell_y) &= \pi_{\varphi}(d\ell_x, d\ell_y)\\
&= \{\ell_x, \ell_y\}_{\varphi}\\
&= \ell_{[x,y]_{\varphi}}.
\end{aligned}$$

4. Using the Golden relation:

$$\begin{aligned}
X_{\varphi^2}^{x} &= \pi^{\sharp} \circ (\varphi^{2})^{*}(d\ell_x)\\
&= \pi^{\sharp} \circ (\varphi + id)^{*}(d\ell_x)\\
&= \pi^{\sharp} \circ (\varphi^{*} + id^{*})(d\ell_x)\\
&= \pi^{\sharp} \circ \varphi^{*}(d\ell_x) + \pi^{\sharp} \circ id^{*}(d\ell_x)\\
&= X_{\varphi}^{x} + X_{id}^{x}
\end{aligned}$$

5. By induction. For $n = 0$:

$$X_{\varphi^2}^{x} = X_{\varphi^1}^{x} + X_{\varphi^0}^{x}$$

which is part (4). Assume true for $n = k$:

$$X_{\varphi^{k+2}}^{x} = X_{\varphi^{k+1}}^{x} + X_{\varphi^{k}}^{x}$$

Then for $n = k + 1$:

$$\begin{aligned}
X_{\varphi^{k+3}}^{x} &= \pi^{\sharp} \circ (\varphi^{k+3})^{*}(d\ell_x)\\
&= \pi^{\sharp} \circ (\varphi^{k+2} + \varphi^{k+1})^{*}(d\ell_x) \quad \text{(Golden relation)}\\
&= \pi^{\sharp} \circ (\varphi^{k+2})^{*}(d\ell_x) + \pi^{\sharp} \circ (\varphi^{k+1})^{*}(d\ell_x)\\
&= X_{\varphi^{k+2}}^{x} + X_{\varphi^{k+1}}^{x}
\end{aligned}$$

□

Theorem 4.6. *Let $(\mathfrak{g}, \pi_\varphi, \varphi)$ be a linear Golden Poisson algebra, and assume that φ is a Poisson automorphism, i.e.*

$$(\varphi^*)\pi_\varphi = \pi_\varphi.$$

For each $n \geq 0$, define the linear Poisson tensor

$$\pi_n := (\varphi^n)_* \pi_\varphi.$$

Then:

1. *π_n is a linear Poisson tensor for all n.*
2. *One has the Fibonacci decomposition:*

   $$\pi_n = F_n \, \pi_1 + F_{n-1} \, \pi_0,$$

   where $\pi_1 = \pi_\varphi$ and $\pi_0 = \pi_{id}$.
3. *The tensors π_n are pairwise compatible:*

   $$[\pi_m, \pi_n]_S = 0 \quad \forall m, n,$$

   where $[\cdot, \cdot]_S$ is the Schouten bracket.

Proof. 1. Since φ is a Poisson automorphism, $(\varphi^*)\pi_\varphi = \pi_\varphi$ implies that φ preserves the Poisson structure. Therefore, for any n, the pushforward $(\varphi^n)_*\pi_\varphi$ is also a Poisson tensor.

2. We prove by induction that $\varphi^n = F_n\varphi + F_{n-1}id$.

   · If $n = 1$, $\varphi^1 = \varphi = 1 \cdot \varphi + 0 \cdot id = F_1\varphi + F_0 id$.

   · If $n = 2$ $\varphi^2 = \varphi + id = 1 \cdot \varphi + 1 \cdot id = F_2\varphi + F_1 id$.

   Assume that $\varphi^k = F_k\varphi + F_{k-1}id$ and $\varphi^{k+1} = F_{k+1}\varphi + F_k id$. Then:

$$\begin{aligned} \varphi^{k+2} &= \varphi^{k+1} \circ \varphi \\ &= (F_{k+1}\varphi + F_k id) \circ \varphi \\ &= F_{k+1}\varphi^2 + F_k\varphi \\ &= F_{k+1}(\varphi + id) + F_k\varphi \\ &= (F_{k+1} + F_k)\varphi + F_{k+1}id \\ &= F_{k+2}\varphi + F_{k+1}id \end{aligned}$$

   Now, by linearity of the pushforward:

$$\begin{aligned} \pi_n &= (\varphi^n)_*\pi_\varphi \\ &= (F_n\varphi + F_{n-1}id)_*\pi_\varphi \\ &= F_n(\varphi_*)\pi_\varphi + F_{n-1}(id_*)\pi_\varphi \\ &= F_n\pi_\varphi + F_{n-1}\pi_{id} \\ &= F_n\pi_1 + F_{n-1}\pi_0 \end{aligned}$$

