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Approximation by diffusion of semiconductor Boltzmann equation

Patrick Atiofack Fouegap, David Dongo and Jean Louis Woukeng

Abstract. In this paper, we deal with the asymptotic behaviour of a semiconductor Boltzmann equation by using the sigma convergence method. We first prove that the scaled model is well-posed in the usual Lebesgue space of square integrable functions, and we perform the a priori estimates. Then, assuming that the coefficients of the model are highly oscillating in space variable, we show that in the non-vanishing flux case, the homogenized problem is equivalent at first order, to a hyperbolic process modified by a perturbation of viscosity, and the diffusion term appears at second order. In the vanishing flux case, we obtain a diffusion model.

Keywords. Homogenization, semiconductor Boltzmann equation, global weak solution, algebras with mean value, Σ-convergence

1 Introduction

Number of physical phenomena are approximated by partial differential equation (PDEs). By so doing, the study of these PDEs provides a better understanding of the corresponding phenomena. Among those models, we distinguish the following semiconductor Boltzmann equation

$$\frac{\partial f}{\partial t}(t,x,v) + v \cdot \nabla_x f(t,x,v) - \nabla_x \Phi(t,x) \cdot \nabla_v f(t,x,v) = \mathcal{Q} f(t,x,v). \tag{1.1}$$

It's an Integral Partial Differential Equation, governing the evolution of a cluster of particles of density f, positive unknown function of time variable $t \in \mathbb{R}_t^+$, the spatial position $x \in \mathbb{R}_x^d$ and the speed $v \in V$ of the considered particles, where V is the whole space $\mathbb{R}_v^d$ or its subset. The space $\mathbb{R}_\nu^d$ stands for space $\mathbb{R}^d$ (integer $d \geq 1$) with generic variable ν. The particles are submitted to a d-dimensional electric field $E(t,x) = -\nabla_x \Phi(t,x)$ due to the scalar potential $\Phi(t,x)$ and the collisions are modeled by the term $\mathcal{Q}f$, known as collision operator.

Equation (1.1) is a standard model of particles transport, which has a wide field of applications such as electrons moving in a semiconductor material [11, 26, 40, 24, 12, 36], radiative transfer or neutrons moving in a nuclear reactor [14, 6, 3, 8, 5], gas discharges [51]. In this work, we neglect the collisions between the particles and consider only the interactions that particles may undergo from their moving medium [22]. This corresponds to the case where the collision

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Corresponding author: David Dongo.

operator is linear (see (1.6)). The media being in general inhomogeneous, the behavior of the phenomenon then essentially depends on the manner in which the media are structured. To start with the model under consideration, we introduce the following change of variables

$$t' = \varepsilon t, \;\; x' = \varepsilon x, \tag{1.2}$$

and rescale the distribution function $f(t,x,v)$ and potential $\Phi(t,x)$ as follows:

$$f_\varepsilon(t',x',v) = f(t,x,v), \;\; \Phi^\varepsilon(t',x') = \Phi(t,x), \tag{1.3}$$

where the small parameter ε is as usual, the mean free path of particles. The function f_ε is now a solution of (we have skipped the primes)

$$\varepsilon\frac{\partial f_\varepsilon}{\partial t}(t,x,v) + \varepsilon v\cdot\nabla_x f_\varepsilon(t,x,v) - \varepsilon\nabla_x\Phi^\varepsilon(t,x)\cdot\nabla_v f_\varepsilon(t,x,v) = \mathcal{Q}^\varepsilon f_\varepsilon(t,x,v), \tag{1.4}$$

The rescaled model (1.4) corresponds to the diffusion approximation of the semiconductor Boltzmann equation (1.1), when both the collision and the electric field have variation at the mean free path scale ε [12], and where there is no time scaling [42]. More precisely, we consider the following asymptotic Cauchy Problem (CP):

$$\begin{cases} \frac{\partial f_\varepsilon}{\partial t}(t,x,v) + v\cdot\nabla_x f_\varepsilon(t,x,v) + E^\varepsilon(t,x)\cdot\nabla_v f_\varepsilon(t,x,v) = \frac{1}{\varepsilon}\mathcal{Q}^\varepsilon f_\varepsilon(t,x,v) & \text{in} \quad \mathbb{R}_t^+\times\mathbb{R}_x^d\times\mathbb{R}_v^d \\ f_\varepsilon(0,x,v) = f^0(x,v) & \text{in} \quad \mathbb{R}_x^d\times\mathbb{R}_v^d, \end{cases} \tag{1.5}$$

where $f_\varepsilon(t,x,v)$ is the density of particles occupying at time t, the position x and having a velocity v. $E^\varepsilon(t,x) = E(t,\frac{x}{\varepsilon}) = -\nabla_x\Phi(t,\frac{x}{\varepsilon})$ is the d-dimensional electric field, solution of Poisson equation and supposed to be given. Here, $\mathcal{Q}^\varepsilon$ designates the collision operator, which here is a linear integral operator acting on the v variable in f_ε and defined by

$$\mathcal{Q}^\varepsilon f_\varepsilon(t,x,v)) = \int_{\mathbb{R}_w^d}\sigma^\varepsilon(t,x,v,w)(f_\varepsilon(t,x,w) - f_\varepsilon(t,x,v))d\mu(w). \tag{1.6}$$

In (1.6), $\sigma^\varepsilon(t,x,v,w) = \sigma(t,x,\frac{x}{\varepsilon},v,w)$ is a nonnegative function defined on $\mathbb{R}_t^+\times\mathbb{R}_x^d\times\mathbb{R}_v^d\times\mathbb{R}_w^d$ called transition rate, modeling the properties of the background medium and supposed also to be given. We assume (see *e.g.* [5, 34]) that it satisfies the semi-detailed balance condition

$$\int_{\mathbb{R}_w^d}\sigma^\varepsilon(t,x,v,w)d\mu(w) = \int_{\mathbb{R}_w^d}\sigma^\varepsilon(t,x,w,v)d\mu(w), \tag{1.7}$$

and define the scattering rate

$$\Sigma^\varepsilon(t,x,v) = \int_{\mathbb{R}_w^d}\sigma^\varepsilon(t,x,v,w)d\mu(w), \tag{1.8}$$

so that the kernel $\sigma^\epsilon(t,x,v,w)$ (up to the multiplicative coefficient $\frac{1}{\Sigma}$) measures the probability of a transition from velocity w to velocity v for particles located at the position x and at time t. Eventually one has:

$$\mathcal{Q}^\varepsilon f_\varepsilon(t,x,v)) = K^\varepsilon f_\varepsilon(t,x,v) - \Sigma^\varepsilon(t,x,v)f_\varepsilon(t,x,v), \tag{1.9}$$

where

$$K^\varepsilon f_\varepsilon(t,x,v) = \int_{\mathbb{R}_w^d}\sigma^\varepsilon(t,x,v,w)f_\varepsilon(t,x,w)d\mu(w) \tag{1.10}$$

and designates the quantity of particles passing from the speed w to v, at time t and at the position x. $f_0(x,v) > 0$ is the initial distribution of particles and the parameter ε measures the level of heterogeneity of the medium.

The Cauchy problem and the initial boundary value problem for the Boltzmann equation have been extensively studied in the literature, in the absence of external force [5, 9, 20] or with external force [45, 10, 29]. Most of these works deal with classical or mild solution in L^1 space, with a relatively strong hypothesis. In this work, we will be interested to the solution in L^2 space, in the weak sense of distributions, under relatively weaker assumptions about the parameters of the model. It should be noted that, L^2-solution has more interesting properties than L^1 one, in physical point of view, and is adapted to Σ-convergence method as it would be seen in Section 3.

To our knowledge, there is still little studies on homogenization and diffusion approximation of semiconductor Boltzmann equation, in particular for the case without time scaling, in setting more general than periodic one. In [12, 36], authors studied the case of Diffusion limit of a semiconductor Boltzmann-Poisson by using the Hilbert formal expansion. The case of linear Boltzmann without time scaling is treated in [42], with a very strong hypothesis of regularity. We can also mention papers [7, 5, 13, 28] which investigate the asymptotic behavior of a linear Boltzmann equation without external force, by two scale method. In all these works, hypotheses were stated in the periodic setting. Here, we use the Σ-convergence method to handle the more general case including asymptotic periodic, almost periodic, asymptotic almost periodic, and others.

The rest of the paper is organized as follows: Section 2 deals with existence, uniqueness and a priori estimates for the weak solution of problem (1.5). In Section 3, we present some fundamental results about the concept of Σ-convergence. Section 4 deals with the corrector result. Finally, Section 5 is devoted to the state and proof of the main homogenization result (see Theorem 5.1).

2 Existence, Uniqueness and a Priori Estimates for the Solution of Semiconductor Boltzmann Equation

We assume that:

$$\sigma^\varepsilon \in C\left([0,\infty); C_0\left(\mathbb{R}^d_x; L^2\left(\mathbb{R}^d_v \times \mathbb{R}^d_w\right)\right)\right), \tag{2.1}$$

$$E^\varepsilon \in \left[C\left([0,\infty); L^2_{loc}\left(\mathbb{R}^d_x\right)\right)\right]^d, \tag{2.2}$$

$$0 \leq f^0 \in L^2\left(\mathbb{R}^d_x \times \mathbb{R}^d_v\right). \tag{2.3}$$

The following Theorem is the first main result of this work.

Theorem 2.1. *Let $\varepsilon > 0$ be fixed. We suppose that σ^ε, E^ε and f^0 satisfy the conditions (2.1)-(2.3) as well as the semi-detailed balance condition (1.7). Then, the Cauchy problem (1.5) admits (in the sense of distribution), a unique global weak solution*

$$f_\varepsilon \in C^1\left([0,\infty); L^2\left(\mathbb{R}^d_x \times \mathbb{R}^d_v\right)\right). \tag{2.4}$$

Furthermore, the sequence of solution $(f_\varepsilon)_{\varepsilon>0}$ is nonnegative and satisfies for any positive constant T, the following uniform estimates:

$$\|f_\varepsilon\|^2_{L^2([0;T]\times\mathbb{R}^d_x\times\mathbb{R}^d_v)} \leq T\|f^0\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \tag{2.5}$$

and

$$\|f_\varepsilon\|^2_{L^\infty(0,\infty;L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v))} \leq \|f^0\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)}. \tag{2.6}$$

Otherwise, the solution f_ε depends continuously on the initial value f^0 in the following sense: let $f^0,\ f_n^0 \in L^2\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)$, $n = 1, 2, \ldots$, and let $f_\varepsilon,\ f_{\varepsilon,n}$ be the unique weak solutions of (1.5), respectively with $f_\varepsilon(0,x,v) = f^0(x,v)$ and $f_{\varepsilon,n}(0,x,v) = f_n^0(x,v)$. Then, if $\|f_n^0 - f^0\|_{L^2(\mathbb{R}_x^d\times\mathbb{R}_v^d)} \longrightarrow 0$ as $n \longrightarrow \infty$, then $\|f_{\varepsilon,n}(t) - f_\varepsilon(t)\|_{L^2(\mathbb{R}_x^d\times\mathbb{R}_v^d)} \longrightarrow 0$ uniformly in t.

For the proof of this theorem, we will need the following intermediate result.

Lemma 2.1. *Let $s > d+1$ be an integer. We suppose that σ, E and g^0 are given and satisfy the following conditions:*

$$g^0 \in C_0^s\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right), \tag{2.7}$$

$$E \in \left[C\left([0,\infty); C_0^s\left(\mathbb{R}_x^d\right)\right)\right]^d, \tag{2.8}$$

$$\sigma \in C\left([0,\infty); C_0^s\left(\mathbb{R}_x^d \times \mathbb{R}_v^d \times \mathbb{R}_w^d\right)\right). \tag{2.9}$$

Then, the following Cauchy problem

$$\begin{cases} \frac{\partial f}{\partial t}(t,x,v) + v\cdot\nabla_x f(t,x,v) + E(t,x)\cdot\nabla_v f(t,x,v) = \mathcal{Q}f(t,x,v) & \text{in} \quad [0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v^d, \\ f(0,x,v) = g^0(x,v) & \text{in} \quad \mathbb{R}_x^d\times\mathbb{R}_v^d, \end{cases} \tag{2.10}$$

where $\mathcal{Q}f(t,x,v) = \int_{\mathbb{R}_w^d} \sigma(t,x,v,w) f(t,x,w)dw - \Sigma(t,x,v)f(t,x,v)$; σ and Σ satisfying

$$\int_{\mathbb{R}_w^d} \sigma(t,x,v,w)dw = \Sigma(t,x,v), \tag{2.11}$$

admits a unique global classic solution

$$f \in C^1\left([0,\infty); H^s\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right) \bigcap C^1\left([0,\infty); H^{s-1}\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right). \tag{2.12}$$

The proof of this Lemma is done in two steps. We first consider the problem in a bounded interval of time ($t\in[0,T]$, $T>0$), that we solve by the approach of [32, Theorem II, P. 195] to obtain a local solution in time. Then, thanks to an idea copy from the proof of [39, Theorem 6.2.], we extend this solution to the whole interval $[0,\infty)$.

Proof of Lemma 2.1.

Step 1. Proof of the local existence

This step follows straightforward from the application of [32, Theorem II, P. 195]. Thus, we just give a sketch of its proof. Let us first note that, following the idea of the proof in [47, Theorem 5.1, P. 36], we can without losing of generality, replace the problem (2.10) of Lemma 2.1 by:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x,v) + \alpha(v)\cdot\nabla_x f(t,x,v) + E(t,x)\cdot\nabla_v f(t,x,v) = \mathcal{Q}f(t,x,v) & \text{in} \quad [0,T]\times\mathbb{R}_x^d\times\mathbb{R}_v^d, \\ f(0,x,v) = g^0(x,v) & \text{in} \quad \mathbb{R}_x^d\times\mathbb{R}_v^d, \end{cases} \tag{2.13}$$

where T is a positive constant and

$$\alpha(v) = (\alpha_i(v))_{i=1}^d = \begin{cases} v & \text{if } |v| \le 2R, \\ 0 & \text{if } |v| \ge 3R, \end{cases} \tag{2.14}$$

that is $\alpha_i(v) = v_i$ for all $i = 1,2,\ldots,d$, $v\in\mathbb{R}^d$ and $|v|\le 2R$; R being a positive constant such that $\mathrm{Supp}(g^0) \subset \left\{(x,v)\in\mathbb{R}_x^d\times\mathbb{R}_v^d / |x|\le R,\ |v|\le R\right\}$.

It is then sufficient to show that under the hypothesis of Lemma 2.1, the problem (2.13) satisfies the conditions (4.2) – (4.9) of [32, Theorem II, P. 195]. This is let to the reader.

Hence, [32, Theorem II] insures the existence of T', $0 < T' \leq T$, such that the CP (2.10), admits a unique solution

$$f \in C^1\left([0,T']; H^s\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)\right) \bigcap C^1\left([0,T']; H^{s-1}\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)\right). \tag{2.15}$$

It then remains to show that the above solution f is global in time.

Step 2. Proof of the global existence

Here, we prove that the local solution obtained in (2.15) is, in fact, a global solution. To that end, let us denote by $[0,T^*]$, where $T^* > 0$, the maximal domain (of time t) where the solution of Cauchy problem (2.19) exists, and is denoted by $\tilde{f}$. In other words we have

$$\begin{cases} \frac{\partial \tilde{f}}{\partial t}(t,x,v) + v \cdot \nabla_x \tilde{f}(t,x,v) + E(t,x) \cdot \nabla_v \tilde{f}(t,x,v) = \mathcal{Q}\tilde{f}(t,x,v) & \text{in} \quad [0,T^*] \times \mathbb{R}_x^d \times \mathbb{R}_v^d, \\ \tilde{f}(0,x,v) = f^0(x,v) & \text{in} \quad \mathbb{R}_x^d \times \mathbb{R}_v^d. \end{cases} \tag{2.16}$$

If $T^* = +\infty$, the problem is solved. We are going to show that, if we suppose that $T^* < +\infty$, then the solution $\tilde{f}$ can be extended beyond T^*, which contradicts the maximality of T^*. In fact, supposing $0 < T^* < +\infty$ and let $t_0 \in [0,T^*]$. Thanks to [32, Theorem II], where 0 is just replaced by t_0 (up to replace the unknown function $f(t,x,v)$ by $g(t,x,v) = f(t+t_0,x,v)$), there exists a strictly positive number $\delta > 0$, independent of t_0, such that the following CP in f

$$\begin{cases} \frac{\partial f}{\partial t} + v \cdot \nabla_x f + E \cdot \nabla_v f = \mathcal{Q} f & \text{in} \quad [t_0, t_0+\delta] \times \mathbb{R}_x^d \times \mathbb{R}_v^d, \\ f(t_0,x,v) = \tilde{f}(t_0,x,v) & \text{in} \quad \mathbb{R}_x^d \times \mathbb{R}_v^d, \end{cases} \tag{2.17}$$

has a unique solution $f \in C^1\left([t_0,t_0+\delta]; H^s\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)\right) \bigcap C^1\left([t_0,t_0+\delta]; H^{s-1}\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)\right)$. Then, by taking t_0 sufficiently close to T^*, for example, t_0 such that $0 < T^* - t_0 < \dfrac{\delta}{2}$; we have $T^* < t_0 + \dfrac{\delta}{2}$. Hence, we can extend the solution $\tilde{f}$ to $[0; t_0+\delta]$ which contains strictly $[0; T^*]$, and this contradict the maximality of T^*. Hence we have

$$f \in C^1\left(0,\infty; H^s\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)\right) \bigcap C^1\left(0,\infty; H^{s-1}\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right)\right), \tag{2.18}$$

and therefore Lemma 2.1. □

Remark 1. As in the proof of [47, Theorem 5.1, P. 36], we can show that the assumption about the compactness of the support of f^0 leads to a solution f which is also compactly supported. But this property of f is not necessary for the rest of our work.

We can prove now the main Theorem 2.1, thanks to Lemma 2.1. The proof follows with some deferences, the approach of Strong-weak solution (see e.g. [25] and references therein), whose the global idea is to prove the uniqueness of weak solution using strong solutions [25].

Proof of Theorem 2.1. The proof is done in three steps.

Step 1: Existence of the weak solution.

We use here the natural density of spaces $\mathcal{C}_0^s$ in Lebesgue L^p $(1 \leq p < \infty)$. Let us consider for each fixed $\varepsilon > 0$, the triplet

$$\left(E^\varepsilon, \frac{1}{\varepsilon}\sigma^\varepsilon, f^0\right) \in \left[C\left([0,\infty); L^2_{loc}\left(\mathbb{R}_x^d\right)\right)\right]^d \times C\left([0,\infty); C_0\left(\mathbb{R}_x^d; L^2\left(\mathbb{R}_v^d \times \mathbb{R}_w^d\right)\right)\right) \times L^2\left(\mathbb{R}_x^d \times \mathbb{R}_v^d\right),$$

as given by the conditions (2.1)-(2.3) of Theorem 2.1. For $s \in \mathbb{N}$ (the nonnegative integers), we have the following continuous and dense inclusion

$$\left[C\left([0,\infty);C_0^s\left(\mathbb{R}_x^d\right)\right)\right]^d \times C\left([0,\infty);C_0^s\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\times\mathbb{R}_w^d\right)\right)\times C_0^s\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right) \hookrightarrow$$
$$\left[C\left([0,\infty);L_{loc}^2\left(\mathbb{R}_x^d\right)\right)\right]^d \times C\left([0,\infty);C_0\left(\mathbb{R}_x^d;L^2\left(\mathbb{R}_v^d\times\mathbb{R}_w^d\right)\right)\right)\times L^2\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right).$$

Thus, we can find a sequence $\left(E_n^\varepsilon,\sigma_n^\varepsilon,g_n^0\right)_{n\in\mathbb{N}}$ in the space

$$\left[C\left([0,\infty);C_0^s\left(\mathbb{R}_x^d\right)\right)\right]^d \times C\left([0,\infty);C_0^s\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\times\mathbb{R}_w^d\right)\right)\times C_0^s\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right),$$

which strongly converges to $(E^\varepsilon,\frac{1}{\varepsilon}\sigma^\varepsilon,f^0)$. For each element $(E_n^\varepsilon,\sigma_n^\varepsilon,g_n^0)$ of that sequence, Lemma 2.1 insures the existence of a unique $f_{\varepsilon,n}\in C^1\left([0,\infty);H_0^{s-1}\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right)$, solution of the Cauchy problem

$$\left\{\begin{array}{lll} \frac{\partial f_{\varepsilon,n}}{\partial t}+v\cdot\nabla_x f_{\varepsilon,n}+E_n^\varepsilon\cdot\nabla_v f_{\varepsilon,n}=\mathcal{Q}_n^\varepsilon f_{\varepsilon,n} & \text{in} & [0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v^d, \\ f_{\varepsilon,n}(0,x,v)=g_n^0(x,v) & \text{in} & \mathbb{R}_x^d\times\mathbb{R}_v^d, \end{array}\right. \tag{2.19}$$

with

$$\mathcal{Q}_n^\varepsilon f_{\varepsilon,n}(t,x,v)=\int_{\mathbb{R}_v^d}\sigma_n^\varepsilon(t,x,v,w)f_{\varepsilon,n}(t,x,w)dw-\Sigma_n^\varepsilon(t,x,v)f_{\varepsilon,n}(t,x,v) \tag{2.20}$$

and

$$\Sigma_n^\varepsilon(t,x,v)=\int_{\mathbb{R}_w^d}\sigma_n^\varepsilon(t,x,v,w)dw. \tag{2.21}$$

Hence one deduces the existence of a bounded sequence $(f_{\varepsilon,n})_{n\in\mathbb{N}}$ in $C^1\left([0,\infty);H_0^{s-1}\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right)$, *i.e.*, also bounded in $C^1\left([0,\infty);L^2\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right)$, for which each element solves the Cauchy problem (2.19). According to Banach-Alaoglu Theorem [15], we can therefore extract a subsequence still noted by $(f_{\varepsilon,n})_{n\in\mathbb{N}}$, which weakly converges in $C^1\left([0,\infty);L^2\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right)$ towards a unique element $f_\varepsilon\in C^1\left([0,\infty);L^2\left(\mathbb{R}_x^d\times\mathbb{R}_v^d\right)\right)$.

Let us show that f_ε is the weak solution of our Cauchy problem (1.5). For that, let $\varphi\in C_0^1\left([0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v^d\right)$ be a test function. Multiplying the first equation of (2.19) above by φ and integrating over $[0,\infty)\times\mathbb{R}_x^d\times\mathbb{R}_v^d$, using the fact that φ has compact support, the semi-detailed balance condition (1.7) and the initial condition in (2.19), we have

$$\begin{aligned} &-\int_0^\infty\int_{\mathbb{R}_{x,v}^{2d}} f_{\varepsilon,n}\left(\frac{\partial\varphi}{\partial t}+v\cdot\nabla_x\varphi+E_n^\varepsilon\cdot\nabla_v\varphi+(\mathcal{Q}_n^\varepsilon)^*\varphi\right)(t,x,v)dvdxdt \\ &=\int_{\mathbb{R}_{x,v}^{2d}}\varphi(0,x,v)g_n^0(x,v)dvdx, \end{aligned} \tag{2.22}$$

where $\mathbb{R}_{x,v}^{2d}=\mathbb{R}_x^d\times\mathbb{R}_v^d$ and $(\mathcal{Q}_n^\varepsilon)^*$ is the adjoint of $\mathcal{Q}_n^\varepsilon$ defined by

$$(\mathcal{Q}_n^\varepsilon)^*\varphi(t,x,v)=\int_{\mathbb{R}_v^d}\sigma_n^\varepsilon(t,x,w,v)\varphi(t,x,w)dw-\Sigma_n^\varepsilon(t,x,v)\varphi(t,x,v)$$

and

$$\Sigma_n^\varepsilon(t,x,v)=\int_{\mathbb{R}_w^d}\sigma_n^\varepsilon(t,x,v,w)dw.$$

Passing to the limit in (2.22) as $n\mapsto\infty$, since $(E_n^\varepsilon,\sigma_n^\varepsilon,g_n^0)_{n\in\mathbb{N}}$ strongly converges to $(E^\varepsilon,\frac{1}{\varepsilon}\sigma^\varepsilon,f^0)$ we then obtain

$$-\int_0^\infty\int_{\mathbb{R}_{x,v}^{2d}} f_\varepsilon(t,x,v)\left(\frac{\partial\varphi}{\partial t}+v\cdot\nabla_x\varphi+E^\varepsilon\cdot\nabla_v\varphi+\frac{1}{\varepsilon}(\mathcal{Q}^\varepsilon)^*\varphi\right)(t,x,v)dvdxdt \tag{2.23}$$

$$= \int_{\mathbb{R}^{2d}_{x,v}} \varphi(0,x,v) f^0(x,v) dv dx, \qquad \forall \varphi \in C^1_0\left([0,\infty) \times \mathbb{R}^d_x \times \mathbb{R}^d_v\right).$$

That is $f_\varepsilon \in C^1\left([0,\infty); L^2\left(\mathbb{R}^d_x \times \mathbb{R}^d_v\right)\right)$ is a weak solution of (1.5).