3. We need to show $[\pi_m, \pi_n]_S = 0$ for all m, n. Using the Fibonacci decomposition:

$$\pi_m = F_m\pi_\varphi + F_{m-1}\pi_{id}$$
$$\pi_n = F_n\pi_\varphi + F_{n-1}\pi_{id}$$

By bilinearity of the Schouten bracket:

$$\begin{aligned}[\pi_m, \pi_n]_S &= [F_m\pi_\varphi + F_{m-1}\pi_{id}, F_n\pi_\varphi + F_{n-1}\pi_{id}]_S \\ &= F_mF_n[\pi_\varphi, \pi_\varphi]_S + F_mF_{n-1}[\pi_\varphi, \pi_{id}]_S \\ &\quad +F_{m-1}F_n[\pi_{id}, \pi_\varphi]_S + F_{m-1}F_{n-1}[\pi_{id}, \pi_{id}]_S \\ &= 0,\end{aligned}$$

since $[\pi_\varphi, \pi_\varphi]_S = 0$ (by Jacobi identity), $[\pi_{id}, \pi_{id}]_S = 0$ (by Jacobi identity) and $[\pi_\varphi, \pi_{id}]_S = [\pi_{id}, \pi_\varphi]_S = 0$ (by compatibility).

□

Remark 10. This provides an infinite Golden-Fibonacci hierarchy of Poisson structures. Here the Golden property makes the recursion explicit and canonical.

Theorem 4.7. *Let $(\mathfrak{g}, \pi_\varphi, \varphi)$ be a linear Golden Poisson algebra and let X^x_ψ denote the linear Hamiltonian vector field associated to x via the operator φ. Then:*

1. *For all $n \geq 0$,*
$$X^x_{\varphi^n} = F_n\, X^x_\varphi + F_{n-1}\, X^x_{id}.$$
2. *The recursion operator*
$$\mathcal{R}(X^x_{\varphi^n}) := X^x_{\varphi^{n+1}}$$
satisfies the Golden identity:
$$\mathcal{R}^2 = \mathcal{R} + id.$$
3. *If $\{X^x_{\varphi^n}, X^x_{\varphi^m}\} = 0$ for some x, then the Golden family $\{X^x_{\varphi^n}\}_{n\geq 0}$ defines an integrable hierarchy of commuting flows.*

Proof. 1. Using the Fibonacci decomposition $\varphi^n = F_n\varphi + F_{n-1}id$:

$$\begin{aligned}X^x_{\varphi^n} &= \pi^\sharp_\varphi \circ (\varphi^n)^*(d\ell_x) \\ &= \pi^\sharp_\varphi \circ (F_n\varphi + F_{n-1}id)^*(d\ell_x) \\ &= \pi^\sharp_\varphi \circ (F_n\varphi^* + F_{n-1}id^*)(d\ell_x) \\ &= F_n\pi^\sharp_\varphi \circ \varphi^*(d\ell_x) + F_{n-1}\pi^\sharp_\varphi \circ id^*(d\ell_x) \\ &= F_nX^x_\varphi + F_{n-1}X^x_{id}\end{aligned}$$

2. For any $X^x_{\varphi^n}$:

$$\begin{aligned}\mathcal{R}^2(X^x_{\varphi^n}) &= \mathcal{R}(\mathcal{R}(X^x_{\varphi^n})) \\ &= \mathcal{R}(X^x_{\varphi^{n+1}}) \\ &= X^x_{\varphi^{n+2}} \\ &= X^x_{\varphi^{n+1}} + X^x_{\varphi^n} \quad \text{(by Fibonacci relation)} \\ &= \mathcal{R}(X^x_{\varphi^n}) + X^x_{\varphi^n}\end{aligned}$$

So $\mathcal{R}^2 = \mathcal{R} + id$.

3. If $[X^x_{\varphi^n}, X^x_{\varphi^m}] = 0$ for some x, then by linearity and the Fibonacci decomposition, all Hamiltonian fields in the family $\{X^x_{\varphi^k}\}_{k\geq 0}$ commute pairwise, forming an integrable hierarchy.

□

Theorem 4.8. *Let C be a Casimir function for π_φ. Define*

$$C_n := C \circ (\varphi^*)^n.$$

Then:

1. *Each C_n is a Casimir of π_φ.*
2. *The sequence (C_n) satisfies the Golden recurrence:*

$$C_{n+2} = C_{n+1} + C_n.$$

3. *Each C_n admits the Fibonacci decomposition:*

$$C_n = F_n C_1 + F_{n-1} C_0.$$

Proof. 1. Since φ^* is a Poisson automorphism (because φ is), it preserves Poisson brackets. If C is Casimir, then for any function F: $\{C, F\}_\varphi = 0$. Then:

$$\{C \circ \varphi^*, F\}_\varphi = \{C, \varphi_* F\}_\varphi \circ \varphi^* = 0$$

So $C \circ \varphi^*$ is also Casimir. By induction, each $C_n = C \circ (\varphi^*)^n$ is Casimir.

2. Using the Golden relation $(\varphi^*)^2 = \varphi^* + id^*$:

$$\begin{aligned} C_{n+2} &= C \circ (\varphi^*)^{n+2} \\ &= C \circ (\varphi^*)^{n+1} \circ \varphi^* \\ &= C \circ (\varphi^*)^n \circ (\varphi^*)^2 \\ &= C \circ (\varphi^*)^n \circ (\varphi^* + id^*) \\ &= C \circ (\varphi^*)^{n+1} + C \circ (\varphi^*)^n \\ &= C_{n+1} + C_n \end{aligned}$$

3. We will show by induction.

   - If $n = 0$, $C_0 = C = 0 \cdot C_1 + 1 \cdot C_0 = F_0 C_1 + F_{-1} C_0$
   - If $n = 1$, $C_1 = C \circ \varphi^* = 1 \cdot C_1 + 0 \cdot C_0 = F_1 C_1 + F_0 C_0$

   Assume $C_k = F_k C_1 + F_{k-1} C_0$ and $C_{k+1} = F_{k+1} C_1 + F_k C_0$. Then:

$$\begin{aligned} C_{k+2} &= C_{k+1} + C_k \\ &= (F_{k+1} C_1 + F_k C_0) + (F_k C_1 + F_{k-1} C_0) \\ &= (F_{k+1} + F_k) C_1 + (F_k + F_{k-1}) C_0 \\ &= F_{k+2} C_1 + F_{k+1} C_0 \end{aligned}$$

□

Theorem 4.9. *Let $\zeta : (\mathfrak{g}, \pi_\varphi, \varphi) \to (\mathfrak{g}', \pi'_{\varphi'}, \varphi')$ be a linear Golden Poisson morphism. Then:*

1. *$\zeta_*(X^x_{\varphi^n}) = X^{\zeta(x)}_{\varphi'^n}$ for all $n \geq 0$.*
2. *If C' is a Casimir of $\pi'_{\varphi'}$, then $C' \circ \zeta^*$ is a Casimir of π_φ.*
3. *The Poisson hierarchies satisfy $\zeta_*(\pi_n) = \pi'_n$.*

Proof. 1. Since $\zeta \circ \varphi^n = \varphi'^n \circ \zeta$ (by iterating condition (2) of Definition 40), one has:

$$\begin{aligned} \zeta_*(X^x_{\varphi^n}) &= \zeta_*\left(\pi^\sharp_\varphi \circ (\varphi^n)^*(d\ell_x)\right) \\ &= \zeta_* \circ \pi^\sharp_\varphi \circ (\varphi^n)^*(d\ell_x) \\ &= \pi'^\sharp \circ \zeta_* \circ (\varphi^n)^*(d\ell_x) \\ &= \pi'^\sharp \circ (\varphi'^n)^* \circ \zeta_*(d\ell_x) \\ &= \pi'^\sharp \circ (\varphi'^n)^*(d\ell_{\zeta(x)}) \\ &= X^{\zeta(x)}_{\varphi'^n} \end{aligned}$$

2. If C' is Casimir for $\pi'_{\varphi'}$, then for any $F \in C^\infty(\mathfrak{g}^*)$:

$$\begin{aligned} \{C' \circ \zeta^*, F\}_\varphi &= \pi_\varphi(d(C' \circ \zeta^*), dF) \\ &= \pi_\varphi(\zeta^*(dC'), dF) \\ &= \zeta^*(\pi'_{\varphi'}(dC', \zeta_* dF)) \quad \text{(by condition (1) of Definition 40)} \\ &= \zeta^*(\{C', \zeta_* F\}_{\varphi'}) = 0 \end{aligned}$$

So $C' \circ \zeta^*$ is Casimir for π_φ.