Step 2: Uniqueness of the weak solution

To show the uniqueness, let us consider two solutions f_1 and f_2 of problem (1.5). Then, their difference $g = f_1 - f_2$ is solution of the following problem:

$$\begin{cases} \frac{\partial g}{\partial t} + v \cdot \nabla_x g + E^\varepsilon \cdot \nabla_v g = \frac{1}{\varepsilon} \mathcal{Q}^\varepsilon g & \text{in} \quad [0,\infty) \times \mathbb{R}^d_x \times \mathbb{R}^d_v, \\ g = 0 & \text{in} \quad \mathbb{R}^d_x \times \mathbb{R}^d_v. \end{cases} \tag{2.24}$$

We then multiply the first equation of (2.24) by g and integrate the obtained equation over $\mathbb{R}^d_x \times \mathbb{R}^d_v$ to have

$$\frac{1}{2} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_x} \left(\frac{\partial |g|^2}{\partial t} + v \cdot \nabla_x |g|^2 + E^\varepsilon \cdot \nabla_v |g|^2 \right) dx dv = \frac{1}{\varepsilon} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_x} (g \mathcal{Q}^\varepsilon g) dx dv. \tag{2.25}$$

Now, up to suppose without lost of generality that g has compact support, integrating by parts the second term at the left hand side of (2.25), we easily show that

$$\int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_x} v \cdot \nabla_x |g|^2 dv dx \quad = \quad 0. \tag{2.26}$$

In the same way, we show that

$$\int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_x} E^\varepsilon(t,x) \cdot \nabla_v |g|^2(t,x,v) dx dv = 0. \tag{2.27}$$

For the term at right hand side of (2.25), thanks to semi-detailed balance condition (1.7), we apply [5, Lemma 2.1., d)] to have

$$\int_{\mathbb{R}^{2d}_{v,x}} (g \mathcal{L}^\varepsilon g)(t,x,v) dx dv = -\frac{1}{2} \int_{\mathbb{R}^{2d}_{v,x}} \int_{\mathbb{R}^d_w} \sigma^\varepsilon(t,x,v,w)(g(t,x,v) - g(t,x,w))^2 dw dx dv. \tag{2.28}$$

Taking into account (2.26)-(2.28), the energy equation (2.25) becomes

$$\frac{\partial}{\partial t}(\|g(t)\|^2_{L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v)}) = -\frac{1}{\varepsilon} \int_{\mathbb{R}^d_v} \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_w} \sigma^\varepsilon(t,x,v,w)(g(t,x,v) - g(t,x,w))^2 dw dx dv \leq 0. \tag{2.29}$$

Now, integrating inequality (2.29) over $[0;t[$ $(t>0)$, in the view of initial condition $g(0,x,v) = 0$, we have $\|g(t)\|^2_{L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v)} \leq 0$, which implies that

$$f_1 - f_2 = g = 0 \text{ for almost every } \ (t,x,v) \in [0,\infty) \times \mathbb{R}^d_x \times \mathbb{R}^d_v.$$

Whence the uniqueness of the solution of (1.5).

Step 3: Uniform estimate, positivity and stability for the global weak solution.

1) As regard the a priori estimate (2.5), let just multiply the first equality of (1.5) by f_ε and integrate the equation obtained over $\mathbb{R}^d_x \times \mathbb{R}^d_v$ to have

$$\frac{1}{2} \int_{\mathbb{R}^{2d}_{v,x}} \left(\frac{\partial |f_\varepsilon|^2}{\partial t} + v \cdot \nabla_x |f_\varepsilon|^2 + E^\varepsilon \cdot \nabla_v |f_\varepsilon|^2 \right) dx dv = \frac{1}{\varepsilon} \int_{\mathbb{R}^{2d}_{v,x}} f_\varepsilon (\mathcal{Q}^\varepsilon f_\varepsilon) dx dv. \tag{2.30}$$

Now, we can proceed as at previous step for equalities (2.26), (2.27) and (2.28), where, g is replaced by f_ε, so that equation (2.30) above becomes

$$\int_{\mathbb{R}^{2d}_{v,x}} \frac{\partial |f_\varepsilon|^2}{\partial t}(t,x,v)dxdv = -\frac{1}{\varepsilon}\int_{\mathbb{R}^{2d}_{v,x}}\int_{\mathbb{R}^d_w} \sigma^\varepsilon(t,x,v,w)(f_\varepsilon(t,x,v) - f_\varepsilon(t,x,w))^2 dwdxdv,$$

which implies that

$$\frac{\partial}{\partial t}\|f_\varepsilon(t)\|^2_{L^2(\mathbb{R}^{2d}_{x,v})} \leq 0. \tag{2.31}$$

Hence, integrating this last inequality on $t \in [0,\tau]$ $(\tau > 0)$, we have

$$\|f_\varepsilon(\tau)\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \leq \|f^0\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)};$$

thereby implying the inequality

$$\|f_\varepsilon\|^2_{L^\infty([0,\infty);L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v))} \leq \|f^0\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)}.$$

Finally, for any constant $T > 0$, integrating (2.31) over $t \in [0,T]$, we obtain the estimation

$$\|f_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^d_x\times\mathbb{R}^d_v)} \leq T\|f^0\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)}.$$

2) Let us show now that the solutions f_ε remains nonnegative for nonnegative initial data. Let us use $f_\varepsilon^- = \min(0, f_\varepsilon)$ as a test function, to multiply the first equality of (1.5) and perform the same calculations as for (2.25) -(2.27) above, while using [5, Lemma 2.1., d)] once more. We obtain the following equality similar to (2.28):

$$\begin{aligned}\int_{\mathbb{R}^{2d}_{v,x}} \frac{\partial |f_\varepsilon^-|^2}{\partial \tau}(\tau,x,v)dxdv \;\;&=\;\; -\frac{1}{\varepsilon}\int_{\mathbb{R}^{2d}_{v,x}}\int_{\mathbb{R}^d_w}\sigma^\varepsilon(\tau,x,v,w)\left(f_\varepsilon^-(\tau,x,v) - f_\varepsilon^-(\tau,x,w)\right)^2 dwdxdv \\ &\quad +\frac{2}{\varepsilon}\int_{\mathbb{R}^{2d}_{v,x}}\int_{\mathbb{R}^d_w}\sigma^\varepsilon(\tau,x,v,w)f_\varepsilon^-(\tau,x,v)f_\varepsilon^+(\tau,x,w)dwdxdv,\end{aligned}$$

where $f_\varepsilon^+ = \max(0, f_\varepsilon)$ and since $f_\varepsilon = f_\varepsilon^- + f_\varepsilon^+$. Finally, integrating this last equality on $\tau \in [0,t]$, we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^{2d}_{v,x}}\int_{\mathbb{R}^d_w}\sigma^\varepsilon(\tau,x,v,w)\left(f_\varepsilon^-(\tau,x,v) - f_\varepsilon^-(\tau,x,w)\right)^2 dwdxdvd\tau + \int_{\mathbb{R}^{2d}_{v,x}}|f_\varepsilon^-|^2(t,x,v)dxdv \\ &-\frac{2}{\varepsilon}\int_0^t\int_{\mathbb{R}^{2d}_{v,x}}\int_{\mathbb{R}^d_w}\sigma^\varepsilon(\tau,x,v,w)f_\varepsilon^-(\tau,x,v)f_\varepsilon^+(\tau,x,w)dwdxdvd\tau = \int_{\mathbb{R}^{2d}_{v,x}}|(f^0)^-|^2(x,v)dxdv.\end{aligned} \tag{2.32}$$

Since σ^ε is nonnegative, all terms in the above left-hand-side are nonnegative. The hypothesis (2.3) on the nonnegative initial data f^0 implies that the right-hand-side vanishes. Accordingly, each term at right-hand-side vanishes. In particular, we have

$$\int_{\mathbb{R}^{2d}_{v,x}}|f_\varepsilon^-|^2(t,x,v)dxdv = 0,$$

proving that $f_\varepsilon^-(t,x,v) = 0$ for a.e. (t,x,v). Thus the solution f_ε stays non negative at all times, since it is continuous with respect to t.

3) To end this proof, let show that the solution f_ε is continuous with respect to the initial data f^0. Indeed, let $(f_n^0)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d_x \times \mathbb{R}^d_v)$. For a fixed n and ε, let $f_{\varepsilon,n}$ be the unique weak solutions of (1.5) with $f_{\varepsilon,n}(0,x,v) = f_n^0(x,v)$. As previously, $g = f_\varepsilon - f_{\varepsilon,n}$ is solution of (1.5) with $g(0,x,v) = f_\varepsilon(0,x,v) - f_{\varepsilon,n}(0,x,v) = f^0(x,v) - f_n^0(x,v)$. Thus, performing the same calculations

as for (2.25)-(2.29), we obtain an inequality similar to (2.31), where f_ε is replaced by $f_\varepsilon - f_{\varepsilon,n}$, that is

$$\frac{\partial}{\partial\tau}\|(f_\varepsilon - f_{\varepsilon,n})(\tau)\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \le 0. \tag{2.33}$$

Hence, integrating this last inequality on $\tau \in [0,t]$ $(t>0)$, we have

$$\|(f_\varepsilon - f_{\varepsilon,n})(t)\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \le \|f^0 - f^0_n\|^2_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)}.$$

This proves that: if $\|f^0_n - f^0\|_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \to 0$ as $n\to\infty$, then $\|f_{\varepsilon,n}(t) - f_\varepsilon(t)\|_{L^2(\mathbb{R}^d_x\times\mathbb{R}^d_v)} \to 0$ uniformly in t. Hence, we have the stability of f_ε with respect to the initial data f^0. □

In the sequel of this work, for practical measures, we will consider that the velocities v of particles lie in V, rather than $\mathbb{R}^d_v$ and the time in t in a compact interval $[0,T]$, $T>0$ being any given constant and V a subset of $\mathbb{R}^d_v$ which properties will be specified. Hence, up to consider a restriction of f_ε to $[0,T]\times\mathbb{R}^d_x\times V$, the results of this section will still be valid.

3 Algebras with mean value and Σ-convergence

In this section we recall the main properties and some basic facts about the concept of Σ-convergence which is closely related to that of algebras with mean value (algebra wmv, in short), also abodes here. We refer the reader to [37, 48, 49] for more details regarding most of the results of this section.

In the sequel of this work, we will often set $\mathbb{R}^{1+d}_T = [0,T]\times\mathbb{R}^d_x$.

3.1 Algebras with mean value

Let A be an algebra with mean value (algebra wmv, for short) on $\mathbb{R}^d$, that is, a closed subalgebra of the Banach algebra BUC($\mathbb{R}^d$) (of bounded uniformly continuous real-valued functions on $\mathbb{R}^d$) that contains the constants, is close under complex conjugation ($\overline{u}\in A$ whenever $u\in A$), is translation invariant ($\tau_a u = u(\cdot + a)\in A$ for any $u\in A$ and $a\in\mathbb{R}^d$) and is such that any of its elements possesses a mean value in the following sense: for every $u\in A$,

$$M(u) = \lim_{R\to\infty} \fint_{B_R} u(y)dy \tag{3.1}$$

where B_R stands for the open ball in $\mathbb{R}^d$ of radius R centered at the origin,q and $\fint_{B_R} = \frac{1}{|B_R|}\int_{B_R}$.

Let $u\in$ BUC($\mathbb{R}^d$) and assume that $M(u)$ exists. Then, defining the sequence $(u^\varepsilon)_{\varepsilon>0}\subset$ BUC($\mathbb{R}^d$) by $u^\varepsilon(x) = u(\frac{x}{\varepsilon})$ for $x\in\mathbb{R}^d$, we have

$$u^\varepsilon \to M(u) \text{ in } L^\infty(\mathbb{R}^d)\text{-weak} * \text{ as } \varepsilon\to 0.$$

This is an easy consequence of the fact that the set of finite linear combinations of the characteristic functions of open balls in $\mathbb{R}^d$ is dense in $L^1(\mathbb{R}^d)$.

Let A be an algebra with mean value. We define the space A^∞ by

$$A^\infty = \left\{u\in A : D^\alpha_y u\in A \text{ for every } \alpha = (\alpha_1,...,\alpha_d)\in\mathbb{N}^d\right\},$$

where $D^\alpha_y = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1}...\partial y_d^{\alpha_d}}$. Then endowed with the family of norms $|||\cdot|||_m$ defined by $|||u|||_m = \sup_{|\alpha|\le m}\sup_{y\in\mathbb{R}^d}|D^\alpha_y u|$, A^∞ is a Fréchet space.

In order to define the generalized Besicovitch space, we first need to define the Marcinkiewicz space $\mathfrak{M}^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$, which is the space of functions $u \in L^p_{loc}(\mathbb{R}^d)$ satisfying

$$\limsup_{R\to\infty} \fint_{B_R} |u(y)|^p \, dy < \infty.$$

Endowed with the seminorm

$$\|u\|_p = \limsup_{R\to\infty} \left(\fint_{B_R} |u(y)|^p \, dy \right)^{\frac{1}{p}},$$

$\mathfrak{M}^p(\mathbb{R}^d)$ is a complete seminormed space. Next we define the generalized Besicovitch space $B^p_A(\mathbb{R}^d)$ $(1 \leq p < \infty)$ associated to the algebra with mean value A as the closure in $\mathfrak{M}^p(\mathbb{R}^d)$ of A with respect to $\|\cdot\|_p$. It is easy to see that for $f \in A$ and $0 < p < \infty$, $|f|^p \in A$, so that

$$\|f\|_p = \left(\lim_{R\to\infty} \fint_{B_R} |f(y)|^p \right)^{\frac{1}{p}} \equiv (M(|f|^p))^{\frac{1}{p}}. \tag{3.2}$$

The equality (3.2) extends by continuity to any $f \in B^p_A(\mathbb{R}^d)$. Equipped with the seminorm (3.2), $B^p_A(\mathbb{R}^d)$ is a Fréchet *i.e.* complete local convex space, whose topology is defined by the family of seminorm $|||\cdot|||_m$ mentioned above. We refer the reader to [43, 48, 49] for further details about these spaces. Namely, the following holds true:

(1) The space $\mathcal{B}^p_A(\mathbb{R}^d) = B^p_A(\mathbb{R}^d)/\mathcal{N}$, where $\mathcal{N} = \{u \in B^p_A(\mathbb{R}^d) : \|u\|_p = 0\}$, is a Banach space under the norm $\|u + \mathcal{N}\|_p = \|u\|_p$ for $u \in B^p_A(\mathbb{R}^d)$.

(2) The mean value $M : A \to \mathbb{R}$ extends by continuity to a continuous linear mapping (still denoted by M) on $B^p_A(\mathbb{R}^d)$. Furthermore, considered as defined on $B^p_A(\mathbb{R}^d)$, M extends in a natural way to $\mathcal{B}^p_A(\mathbb{R}^d)$ as follows: for $u = v + \mathcal{N} \in \mathcal{B}^p_A(\mathbb{R}^d)$, we set $M(u) := M(v)$. This is well defined since $M(v) = 0$ for any $v \in \mathcal{N}$.

In the current work, we will deal with the concept of *ergodic* algebras with mean value. A function $u \in \mathcal{B}^1_A(\mathbb{R}^d)$ is said to be *invariant* if for any $y \in \mathbb{R}^d$, $\|u(\cdot + y) - u\|_1 = 0$. This being so, an algebra with mean value A is ergodic if every invariant function u is constant in $\mathcal{B}^1_A(\mathbb{R}^d)$, i.e. if $\|u(\cdot + y) - u\|_1 = 0$ for any $y \in \mathbb{R}^d$, then $\|u - c\|_1 = 0$ where c is a constant. We assume that all the algebras with mean value used in the sequel are ergodic.

We also define the notion of vector-valued algebra with mean value. Indeed, for $\mathbb{G}$ a Banach space, we denote by $\mathrm{BUC}(\mathbb{R}^d; \mathbb{G})$ the Banach space of bounded uniformly continuous functions $u : \mathbb{R}^d \to \mathbb{G}$, endowed with the norm

$$\|u\|_\infty = \sup_{y\in\mathbb{R}^d} \|u(y)\|_{\mathbb{G}}$$

where $\|\cdot\|_{\mathbb{G}}$ stands for the norm in $\mathbb{G}$. Let A be an algebra with mean value on $\mathbb{R}^d$. We denote by $A \otimes \mathbb{G}$ the usual space of functions of the form

$$\sum_{finite} u_i \otimes e_i \text{ with } u_i \in A \text{ and } e_i \in \mathbb{G}$$

where $(u_i \otimes e_i)(y) = u_i(y)e_i$ for $y \in \mathbb{R}^d$. We then define the vector-valued algebra wmv $A(\mathbb{R}^d; \mathbb{G})$ as the closure of $A \otimes \mathbb{G}$ in $\mathrm{BUC}(\mathbb{R}^\mathrm{d}; \mathbb{G})$.

Now, let $f \in A(\mathbb{R}^d;\mathbb{G})$. Then, defining $\|f\|_{\mathbb{G}}$ by $\|f\|_{\mathbb{G}}(y) = \|f(y)\|_{\mathbb{G}}$ ($y \in \mathbb{R}^d$), we have that $\|f\|_{\mathbb{G}} \in A$. Similarly, we can define (for $0 < p < \infty$) the function $\|f\|_{\mathbb{G}}^p$, and $\|f\|_{\mathbb{G}}^p \in A$. This allows us to define the Besicovitch seminorm on $A(\mathbb{R}^d;\mathbb{G})$ as follows : for $1 \leq p < \infty$,

$$\|f\|_{p,\mathbb{G}} = \left(\lim_{R\to\infty} \fint_{B_R} \|f(y)\|_{\mathbb{G}}^p \, dy\right)^{\frac{1}{p}} \equiv (M(\|f\|_{\mathbb{G}}^p))^{\frac{1}{p}} \text{ for } f \in A(\mathbb{R}^d;\mathbb{G}).$$

Now, we can define the Besicovitch space $B_A^p(\mathbb{R}^d;\mathbb{G})$ as the completion of $A(\mathbb{R}^d;\mathbb{G})$ with respect to $\|\cdot\|_{p,\mathbb{G}}$. The space $B_A^p(\mathbb{R}^d;\mathbb{G})$ is a complete seminormed subspace of $L_{loc}^p(\mathbb{R}^d;\mathbb{G})$, and the following holds true:

1. The space $\mathcal{B}_A^p(\mathbb{R}^d;\mathbb{G}) = B_A^p(\mathbb{R}^d;\mathbb{G})/\mathcal{N}$, where $\mathcal{N} = \{u \in B_A^p(\mathbb{R}^d;\mathbb{G}) : \|u\|_{p,\mathbb{G}} = 0\}$, is a Banach space under the norm $\|u + \mathcal{N}\|_{p,\mathbb{G}} = \|u\|_{p,\mathbb{G}}$ for $u \in B_A^p(\mathbb{R}^d;\mathbb{G})$.

2. The mean value $M : A(\mathbb{R}^d;\mathbb{G}) \to \mathbb{G}$ extends by continuity to a continuous linear mapping (still denoted by M) on $B_A^p(\mathbb{R}^d;\mathbb{G})$ satisfying

   $$L(M(u)) = M(L(u)) \text{ for all } L \in \mathbb{G}' \text{ and } u \in B_A^p(\mathbb{R}^d;\mathbb{G}).$$

   Moreover, for $u \in B_A^p(\mathbb{R}^d;\mathbb{G})$ we have

   $$\|u\|_{p,\mathbb{G}} = [M(\|u\|_{\mathbb{G}}^p)]^{\frac{1}{p}} \equiv \left[\lim_{R\to\infty} \fint_{B_R} \|u(y)\|_{\mathbb{G}}^p \, dy\right]^{\frac{1}{p}},$$

   and for $u \in \mathcal{N}$, one has $M(u) = 0$. It is to be noted that $\mathcal{B}_A^2(\mathbb{R}^d;H)$ (when $\mathbb{G} = H$ is a Hilbert space) is a Hilbert space with inner product

   $$(u,v)_2 = M[(u,v)_H] \text{ for } u,v \in \mathcal{B}_A^2(\mathbb{R}^d;H), \tag{3.3}$$

   $(,)_H$ denoting the inner product in H.

If in particular $\mathbb{G} = \mathbb{R}$, then $B_A^p(\mathbb{R}^d) := B_A^p(\mathbb{R}^d;\mathbb{R})$ and $\mathcal{B}_A^p(\mathbb{R}^d) := \mathcal{B}_A^p(\mathbb{R}^d;\mathbb{R})$. The mean value extends in a natural way to $\mathcal{B}_A^p(\mathbb{R}^d)$ as follows: for $u = v + \mathcal{N} \in \mathcal{B}_A^p(\mathbb{R}^d)$, we set $M(u) := M(v)$; which his is well-defined since $M(w) = 0$ for any $w \in \mathcal{N}$. The Besicovitch seminorm in $B_A^p(\mathbb{R}^d)$ is merely denoted by $\|\cdot\|_p$, and we have $B_A^q(\mathbb{R}^d) \subset B_A^p(\mathbb{R}^d)$ for $1 \leq p \leq q < \infty$.

For special purposes, we consider the case where $A = AP(\mathbb{R}^d)$ is the algebra of almost periodic continuous complex functions on $\mathbb{R}^d$ and $\mathbb{H} = L^2(V)$ the usual Lebesgue space of complex-valued square integrable functions; V being a subset of $\mathbb{R}^d$ endowed with de measure $d\mu$. The L^2-valued algebra wmv $A(\mathbb{R}^d;\mathbb{H}) = A(\mathbb{R}^d;L^2(V))$ is now the closure in the L^2-valued algebra of bounded uniformly continuous functions $\mathrm{BUC}(\mathbb{R}^d;L^2(V))$ of the L^2-valued trigonometric polynomial $\mathcal{P}(L^2(V))$ defined by: $P \in \mathcal{P}(L^2(V))$ iff

$$P(x,v) = \sum_{j=1}^{m} c_j(v)e^{i\lambda_j\cdot x}, \quad \forall (x,v) \in \mathbb{R}^d \times V, \tag{3.4}$$

where $c_j \in L^2(V)$, the $\lambda_j \in \mathbb{R}^d$ are distinct ($\lambda_j \neq \lambda_k$ if $j \neq k$), $i^2 = -1$ and $m \in \mathbb{N}$ is finite. For each P of form (3.4) above, we define its spectrum by

$$\mathcal{S}p(f) = \left\{\lambda_j \in \mathbb{R}^d : c_j \neq 0\right\}. \tag{3.5}$$

The L^2-valued generalized Besicovitch space of almost periodic functions $B^2_{ap}(\mathbb{R}^d; L^2(V))$ is defined in the *sens of Bohr* by: $f \in B^2_{ap}(\mathbb{R}^d; L^2(V))$ iff there exists a sequence of L^2-valued trigonometric polynomials $(P_n)_{n\in\mathbb{N}} \subset \mathcal{P}\left(L^2(V)\right)$ such that

$$\|f\|_{B^2_{ap}(\mathbb{R}^d;L^2(V))} = \lim_{n\to\infty} \|P_n\|_{B^2_{ap}(\mathbb{R}^d;L^2(V))} = \lim_{n\to\infty} \left(\lim_{R\to\infty} ⨍_{B_R} \int_V P_n^2(x,v)d\mu(v)dx \right)^{\frac{1}{2}}. \quad (3.6)$$

This defines a seminorm on $B^2_{ap}(\mathbb{R}^d; L^2(V))$.

The spaces $B^2_{ap}(\mathbb{R}^d)$ for $\mathbb{H} = \mathbb{R}$ and the Banach $\mathcal{B}^2_{ap}(\mathbb{R}^d; L^2(V)) = B^2_{ap}(\mathbb{R}^d; L^2(V))/\mathcal{N}$ are defined similarly as above and equality (3.6) also defines a norm on the Banach $\mathcal{B}^2_{ap}(\mathbb{R}^d; L^2(V))$. In the sequel, $\|\cdot\|_{B^2_{ap}(\mathbb{R}^d;L^2(V))}$ will simply be noted $\|\cdot\|_{2,2}$ and $\|\cdot\|_{B^2_{ap}(\mathbb{R}^d)}$ will be noted $\|\cdot\|_2$. We recall the following usual map:

$$\lambda \mapsto a(\lambda; P) = \lim_{R\to\infty} ⨍_{B_R} P(x)e^{-i\lambda\cdot x}dx = \begin{cases} c_j \text{ if } \lambda = \lambda_j \in \mathcal{S}p(P) \\ 0 \text{ if not} \end{cases}, \quad (3.7)$$

called the *Bohr transform* of $P \in \mathcal{P}(L^2(V))$. By so doing, we define the *Bohr transform* of all $f \in B^q_{ap}(\mathbb{R}^d; L^2(V))$ by:

$$\lambda \mapsto a(\lambda; f) = \lim_{R\to\infty} ⨍_{B_R} f(x)e^{-i\lambda\cdot x}dx = \lim_{n\to\infty} a(\lambda; P_n),$$

for $(P_n)_{n\in\mathbb{N}} \subset \mathcal{P}(L^2(V))$ and converging towards f in the sense of equality (3.6). We then call *spectrum* of $f \in B^2_{ap}(\mathbb{R}^d; L^2(V))$, the subset of $\mathbb{R}^d$ defined by

$$\mathcal{S}p(f) = \left\{\lambda \in \mathbb{R}^d : a(\lambda; f) \neq 0\right\}. \quad (3.8)$$

Now, let Λ be a countable subset of $\mathbb{R}^d$ satisfying the so called β-condition:

$$\sum_{\lambda\in\Lambda} \frac{1}{|\lambda|^\gamma} \begin{cases} < \infty & \text{for} \quad \gamma > \beta, \\ = \infty & \text{for} \quad \gamma \leq \beta. \end{cases} \quad (3.9)$$

Such condition is *e.g.*, satisfied for $\Lambda = \mathbb{Z}^d\backslash\{0\}$ and $\beta = d$ [4, 16]. Hence, we define

$$B^2_{ap}(\Lambda; L^2(V)) = \left\{f \in B^2_{ap}(\mathbb{R}^d; L^2(V)) : \mathcal{S}p(f) \subset \Lambda\right\}. \quad (3.10)$$

$B^2_{ap}(\Lambda; L^2(V))$ is a separable subspace of $B^2_{ap}(\mathbb{R}^d; L^2(V))$ [16, 4] associated to the $L^2(V)$-valued algebra wmv $AP(\Lambda; L^2(V)) = \left\{f \in AP(\mathbb{R}^d; L^2(V)) : \mathcal{S}p(f) \subset \Lambda\right\}$ and all the properties stated above are still valid in $B^2_{ap}(\Lambda; L^2(V))$. The Banach space $\mathcal{B}^2_{ap}(\Lambda; L^2(V))$ is defined in the same way as previously, as well as the real-valued spaces $AP(\Lambda)$, $B^2_{ap}(\Lambda)$ and $\mathcal{B}^2_{ap}(\Lambda)$, which are merely particular and simplest cases of previous ones.

3.2 The Σ-convergence method

In what follows, the notations are those of the previous subsections.

Let A be an algebra wmv on $\mathbb{R}^d_y$. We know that A is a subalgebra of the $\mathcal{C}^*$-algebra of bounded uniformly continuous functions $\mathrm{BUC}(\mathbb{R}^d_y)$. The generic element of $\mathbb{R}^{1+d}_T$ is denoted by (t,x) while any function in A is of variable $y \in \mathbb{R}^d_y$. For a function $u \in L^p(\mathbb{R}^d_T \times V; B^p_A(\mathbb{R}^d))$ we denote by $u(t,x,\cdot,v)$ (for a.e. $(t,x,v) \in \mathbb{R}^{1+d}_T \times V$) the function defined by

$$u(t,x,\cdot,v)(y) = u(t,x,y,v) \text{ for a.e. } y \in \mathbb{R}^d.$$

Then $u(t,x,\cdot,v) \in B^p_A(\mathbb{R}^d)$, so that the mean value of $u(t,x,\cdot,v)$ is defined accordingly.