3. Using naturality of the pushforward:

$$\begin{aligned} \zeta_*(\pi_n) &= \zeta_*((\varphi^n)_* \pi_\varphi) \\ &= (\varphi'^n)_* \zeta_*(\pi_\varphi) \quad (\text{since } \zeta \circ \varphi^n = \varphi'^n \circ \zeta) \\ &= (\varphi'^n)_* \pi'_{\varphi'} \\ &= \pi'_n \end{aligned}$$

□

Proposition 4.13. *Let $(\mathfrak{g}, \pi_\varphi, \varphi)$ be a linear Golden Poisson algebra. The set of Casimir functions is a module over the Golden ring $\mathbb{Z}[\phi]$, where $\phi = \frac{1+\sqrt{5}}{2}$.*

Proposition 4.14. *Let $\zeta : (\mathfrak{g}, \pi_\varphi, \varphi) \to (\mathfrak{h}, \pi_\psi, \psi)$ be a linear Golden Poisson algebra morphism. Then for all $x \in \mathfrak{g}$:*

$$X^{\zeta(x)}_\psi \circ \zeta^* = \zeta^* \circ X^x_\varphi.$$

Proof. Let $f \in C^\infty(\mathfrak{h}^*)$ and $\alpha \in \mathfrak{g}^*$. We compute:

$$\begin{aligned} (X^{\zeta(x)}_\psi \circ \zeta^*)(f)(\alpha) &= X^{\zeta(x)}_\psi(f)(\zeta^*(\alpha)) \\ &= \pi'_\psi(df, d\ell_{\zeta(x)})(\zeta^*(\alpha)) \quad (\text{since } X^h_\psi(g) = \pi'_\psi(dg, d\ell_h)) \\ &= \pi'_\psi(df, d\ell_{\zeta(x)})(\zeta^*(\alpha)). \end{aligned}$$

On the other hand:

$$
\begin{aligned}
(\zeta^* \circ X_\varphi^x)(f)(\alpha) &= \zeta^*(X_\varphi^x(f))(\alpha) \\
&= X_\varphi^x(f)(\zeta^*(\alpha)) \\
&= \pi_\varphi(d(\zeta^* f), d\ell_x)(\zeta^*(\alpha)).
\end{aligned}
$$

To conclude, we use condition (1) of Definition 40, which implies that ζ^* is a Poisson morphism, i.e.:

$$\{\zeta^* f, \zeta^* g\}_\varphi = \zeta^*\{f, g\}_\psi.$$

Applying this property with $g = \ell_{\zeta(x)}$ and using the fact that $\zeta^*\ell_{\zeta(x)} = \ell_x$, we obtain the desired equality. ☐

4.4.4 Symplectic foliation of the linear Golden Poisson structure

Define the distribution:

$$D_\varphi^\xi = \operatorname{span}\{X_\varphi^x(\xi) = \pi_\varphi^\sharp \circ \varphi^*(d\ell_x) \mid x \in \mathfrak{g}\} \subseteq T_\xi \mathfrak{g}^*, \quad \xi \in \mathfrak{g}^*.$$

Definition 42. The distribution D_φ is called the characteristic distribution of the linear Poisson manifold $(\mathfrak{g}^*, \pi_\varphi)$.

Theorem 4.10. *Let $(\mathfrak{g}, \pi_\varphi, \varphi)$ be a linear Golden Poisson algebra. Then the distribution D_ϕ is linear, involutive and integrable.*

Proof. Since π_φ is a Poisson structure, for all $x, y \in \mathfrak{g}$ we have:

$$[X_\varphi^{\ell_x}, X_\varphi^{\ell_y}] = X_\varphi^{\{\ell_x, \ell_y\}_\varphi} = X_\varphi^{\ell_{[x,y]_\varphi}} \in D_\varphi,$$

which shows that D_φ is involutive. By the Stefan-Sussmann theorem, any involutive distribution locally spanned by vector fields integrates to maximal immersed submanifolds $S(\xi)$. Therefore, D_φ is integrable. ☐

Remark 11. The integral leaves of D_φ coincide with the Golden symplectic leaves of $(\mathfrak{g}^*, \pi_\varphi)$.

Acknowledgments

The authors are grateful to the anonymous referees for their constructive comments and helpful suggestions.

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Hinamari Mang-massou Emmanuel Higher Teacher Training School of Bongor, Department of Mathematics and Computer Science, University of Bongor, Chad

E-mail: hinamariemmanuel@gmail.com

Wankap Nono Georges Florian Department of Mathematics and Computer Science, Faculty of Science, University of Ngaoundéré, PO.BOX 454 Ngaoundéré, Cameroon

E-mail: georgywan@yahoo.fr

Djagwa Dehainsala Faculty of Applied and Exact Sciences, University of N'djamena, Chad

E-mail: djagwa73@gmail.com

Kilanta Ezekiel Department of Mathematics and Computer Science, Faculty of Science, University of Ngaoundéré, PO.BOX 454 Ngaoundéré, Cameroon

E-mail: ekilanta@yahoo.com