Definition 1. *A sequence* $(u_\varepsilon)_{\varepsilon>0} \subset L^p(\mathbb{R}_T^{1+d} \times V)$ $(1 \leq p < \infty)$ *is weakly* Σ-convergent *in* $L^p(\mathbb{R}_T^{1+d} \times V)$ *to some function* $u_0 \in L^p(\mathbb{R}_T^{1+d} \times V; \mathcal{B}_A^p(\mathbb{R}^d))$ *if as* $\varepsilon \to 0$, *we have*

$$\int_{\mathbb{R}_T^{1+d}\times V} u_\varepsilon(t,x,v)\varphi^\varepsilon(t,x,v)d\mu(v)dxdt \to \int_{\mathbb{R}_T^{1+d}\times V} M(u_0(t,x,\cdot,v)\varphi(t,x,\cdot,v))d\mu(v)dxdt, \quad (3.11)$$

for every $\varphi \in L^{p'}(\mathbb{R}_T^{1+d} \times V; A)$ $(\frac{1}{p'} = 1 - \frac{1}{p})$, *where* $\varphi^\varepsilon(t,x,v) = \varphi(t,x,\frac{x}{\varepsilon},v)$. *We express this by writing* $u_\varepsilon \to u_0$ *in* $L^p(\mathbb{R}_T^{1+d} \times V)$-weak Σ.

In the above definition, if $A = \mathcal{C}_{per}(Y)$ is the algebra of continuous periodic functions on Y, where $Y = (0,1)^d$, then (3.11) reads as

$$\int_{\mathbb{R}_T^{1+d}\times V} u_\varepsilon(t,x,v)\varphi^\varepsilon(t,x,v)d\mu(v)dxdt \to \int_{\mathbb{R}_T^{1+d}\times V\times Y} u_0(t,x,y,v)\varphi(t,x,y,v)dyd\mu(v)dxdt,$$

where $u_0 \in L^p(\mathbb{R}_T^{1+d} \times V \times Y)$.

Remark 2. *The above weak* Σ*-convergence in* $L^p(\mathbb{R}_T^d \times \mathbb{R}^m)$ *implies the weak convergence in* $L^p(\mathbb{R}_T^d \times \mathbb{R}^m)$. *One may show as in* [37] *that, for any* $f \in L^p(\mathbb{R}_T^d \times \mathbb{R}^m; A)$, *the sequence* $(f^\varepsilon)_{\varepsilon>0}$ *weakly* Σ*-converges towards* $f + \mathcal{N}$ *in* $L^p(\mathbb{R}_T^d \times \mathbb{R}^m)$. *It is also possible to show (see* [37, Corollary 4.1]*) that the property (3.11) still holds for* $\varphi \in L^2(\mathbb{R}^d; \mathcal{C}(\mathbb{R}_T^d; B_A^{2,\infty}(\mathbb{R}_y^d)))$, *where* $B_A^{2,\infty} = B_A^2 \cap L^\infty(\mathbb{R}_y^d)$ *is endowed with the* $L^\infty(\mathbb{R}_y^d)$*-norm.*

In the sequel, the letter $(\varepsilon)_{\varepsilon>0}$ will always denote any ordinary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying: $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ when $n \to \infty$.

The following result and its proof are simple adaptation of its counterpart in [38, 37, 48].

Theorem 3.1. *Let* $1 < p < \infty$ *and let* $(u_\varepsilon)_{\varepsilon>0} \subset L^p\left(\mathbb{R}_T^{1+d} \times V\right)$ *be a bounded sequence. Then, there exists a subsequence* $(\varepsilon')_{\varepsilon'>0}$ *of* $(\varepsilon)_{\varepsilon>0}$ *and a function* $u \in L^p\left(\mathbb{R}_T^{1+d} \times V; \mathcal{B}_A^p\left(\mathbb{R}^d\right)\right)$ *such that the sequence* $(u'_\varepsilon)_{\varepsilon'>0}$ *weakly* Σ*-converges in* $L^p\left(\mathbb{R}_T^{1+d} \times V\right)$ *to* u.

4 Corrector results

In this section, we deal with a key result that has already been proven by [28] in a periodic setting and generalized to the context of algebras with mean value in our previous work [23]. We can also mention the work of [1] in which similar result is obtained for velocities belonging to the whole space $\mathbb{R}_v^d$.

To start with, let A be an algebra wmv on $\mathbb{R}_y^d$ as defined in the previous section. We consider the unbounded operator P defined on $L^2(V; B_A^2(\mathbb{R}_y^d))$ by

$$Pu = v \cdot \nabla_y u - \mathcal{Q}u, \quad (4.1)$$

with domain

$$D(P) = \{u \in L^2(V; B_A^2(\mathbb{R}_y^d)) : Pu \in L^2(V; B_A^2(\mathbb{R}_y^d))\},$$

where $\mathcal{Q}$ is the operator defined by

$$\mathcal{Q}f(t,x,y,v) = \int_V \sigma(t,x,y,v,w)(f(y,w) - f(y,v))d\mu(w), \ f \in L^2(V; B_A^2(\mathbb{R}_y^d)), \quad (4.2)$$

with kernel σ satisfying the semi-detailed balance condition (1.7). The equality (4.1) above stands in the weak sense in $L^2(V; B^2_A(\mathbb{R}^d_y))$, that is

$$(Pu, \phi) = \int_V M(-uv \cdot \nabla_y \phi - \phi \mathcal{Q} u) d\mu(v) \text{ for all } \phi \in \mathcal{C}^\infty(V; A^\infty). \tag{4.3}$$

As in our previous work [23, Section 3], we can easily show that the domain of P is given by

$$D(P) = \{u \in L^2(V; B^2_A(\mathbb{R}^d_y)) : v \cdot \nabla_y u \in L^2(V; B^2_A(\mathbb{R}^d_y))\} \tag{4.4}$$

and that P extends by continuity to $\mathcal{P}$, on the Banach space $L^2(V; \mathcal{B}^2_A(\mathbb{R}^d_y))$ throughout the equality

$$\mathcal{P}\widetilde{u} = \widetilde{Pu} \quad \text{for} \quad \tilde{u} = u + \mathcal{N} \text{ with } u \in L^2(V; B^2_A(\mathbb{R}^d_y)), \tag{4.5}$$

where $\widetilde{Pu} = Pu + \mathcal{N}$, with $\mathcal{N} = \{u \in B^2_A(\mathbb{R}^d_y) : \|u\|_2 = 0\}$.

Our goal in this section is to solve the auxiliary problems (4.6) and (4.7) below:

i) The corrector problem:

$$\text{For } g \in L^2(V; B^2_A(\mathbb{R}^d_y)), \text{ look for } f \in D(P) \text{ such that } Pf = g \tag{4.6}$$

and

ii) The dual corrector problem:

$$\text{For } \varphi \in L^2(V; B^2_A(\mathbb{R}^d_y)), \text{ look for } \phi \in D(P^*) \text{ such that } P^*\phi = \varphi, \tag{4.7}$$

where P^* is the adjoint of P defined

$$P^*\phi = -v \cdot \nabla_y \phi - \mathcal{Q}^*\phi.$$

To that end, we consider the integral operator K of kernel σ defined on $L^2(V; B^2_A(\mathbb{R}^d_y))$ (uniformly with respect to $(t, x) \in \mathbb{R}^{1+d}_T$) by

$$Kf(y, v) = \int_V \sigma(y, v, w) f(y, w) d\mu(w) \text{ for } f \in L^2(V; B^2_A(\mathbb{R}^d_y)). \tag{4.8}$$

We may easily show (see Step 1 of the proof [23, Proposition 1.]) that K sends continuously $L^2(V; B^2_A(\mathbb{R}^d_y))$ into itself. We assume that

$$K \text{ is compact from } D(P) \text{ into } L^2(V; B^2_A(\mathbb{R}^d_y)). \tag{4.9}$$

Assumption (4.9) is verified in the periodic setting (see [28]) and can be easily extended to the asymptotic periodic setting. We can also show that it is valid in the almost periodic setting and hence, can be extended to the asymptotic almost periodic framework in the same way as in the asymptotic periodic one. In [23], we just gave the outline of the proof of this result. Here in Appendix, we give a more detailed proof in almost periodic setting (see Proposition 6.1).

Next, let us suppose that the set $(V, d\mu(v))$ satisfies the following assumption:

$$\begin{cases} \mu(V) < \infty. \\ \text{There exists a constant } \kappa > 0 \text{ such that} \\ \mu(\{v \in V : |v \cdot \lambda| \leq \alpha|\lambda|\}) \leq \kappa\alpha \text{ for all } \lambda \in \mathbb{R}^d_* = \mathbb{R}^d \backslash \{0\}. \end{cases} \tag{4.10}$$

Let us say a word about assumption (4.10). For $\alpha > 0$ and $\lambda \in \mathbb{R}^d_*$, we define

$$E_{\alpha,\lambda} = \{v \in V : |v \cdot \lambda| > \alpha|\lambda|\} \text{ and } E^c_{\alpha,\lambda} = V \backslash E_{\alpha,\lambda} = \{v \in V : |v \cdot \lambda| \leq \alpha|\lambda|\}. \tag{4.11}$$

$E^c_{\alpha,\lambda}$ is therefore the set of velocities whose orthogonal projection on the λ-axis is inside the band limited by the two planes $\left\{v \in \mathbb{R}^d : |v \cdot \lambda|/|\lambda| = \pm\alpha\right\}$. The condition: there exists $\kappa > 0$ such that $\mu(E^c_{\alpha,\lambda}) < \kappa\alpha$ for all $\lambda \in \mathbb{R}^d_*$ appearing in hypothesis (4.10) is fulfilled in particular if V is bounded and μ is the Lebesgue measure over V, or V is a sphere and μ the surface measure [19]. Assumption (4.10) is of great interest to verify hypothesis (4.9) in almost periodic framework, as it can be seen (proof of Lemma 6.1) in Appendix.

With all this in mind, problems (4.6) and (4.7) are solved by the following proposition.

Proposition 4.1. *Suppose (4.9) holds true and that $\sigma = \sigma(t, x, \cdot, \cdot, \cdot) \in L^\infty(V \times V; B^{2,\infty}_A(\mathbb{R}^d_y))$. Then the following assertions hold:*

(i) There exists a unique function $F \in L^2(V; B^2_A(\mathbb{R}^d_y))$ satisfying

$$PF = 0, \int_V M(F)d\mu(v) = 1 \text{ and } F > 0 \text{ a.e. in } \mathbb{R}^d_y \times V. \tag{4.12}$$

Furthermore F is the unique solution of the variational equation

$$\int_V M(F(v \cdot \nabla_y \phi(\cdot, v) + \mathcal{Q}^*\phi(\cdot, v)))d\mu(v) = 0 \text{ for all } \phi \in \mathcal{C}^\infty(V; A^\infty). \tag{4.13}$$

Similarly, we have $KerP^* = \operatorname{span}\{1_{\mathbb{R}^d_y \times V}\}$, where $1_{\mathbb{R}^d_y \times V}$ stands for the characteristic function of $\mathbb{R}^d_y \times V$.

(ii) Let $g \in L^2(V; B^2_A(\mathbb{R}^d_y))$. The equation (4.6) admits a unique f in $D(P)$ solution if and only if $\int_V M(g)d\mu(v) = 0$. Moreover it holds that

$$\|f\|_{L^2(V;B^2_A(\mathbb{R}^d_y))} \leq C \|g\|_{L^2(V;B^2_A(\mathbb{R}^d_y))}, \tag{4.14}$$

where $C > 0$ is a constant independent of g.

(iii) Let $\varphi \in L^2(V; B^2_A(\mathbb{R}^d_y))$. The equation (4.7) admits a solution $\phi \in D(P^*)$ if and only if $\int_V M(\varphi F)d\mu(v) = 0$. The condition $\int_V M(\phi)d\mu(v) = 0$ ensures the uniqueness of the solution. Furthermore we have

$$\|\phi\|_{L^2\left(V;B^2_A\left(\mathbb{R}^d_y\right)\right)} \leq C \|\varphi\|_{L^2\left(V;B^2_A\left(\mathbb{R}^d_y\right)\right)}, \tag{4.15}$$

where $C > 0$ is a constant independent of φ.

We proved this result in details in proof of [23, Proposition 1]) (see also proof of [28, Proposition 3.1]). The proof is not repeated here.

Similarly, we also need for the sequel, the lemmas below, showing how the regularity of the coefficients gives rise to the regularity of the solutions of (4.6) and (4.7). The last one takes into account the dependence with respect to the parameter (t, x). Their proofs are copied on that of [23, Lemmas 3.1 and 3.2] (see also [28, Lemmas 3.2 and 3.3]) and are left to the reader. In what follows, A^1 denotes the space $\{u \in A : \nabla u \in (A)^d\}$ while V_ω stands for the subspace V of $\mathbb{R}^d$, with generic variable $\omega \in V$. We then have:

Lemma 4.1. *Suppose that* $\sigma(y,v,w)$ *and* $\frac{\partial\sigma}{\partial y_i}(y,v,w)$ $(1\leq i\leq d)$ *lie in* $\mathcal{C}(V_v\times V_w;L^\infty(\mathbb{R}^d_y))$ $\cap L^\infty(V_v\times V_w\times\mathbb{R}^d_y)$. *Then, for* $g\in\mathcal{C}(V;A^1)$, *the solution of* (4.6) *lies in* $\mathcal{C}(V;A^1)$. *A similar conclusion holds for the adjoint equation* (4.7).

It is worth noticing that derivation of the equation for the equilibrium function F shows similarly that $F\in\mathcal{C}(V;A^1)$.

Lemma 4.2. *Let* $k\in\mathbb{N}^*$. *If* $\sigma(t,x,y,v,w)\in\mathcal{C}^k(\mathbb{R}^{1+d}_T;L^\infty(V_v\times V_w;B^{2,\infty}_A(\mathbb{R}^d_y)))$ *and* $g\in\mathcal{C}^k(\mathbb{R}^{1+d}_T;L^2(V;B^2_A(\mathbb{R}^d_y)))$, *then the solution of* (4.6) *lies in* $\mathcal{C}^k(\mathbb{R}^{1+d}_T;L^2(V;B^2_A(\mathbb{R}^d_y)))$. *A similar conclusion holds for the adjoint equation* (4.7).

In particular, the following consequence of the lemmas above will be useful in the sequel.

Corollary 4.1. *Suppose* (5.1) *holds. Then*

(i) F *and* $\nabla_v F$ *are continuous functions of their arguments, where* F *is determined by* (4.12).

(ii) For any $g\in\mathcal{C}^1_0(\mathbb{R}^{1+d}_T;\mathcal{C}(V;A^1))$, *satisfying the compatibility condition in Proposition* 4.1, *the solution of* $Pf=g$ *is a continuous function of its arguments as well as its first derivative with respect to* x. *A similar conclusion holds for the adjoint equation* $P^*\varphi=\phi$.

5 Homogenization result

As in our previous work related to the topic [23], we state on the collision kernel σ the following homogenization hypothesis

$$\sigma\in\mathcal{B}(\mathbb{R}^{1+d}_T\times V_v\times V_w;B^{2,\infty}_A(\mathbb{R}^d_y)),\tag{5.1}$$

where, A is a given algebra with mean value on $\mathbb{R}^d$ and $B^{2,\infty}_A(\mathbb{R}^d_y)=B^2_A(\mathbb{R}^d_y)\cap L^\infty(\mathbb{R}^d_y)$, as given in Section 2. Hypothesis (5.1) plays a special role in this work. Indeed, when accounting for the specific properties of the medium in which the collisions occur, the analysis of our model becomes more involved and necessitates special attention. These properties are naturally included in the behaviour of the scattering rate function σ, and they influence the overall behaviour of the density function f_ε. They depend on the way the microstructures are distributed in the heterogeneous medium (where the collisions occur). For example, we could assume that the medium is made of micro-structures that are either uniformly distributed inside or almost uniformly distributed, or assume another kind of deterministic distribution. This leads to a function $(y)\mapsto\sigma(y,v,w)$ which is assumed to be either periodic (with respect to (y)) or almost periodic, or even asymptotic almost periodic. Assumption (5.1) includes all these properties. We also assume that the electric field E is highly oscillating in space variable. That is,

$$E\in(B^{2,\infty}_A(\mathbb{R}^d_y))^d.\tag{5.2}$$

Now, let us consider the operator P defined by (4.1) in the previous section. By Proposition 4.1, we known that there exists a unique $F\in\mathcal{C}^1(\mathbb{R}^{1+d}_T;L^2(V;B^2_A(\mathbb{R}^d_y)))$ such that

$$PF=0,\ \int_V M(F(\cdot,v))d\mu(v)=1 \text{ and } F>0 \text{ a.e.};\tag{5.3}$$

We define the following flux:

$$c(t,x)=\int_V M(vF(\cdot,v))d\mu(v).\tag{5.4}$$

The following theorem is our main homogenization result.

Theorem 5.1. *Let A be an algebra with mean value on $\mathbb{R}^d$. Assume (5.1) and (5.2). For each $\varepsilon > 0$, let f_ε be the unique solution of (1.5). Then, there exists $f_0 + \mathcal{N} \in L^2(\mathbb{R}_T^{1+d} \times V; B_A^2(\mathbb{R}^d)/\mathcal{N})$ such that the sequence $(f_\varepsilon)_{\varepsilon>0}$ weakly sigma-converges in $L^2(\mathbb{R}_T^{1+d} \times V)$ towards $f_0 + \mathcal{N}$. Moreover f_0 is nonnegative and has the form $f_0(t,x,y,v) = F(t,x,y,v)\rho_0(t,x)$, where $F \in \mathcal{C}^1(\mathbb{R}_T^{1+d}; L^2(V; B_A^2(\mathbb{R}_y^d)) \cap \ker P)$ is given by (5.3) and ρ_0 satisfies:*

(i) *if $c \neq 0$,*

$$\begin{cases} \frac{\partial \rho_0}{\partial t} +_x (c(t,x)\rho_0) = 0 & \text{in} \quad (0,T) \times \mathbb{R}_x^d, \\ \rho_0(0,x) = \int_V f^0(x,v)d\mu(v) & \text{in} \quad \mathbb{R}_x^d. \end{cases} \tag{5.5}$$

In addition, for all $\phi \in C_0^\infty(\mathbb{R}_T^d)$ satisfying $\frac{\partial \phi}{\partial t} + c(t,x)\cdot\nabla_x\phi = 0$, the following variational equation holds true

$$\left\langle _x \left(A(t,x)\frac{\partial \rho_0}{\partial t} + D(t,x)^{\mathcal{T}}\nabla_x\rho_0 + U(t,x)\rho_0 \right); \phi \right\rangle = 0, \tag{5.6}$$

where, ${}^{\mathcal{T}}\nabla_x\rho_0$ is the transpose of vector $\nabla_x\rho_0 = \left(\frac{\partial\rho_0}{\partial x_1}, \frac{\partial\rho_0}{\partial x_2}, ..., \frac{\partial\rho_0}{\partial x_d}\right)$,

$$\begin{array}{rcll} A(t,x) & = & \int_V M\left[\chi^* F\right] d\mu(v) & = \left(\int_V M\left[\chi_i^* F\right] d\mu(v)\right)_{i=1}^d, \\ D(t,x) & = & \int_V M\left[\chi^* \otimes vF\right] d\mu(v) & = \left(\int_V M\left[\chi_i^* v_j F\right] d\mu(v)\right)_{i,j=1}^d, \end{array}$$

with $(\chi^* \otimes vF) = (\chi_i^* v_j F)_{1\leq i,j\leq d}$,

$$\begin{aligned} U(t,x) &= \int_V M\left[\chi^*\frac{\partial F}{\partial t} + \chi^*(v\cdot\nabla_x F) - FE\cdot\nabla_v\chi^*\right] d\mu(v), \\ &= \left(\int_V M\left[\chi_i^*\frac{\partial F}{\partial t} + \chi_i^*(v\cdot\nabla_x F) - FE\cdot\nabla_v\chi_i^*\right] d\mu(v)\right)_{i=1}^d, \end{aligned}$$

where $\chi^* \in \left[C_0^\infty(\mathbb{R}_T^d \times V; B_A^2(\mathbb{R}_y^d))\right]^d$ is the unique solution of the corrector problem:

$$P^*\chi^* = -(c(t,x) - v) \text{ and } \int_V M[\chi^*(t,x,.,v)]d\mu(v) = 0.$$

(ii) *If $c(t,x) = 0$, the homogenized model is stationary and the solution ρ_0 satisfies*

$$_x\big(D(x)^{\mathcal{T}}\nabla_x\rho_0(x) + U(x)\rho_0(x)\big) = 0, \tag{5.7}$$

where

$$\begin{aligned} D(x) &= \int_V\int_0^T M\left[\chi^*\otimes vF\right] dtd\mu(v) = \left(\int_V\int_0^T M\left[\chi_i^* v_j F\right] d\mu(v)dt\right)_{i,j=1}^d, \\ U(x) &= \int_V\int_0^T M\left[F\frac{\partial\chi^*}{\partial t} + \chi^*(v\cdot\nabla_x F) - FE\cdot\nabla_v\chi^*\right] dtd\mu(v) \\ &= \left(\int_V\int_0^T M\left[F\frac{\partial\chi_i^*}{\partial t} + \chi_i^*(v\cdot\nabla_x F) - FE\cdot\nabla_v\chi_i^*\right] dtd\mu(v)\right)_{i=1}^d \end{aligned}$$

and where χ^* is the unique solution $\left[C_0^\infty(\mathbb{R}_T^d \times V; B_A^2(\mathbb{R}_y^d))\right]^d$ of the corrector problem:

$$P^*\chi^* = -v \text{ and } \int_V M[\chi^*(t,x,.,v)]d\mu(v) = 0. \tag{5.8}$$

P^* being the adjoint operator of P.

Proof of Theorem 5.1. Let first notice that applying the Theorem 3.1, in virtue of a priori estimates (2.5) of theorem 2.1, we infer the existence of a subsequence $(f_{\varepsilon'})_{\varepsilon'>0}$ of $(f_\varepsilon)_{\varepsilon>0}$, and an element $f_0 \in L^2(\mathbb{R}_T^{1+d} \times V; B_A^2(\mathbb{R}_y^d))$, such that as $\varepsilon' \to 0$, one has

$$f_{\varepsilon'} \to f_0 + \mathcal{N} \text{ in } L^2(\mathbb{R}_T^{1+d} \times V) - weak\ \Sigma,$$

where $\mathcal{N} = \left\{ f \in B_A^2(\mathbb{R}_y^d) : \|f\|_2 = M(|f|^2)^{\frac{1}{2}} = 0 \right\}$. We consider now our initial model (1.5). Let $\psi \in \mathcal{C}_0^\infty\left(\mathbb{R}_T^{1+d} \times \mathcal{O}\right) \otimes A^\infty$, where $\mathcal{O} = V \setminus \partial V$ (∂V being the boundary of V). We then define $\psi^\varepsilon = \psi(t, x, \frac{x}{\varepsilon}, v)\ \forall\ (t, x, v) \in \mathbb{R}_T^{1+d} \times V$. Hence, $\psi^\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}_T^{1+d} \times V)$.

Multiplying the first equation of (1.5) by ψ^ε and integrating by parts, using the semi-detailed balance condition (1.7) and the fact that $\nabla_v \cdot E = 0$, we obtain

$$-\int_{\mathbb{R}_T^{d+1} \times V} \left[f_\varepsilon \left(\left(\frac{\partial \psi}{\partial t} \right)^\varepsilon + v \cdot (\nabla_x \psi)^\varepsilon + \frac{1}{\varepsilon} v \cdot (\nabla_y \psi)^\varepsilon + E^\varepsilon \cdot (\nabla_v \psi)^\varepsilon + \frac{1}{\varepsilon} \left(\mathcal{Q}^* \psi \right)^\varepsilon \right) \right] d\mu(v) dx dt = 0.$$

Then, multiplying (5.9) by ε and passing to the limit as $\varepsilon' \to 0$, using (in view of Remark 2) E and σ as test functions, thanks to Theorem 3.1, one has by Σ-convergence

$$\begin{aligned} &\int_{V \times \mathbb{R}_T^{d+1}} \left[f_\varepsilon \left(\varepsilon \left(\frac{\partial \psi}{\partial t} \right)^\varepsilon + \varepsilon v \cdot (\nabla_x \psi)^\varepsilon + v \cdot (\nabla_y \psi)^\varepsilon + \varepsilon E^\varepsilon \cdot (\nabla_v \psi)^\varepsilon + (\mathcal{Q}^* \psi)^\varepsilon \right) \right] dt dx d\mu(v) \\ &\longrightarrow \int_{V \times \mathbb{R}_T^{d+1}} M \left[f_0 \left(v \cdot \nabla_y \psi + \mathcal{Q}^* \psi \right) \right] dt dx d\mu(v) = 0. \end{aligned}$$

We deduce that

$$-\int_{V \times \mathbb{R}_T^{1+d}} M \left[f_0 \left(v \cdot \nabla_y \psi + \mathcal{Q}^* \psi \right) \right] dt dx d\mu(v) = 0. \tag{5.9}$$

Choosing $\psi(t, x, y, v) = \varphi(t, x)\phi(y, v)$, where $\varphi \in \mathcal{C}_0^\infty\left(\mathbb{R}_T^{1+d}\right)$ and $\phi \in A^\infty \otimes \mathbf{C}^\infty(\mathcal{O})$, we obtain that (5.9) is equivalent to:

$$\int_{\mathbb{R}_T^{1+d}} \left(\int_V M \left[f_0(t, x, ., v) \left(v \cdot \nabla_y \phi(., v) \right) + \mathcal{Q}^* \phi(., v) \right] d\mu(v) \right) \varphi(t, x) dx dt = 0.$$

Since φ is arbitrary chosen, the preceding equation leads to

$$\int_V M \left[f(t, x, \cdot, v) \left(v \cdot \nabla_y \phi(., v) + \mathcal{Q}^* \phi(., v) \right) \right] d\mu(v) = 0.$$

Now, let us consider the problem: Find $F \in L^2(V; B_A^2(\mathbb{R}_y^d))$ such that

$$v \cdot \nabla_y F - \mathcal{Q} F = 0 \text{ in } \mathbb{R}_y^d \times V. \tag{5.10}$$

Then, appealing [part (i) of] Proposition 4.1, there exists a unique $F \in L^2(V; B_A^2(\mathbb{R}_y^d))$ with $\int_V M(F) d\mu(v) = 1$, which solves (5.10) and further satisfies the variational formulation

$$\int_V M \left(F(\cdot, v) \left(v \cdot \nabla_y \phi(\cdot, v) + \mathcal{Q}^* \phi(\cdot, v) \right) \right) d\mu(v) = 0 \quad \forall \phi \in A^\infty \otimes \mathcal{C}_0^\infty(V). \tag{5.11}$$

Next, setting

$$\rho_0(t,x) = \int_V M(f_0(t,x,\cdot,v))d\mu(v)$$

and assuming without lost of generality that ρ_0 is not identically zero, we define a function with the property that $\rho_0^{-1}f_0(t,x,\cdot)$ solves (5.11) and $\int_V M\left(\rho_0^{-1}f_0\left(t,x,\cdot,v\right)\right)d\mu(v) = 1$. Invoking the uniqueness of the solution of (5.11) with the further normality condition $\int_V M(F)\ d\mu(v) = 1$ insured by Proposition 4.1, we get readily $F = \rho_0^{-1}f_0$, *i.e.*,

$$f_0(t,x,y,v) = \rho_0(t,x)F(y,v) \text{ for a.e. } (t,x,y,v) \in \mathbb{R}_T^{1+d} \times \mathbb{R}_y^d \times V, \tag{5.12}$$

where F is given by (4.12) and ρ_0 is to be determined. Now, as $f_0 \in L^2(\mathbb{R}_T^{1+d} \times V; B_A^2(\mathbb{R}_y^d)) = L^2(\mathbb{R}_T^{1+d}; L^2(V; B_A^2(\mathbb{R}_y^d)))$ we see that $\rho_0 \in L^2(\mathbb{R}_T^{1+d})$.

Let us still consider the weak formulation (5.9), but with $\psi \in \left[\mathcal{C}_0^\infty(\mathbb{R}_T^{1+d}) \otimes A^\infty \otimes \mathcal{C}_0^\infty(\mathcal{O})\right] \cap KerP^*$. In virtue of statement (i) of Proposition 4.1, $KerP^*$ is spanned by $1_{\mathbb{R}_y^d \times V}$; that is $\psi(t,x,y,v) = \varphi(t,x)$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}_T^{1+d})$. The derivatives in y and v vanish in (5.9) and it becomes

$$\int_{\mathbb{R}_T^{1+d}\times V} f_\varepsilon \left(\left(\frac{\partial\varphi}{\partial t}\right)^\varepsilon + v\cdot(\nabla_x\varphi)^\varepsilon\right)d\mu(v)dxdt = 0.$$

Passing to the limit as $\varepsilon' \to 0$, we obtain by Σ-convergence

$$\int_{\mathbb{R}_T^{1+d}\times V} M\left[f_0\left(\frac{\partial\varphi}{\partial t} + v\cdot\nabla_x\varphi\right)\right]d\mu(v)dxdt = 0.$$

Taking into account (5.12) and (4.12), we can compute this last equation as follows:

$$\begin{aligned}
0 &= \int_{\mathbb{R}_T^{1+d}\times V} M\left[f_0(t,x,\cdot,v)\left(\frac{\partial\varphi}{\partial t}(t,x) + v\cdot\nabla_x\varphi(t,x)\right)\right]d\mu(v)dxdt,\\
0 &= \int_{\mathbb{R}_T^{1+d}\times V} M\left[\rho_0(t,x)F(t,x,\cdot,v)\left(\frac{\partial\varphi}{\partial t}(t,x) + v\cdot\nabla_x\varphi(t,x)\right)\right]d\mu(v)dxdt,\\
0 &= \int_{\mathbb{R}_T^{1+d}} \rho_0\left[\left(\int_V M\left[F(t,x,\cdot,v)\right]d\mu(v)\right)\frac{\partial\varphi}{\partial t} + \left(\int_V M\left[F(t,x,.,v)v\right]d\mu(v)\right)\cdot\nabla_x\varphi\right]dxdt,\\
0 &= \int_{\mathbb{R}_T^{1+d}} \rho_0(t,x)\left(\frac{\partial\varphi}{\partial t}(t,x) + c(t,x)\cdot\nabla_x\varphi(t,x)\right)d\mu(v)dxdt,\\
0 &= \left\langle -\frac{\partial\rho_0}{\partial t} - \nabla_x\cdot(c\rho_0);\varphi\right\rangle,
\end{aligned}$$

since $\int_V M\left[F(t,x,\cdot,v)\right]d\mu(v) = 1$ and $\int_V M\left[F(t,x,\cdot,v)v\right]d\mu(v) = c(t,x)$. This implies in virtue of the arbitrariness of φ that

$$\frac{\partial\rho_0}{\partial t}(t,x) +_x (c(t,x)\rho_0(t,x)) \quad = \quad 0 \text{ for all } (t,x) \in \mathbb{R}_{T,x}^{1+d}. \tag{5.13}$$

To obtain equation (5.6) of Theorem 5.1, we still consider the weak formulation (5.9), that we rewrite in the following form:

$$\int_{\times\mathbb{R}_T^{d+1}V} \left[f_\varepsilon\left(\left(\frac{\partial\psi}{\partial t}\right)^\varepsilon + v\cdot(\nabla_x\psi)^\varepsilon + E^\varepsilon\cdot(\nabla_v\psi)^\varepsilon - \frac{1}{\varepsilon}\left(P^*\psi\right)^\varepsilon\right)\right]dtdxd\mu(v) = 0, \tag{5.14}$$

but this time with

$$\left\{\begin{array}{rl}\psi^{\varepsilon}(t,x,v) & =\frac{1}{\varepsilon}\phi(t,x)+\varphi(t,x,\frac{x}{\varepsilon},v), \quad \phi\in\mathcal{C}_0^{\infty}(\mathbb{R}_T^{d+1}), \\ P^*(\varphi) & =(v-c(t,x))\cdot\nabla_x\phi.\end{array}\right. \tag{5.15}$$

Indeed, in view of (5.4) and (4.12), we have

$$\int_V M\left[(v-c)F\right]d\mu(v) \quad = \quad \int_V M\left[vF\right]d\mu(v)-c\int_V M\left[F\right]d\mu(v)=c-c\times 1=0,$$

so that appealing statement (iii) of Proposition 4.1 and Lemma 4.2, we have the existence of a unique $\chi^*\in\left[\mathcal{C}_0^{\infty}(\mathbb{R}_T^{1+d};B_A^2(\mathbb{R}_y^d))\otimes\mathcal{C}_0^{\infty}(V)\right]^d$ satisfying

$$P^*(\chi^*)=(v-c(t,x)) \text{ and } \int_V M\left[\chi^*\right]d\mu(v)=0.$$

Hence, choosing ϕ so that

$$\frac{\partial\phi}{\partial t}+c(t,x)\cdot\nabla_x\phi=0, \tag{5.16}$$

we can rewrite (5.14) as follows

$$\begin{aligned}0=\int_{V\times\mathbb{R}_T^{d+1}} f_\varepsilon\Bigg[\frac{1}{\varepsilon}\left(\frac{\partial\phi}{\partial t}+c(t,x)\cdot\nabla_x\phi+((v-c(t,x))\cdot\nabla_x\phi-(P^*\varphi)^{\varepsilon})\right)+ \\ \frac{1}{\varepsilon}E^{\varepsilon}\cdot\nabla_v\phi+\frac{1}{\varepsilon^2}P^*\phi+\left(\left(\frac{\partial\varphi}{\partial t}\right)^{\varepsilon}+v\cdot(\nabla_x\varphi)^{\varepsilon}+E^{\varepsilon}\cdot(\nabla_v\varphi)^{\varepsilon}\right)\Bigg]dtdxd\mu(v).\end{aligned} \tag{5.17}$$

But in view of (5.15), (5.16) and as $\phi=\phi(t,x)$, all the terms in $\frac{1}{\varepsilon}$ and $\frac{1}{\varepsilon^2}$ in (5.17) vanish. We then pass to the limit as $\varepsilon'\to 0$, to obtain by Σ-convergence

$$\int_{V\times\mathbb{R}_T^{d+1}} M\left[f_0\left(\frac{\partial\varphi}{\partial t}+v\cdot\nabla_x\varphi+E\cdot\nabla_v\varphi\right)\right]dtdxd\mu(v)=0. \tag{5.18}$$

Furthermore, since $\varphi(t,x,y,v)=\chi^*(t,x,y,v)\cdot\nabla_x\phi(t,x)$ and $f_0=\rho_0(t,x)F(t,x,y,v)$, equation (5.18) reads as

$$\int_{V\times\mathbb{R}_T^{d+1}} M\left[\rho_0 F\left(\frac{\partial}{\partial t}\left(\chi^*\cdot\nabla_x\phi\right)+v\cdot\nabla_x\left(\chi^*\cdot\nabla_x\phi\right)+E\cdot\nabla_v\left(\chi^*\cdot\nabla_x\phi\right)\right)\right]dtdxd\mu(v)=0. \tag{5.19}$$

Let us set $\sum_{j,i=1}^d=\sum_{j=1}^d\sum_{i=1}^d$ and

$$\begin{aligned}I_1 &= \int_{V\times\mathbb{R}_T^{d+1}} M\left[\rho_0 F\frac{\partial}{\partial t}\left(\chi^*\cdot\nabla_x\phi\right)\right]dtdxd\mu(v), \\ I_2 &= \int_{V\times\mathbb{R}_T^{d+1}} M[\rho_0 Fv\cdot\nabla_x\left(\chi^*\cdot\nabla_x\phi\right)]dtdxd\mu(v), \\ I_3 &= \int_{V\times\mathbb{R}_T^{d+1}} M\left[\rho_0 FE\cdot\nabla_v\left(\chi^*\cdot\nabla_x\phi\right)\right]dtdxd\mu(v).\end{aligned}$$

We can compute each term of (5.19) as follows:

$$
\begin{aligned}
I_1 &= \int_{V\times\mathbb{R}_T^{d+1}} M\left[\sum_{i=1}^{d}\rho_0 F\frac{\partial}{\partial t}\left(\chi_i^*\frac{\partial\phi}{\partial x_i}\right)\right] dtdxd\mu(v), \qquad (5.20)\\
&= -\int_V M\left[\sum_{i=1}^{d}\int_{\mathbb{R}_T^{1+d}}\left(\frac{\partial\rho_0}{\partial t}F\chi_i^*\frac{\partial\phi}{\partial x_i}+\frac{\partial F}{\partial t}\rho_0\chi_i^*\frac{\partial\phi}{\partial x_i}\right)dtdx\right]d\mu(v),\\
&= \int_V M\left[\sum_{i=1}^{d}\int_{\mathbb{R}_T^{1+d}}\left(\frac{\partial}{\partial x_i}\left(\chi_i^* F\frac{\partial\rho_0}{\partial t}\right)+\frac{\partial}{\partial x_i}\left(\chi_i^*\frac{\partial F}{\partial t}\rho_0\right)\right)\phi dtdx\right]d\mu(v),\\
&= \int_{\mathbb{R}_T^{1+d}}\sum_{i=1}^{d}\left[\frac{\partial}{\partial x_i}\left(\left(\int_V M\left[\chi_i^* F\right]d\mu(v)\right)\frac{\partial\rho_0}{\partial t}\right)+\frac{\partial}{\partial x_i}\left(\left(\int_V M\left[\chi_i^*\frac{\partial F}{\partial t}\right]d\mu(v)\right)\rho_0\right)\right]\phi dtdx,\\
&= \left\langle_x\left[\left(\int_V M\left[\chi^* F\right]d\mu(v)\right)\frac{\partial\rho_0}{\partial t}+\left(\int_V M\left[\chi^*\frac{\partial F}{\partial t}\right]d\mu(v)\right)\rho_0\right];\phi\right\rangle.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= -\int_V M\left[\sum_{i,j=1}^{d}\int_{\mathbb{R}_T^{1+d}}\left(v_i\frac{\partial\rho_0}{\partial x_i}F\chi_j^*\frac{\partial\phi}{\partial x_j}+v_i\frac{\partial F}{\partial x_i}\rho_0\chi_j^*\frac{\partial\phi}{\partial x_j}\right)dtdx\right]d\mu(v), \qquad (5.21)\\
&= \int_V M\left[\sum_{i,j=1}^{d}\int_{\mathbb{R}_T^{1+d}}\left(\frac{\partial}{\partial x_j}\left(\chi_j^* F v_i\frac{\partial\rho_0}{\partial x_i}\right)+\frac{\partial}{\partial x_j}\left(\chi_j^* v_i\frac{\partial F}{\partial x_i}\rho_0\right)\right)\phi dtdx\right]d\mu(v),\\
&= \int_V M\left[\sum_{j=1}^{d}\int_{\mathbb{R}_T^{1+d}}\left(\sum_{i=1}^{d}\frac{\partial}{\partial x_j}\left(\chi_j^* F\frac{\partial\rho_0}{\partial x_i}\right)+\frac{\partial}{\partial x_j}\left(\chi_j^*\left(\sum_{i=1}^{d}v_i\frac{\partial F}{\partial x_i}\right)\rho_0\right)\right)\phi dtdx\right]d\mu(v),\\
&= \int_{\mathbb{R}_T^{1+d}}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left[\sum_{i=1}^{d}\left(\int_V M\left[\chi_j^* v_i F\right]d\mu(v)\right)\frac{\partial\rho_0}{\partial x_i}+\left(\int_V M\left[\chi_j^*(v\cdot\nabla_x F)\right]d\mu(v)\right)\rho_0\right]\phi dtdx,\\
&= \left\langle_x\left[\left(\int_V M\left[\chi^*\otimes vF\right]d\mu(v)\right)^{\mathcal{T}}\nabla_x\rho_0+\left(\int_V M\left[\chi^*(v\cdot\nabla_x F)\right]d\mu(v)\right)\rho_0\right];\phi\right\rangle;
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= \int_{V\times\mathbb{R}_T^{d+1}} M\left[\sum_{i,j=1}^{d}\rho_0 F E_i\frac{\partial\chi_j^*}{\partial v_i}\frac{\partial\phi}{\partial x_j}\right]dtdxd\mu(v), \qquad (5.22)\\
I_3 &= -\int_V M\left[\sum_{i,j=1}^{d}\int_{\mathbb{R}_T^{1+d}}\left(v_i\frac{\partial}{\partial x_i}\left(FE_i\frac{\partial\chi_j^*}{\partial v_i}\rho_0\right)\right)dtdx\right]\phi d\mu(v),\\
&= -\int_{\mathbb{R}_T^{1+d}}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left[\left(\int_{\mathbb{R}_v^d} M\left[F\sum_{i=1}^{d}E_i\frac{\partial\chi_j^*}{\partial v_i}\right]d\mu(v)\right)\rho_0\right]\phi dtdx,\\
&= -\int_{\mathbb{R}_T^{1+d}}\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left[\left(\int_{\mathbb{R}_v^d} M\left[FE\cdot\nabla_v\chi_j^*\right]d\mu(v)\right)\rho_0\right]\phi dtdx,\\
&= \left\langle_x\left[\left(\int_V M\left[FE\cdot\nabla_v\chi^*\right]d\mu(v)\right)\rho_0\right];\phi\right\rangle.
\end{aligned}
$$

Hence taking into account (5.20), (5.21) and (5.22) we obtain that

$$\left\langle {}_x \left(A(t,x)\frac{\partial \rho_0}{\partial t} + D(t,x)^{\mathcal{T}}\nabla_x \rho_0 + U(t,x)\rho_0 \right) ; \phi \right\rangle = 0, \tag{5.23}$$

for all ϕ satisfying (5.16), with

$$A(t,x) = \int_V M\left[\chi^* F\right] d\mu(v) \quad ; \quad D(t,x) = \int_V M\left[\chi^* \otimes vF\right] d\mu(v) \ ;$$
$$U(t,x) = \int_V M\left[\chi^* \frac{\partial F}{\partial t} + \chi^*(v\cdot\nabla_x F) - FE\cdot\nabla_v \chi^*\right] d\mu(v).$$

For the proof of statement (ii) of Theorem 5.1, let us note that when $c(t,x) = 0$, equation (5.21) implies that $\frac{\partial \rho_0}{\partial t} = 0$ and (5.16) implies $\phi = \phi(x)$. Hence, it's just a particular case of statement (i), so that performing the same calculations as above with ϕ arbitrary chosen in $\mathcal{C}_0^\infty(\mathbb{R}^d_x)$, we obtain this time

$${}_x \left(D(x)^T \nabla_x \rho_0(x) + U(x)\rho_0(x) \right) = 0,$$

where

$$U(x) = \left(\int_0^T \int_V M\left[F\frac{\partial \chi_i^*}{\partial t} + \chi_i^*(v\cdot\nabla_x F) - FE\cdot\nabla_v \chi_i^*\right] d\mu(v)dt \right)_{i=1}^d ,$$

$$D(x) = \left(\int_0^T \int_V M\left[\chi_i^* v_j F\right] d\mu(v)dt \right)_{i,j=1}^d$$

and where χ^* the unique solution $\left[C_0^\infty(\mathbb{R}^d_T \times V; B_A^2(\mathbb{R}^d_y))\right]^d$ of the cell problem:

$$P^*(\chi^*) = -v \text{ and } \int_V M[\chi^*(x,.,v)]d\mu(v) = 0.$$

We come now to the initial condition satisfied by ρ_0 in statement (i). Let us consider the following test function $\psi^\varepsilon(t,x,v) = \eta(t)\phi(x)$, where $\eta \in \mathcal{C}_0^\infty([0,T])$ and $\eta(T) = 0$. That is ψ^ε belongs to $KerP$, and obviously to $Ker\mathcal{Q}^*$. Multiplying the first equation of (1.5) by such test function and integrating over (t,x,v) variables, we can perform each term of equation obtained as follows:

$$\begin{aligned}
\int_{\mathbb{R}^{1+d}_T\times V} \frac{\partial f_\varepsilon}{\partial t}\psi^\varepsilon dt d\mu(v)dx &= \int_{\mathbb{R}^d_x\times V}\left[\int_0^T \frac{\partial f_\varepsilon}{\partial t}\psi^\varepsilon dt\right] d\mu(v)dx, \qquad (5.24)\\
&= \int_{\mathbb{R}^d_x\times V}\left[\int_0^T \left(\frac{\partial}{\partial t}(f_\varepsilon\psi^\varepsilon) - f_\varepsilon\left(\frac{\partial \psi}{\partial t}\right)^\varepsilon\right) dt\right] dxd\mu(v),\\
&= -\eta(0)\int_{\mathbb{R}^d_x\times V} (f_\varepsilon)_{t=0}\phi dxd\mu(v) - \int_{\mathbb{R}^{1+d}_T\times V} \eta' f_\varepsilon \phi dt dx d\mu(v),\\
&= -\eta(0)\int_{\mathbb{R}^d_x\times V} f^0\phi dxd\mu(v) - \int_{\mathbb{R}^{1+d}_T\times V} \eta' f_\varepsilon \phi dt dx d\mu(v) \ ;
\end{aligned}$$

$$\int_{\mathbb{R}_T^{1+d}\times V} \psi^\varepsilon (v\cdot\nabla_x f_\varepsilon) d\mu(v)dxdt = -\int_{\mathbb{R}_T^{1+d}\times V} \eta f_\varepsilon [v\cdot\nabla_x \phi] d\mu(v)dxdt \; ; \tag{5.25}$$

$$\int_{\mathbb{R}_T^{1+d}\times V} \psi^\varepsilon (E^\varepsilon\cdot\nabla_v f_\varepsilon) d\mu(v)dxdt = -\int_{\mathbb{R}_T^{1+d}\times V} \eta f_\varepsilon [E^\varepsilon\cdot\nabla_v \phi] d\mu(v)dxdt = 0 \tag{5.26}$$

and using the semi-detailed balance condition (1.7), we have

$$\int_{\mathbb{R}_T^{1+d}\times V} \psi^\varepsilon (\mathcal{Q}^\varepsilon f_\varepsilon) d\mu(v)dxdt = \int_{\mathbb{R}_T^{1+d}\times V} f_\varepsilon \eta(t) (\mathcal{Q}^\varepsilon)^* \phi d\mu(v)dxdt = 0, \tag{5.27}$$

since $\nabla_v\phi = \mathcal{Q}_\varepsilon^*\phi = 0$. Now, taking into account (5.24)-(5.27), we obtain:

$$\int_{\mathbb{R}_T^{1+d}\times V} f_\varepsilon \left(\eta'\phi + \eta v\cdot(\nabla_x\phi)^\varepsilon\right) d\mu(v)dxdt = \eta(0)\int_{\mathbb{R}_x^d\times V} f^0 \phi d\mu(v)dx.$$

Passing to the limit as $\varepsilon'\to 0$, we obtain

$$\int_{\mathbb{R}_T^{1+d}\times V} M\left[f_0\left(\eta'\phi + \eta v\cdot\nabla_x\phi\right)\right] dtd\mu(v)dx = \eta(0)\int_{\mathbb{R}_x^d\times V} f^0(x,v)\phi(x)dxd\mu(v). \tag{5.28}$$

Integrating by parts the left-hand side of (5.28) over t and x variables, using (4.12), we have

$$\begin{aligned}\int_{\mathbb{R}_T^{1+d}\times V} M[f_0(\eta'\phi + \eta v\cdot\nabla_x\phi)] d\mu(v)dxdt = -\int_{\mathbb{R}_T^{1+d}} \left(\frac{\partial\rho_0}{\partial t} +{}_x \left(c(t,x)\rho_0\right)\right)\eta(t)\phi(x)dxdt \\ + \eta(0)\int_{\mathbb{R}_x^d} \rho_0(0,x)\phi(x)dx,\end{aligned} \tag{5.29}$$

Combining (5.28) and (5.29) in view of (5.13), we are led to

$$\eta(0)\int_{\mathbb{R}_x^d} \rho_0(0,x)\phi(x)dx = \eta(0)\int_{\mathbb{R}_x^d}\left(\int_V f^0(x,v)d\mu(v)\right)\phi(x)dx.$$

This readily gives

$$\rho_0(0,x) = \int_V f^0(x,v)d\mu(v) \text{ in } \mathbb{R}_x^d. \tag{5.30}$$

Combining (5.13) and (5.30), we get (5.5).

Let end the proof by showing that $f_0 = \rho_0 F$ is nonnegative. As $F\geq 0$, it is enough to show that ρ_0 is nonnegative. To that end, we consider the Cauchy problem (5.5) above, rewritten as follows

$$\begin{cases} \frac{\partial\rho_0}{\partial t} + c(t,x)\cdot\nabla_x\rho_0 = -{}_x\left(c(t,x)\right)\rho_0 & \text{in} \quad (0,T)\times\mathbb{R}_x^d \\ \rho_0(0,x) = \int_V f^0(x,v)d\mu(v) & \text{in} \quad \mathbb{R}_x^d. \end{cases} \tag{5.31}$$

We then consider the associated characteristics $s\mapsto (X(s),\rho_0(s,X(s)))$ under the condition $X(t)=x$, where $(X(s),\rho_0(s,X(s))) = (X(s;t,x),\rho_0(s,X(s;t,x)))$. Hence, integrating over the interval $[s_0,t]$ the characteristics system

$$\begin{cases} \frac{dX(s)}{ds} &= c(s,X(s)), \\ \frac{d\rho_0(s,X(s))}{ds} &= -{}_x\left(c\left(s,X(s)\right)\right)\rho_0\left(s,X(s)\right), \end{cases}$$

we obtain:

$$X(t) = X(s_0) + \int_{s_0}^{t} c(s, X(s))ds \quad \text{and} \quad \ln\left(\frac{\rho_0(t, X(t))}{\rho_0(s_0, X(s_0))}\right) = -\int_{s_0}^{t} {}_x\left(c(s, X(s))\right) ds.$$

That is

$$\rho_0(t, X(t)) = \rho_0(s_0, X(s_0)) \exp\left(-\int_{s_0}^{t} {}_x\left(c(s, X(s))\right) ds\right).$$

In particular for $s_0 = 0$ and in view of the condition $X(t) = x$, we are leaded to

$$\rho_0(t, x) = \rho_0(0, X(0)) \exp\left(-\int_{0}^{t} {}_x\left(c(s, X(s))\right) ds\right),$$

which in view of the initial condition of (5.31) implies

$$\rho_0(t, x) = \int_V f^0\left(x - \int_0^t c(s, X(s))ds, v\right) d\mu(v) \exp\left(-\int_0^t {}_x\left(c(s, X(s))\right) ds\right). \tag{5.32}$$

But as f^0 is nonnegative, (5.32) implies that ρ_0 is also nonnegative, whence we have the Theorem. □

6 Appendix

Let us consider the algebra wmv $A = AP(\Lambda)$ and the generalized Besicovitch $B_A^2(\Lambda)$, as defined in Subsection 3.1, where Λ satisfies the β-condition (3.9). We intend to prove the following proposition.

Proposition 6.1. *Let K be the operator defined in (4.8). We have*

(i) *K is compact from $D(P)$ into $L^2(V; B_A^2(\Lambda))$.*

(ii) *The compactness in (i) is still valid for $A = \mathcal{C}_{per}(Y) + \mathcal{C}_0(\Lambda)$ and $A = AP(\Lambda) + \mathcal{C}_0(\Lambda)$, where,*

$$\mathcal{C}_{per}(Y) + \mathcal{C}_0(\Lambda) = \left\{f \in \mathcal{C}_{per}(Y) + \mathcal{C}_0(\mathbb{R}^d) : \mathcal{S}p(f) \subset \Lambda\right\} \text{ and}$$

$$AP(\Lambda) + \mathcal{C}_0(\Lambda) = \left\{f \in AP(\mathbb{R}^d) + \mathcal{C}_0(\mathbb{R}^d) : \mathcal{S}p(f) \subset \Lambda\right\}.$$

$\mathcal{C}_{per}(Y) + \mathcal{C}_0(\mathbb{R}^d)$ and $AP(\mathbb{R}^d) + \mathcal{C}_0(\mathbb{R}^d)$ being respectively the algebra of asymptotic periodic functions and the algebra of asymptotic almost periodic functions.

To prove this proposition, we first state an essential result giving a criterion of compactness in $B_{ap}^2(\mathbb{R}^d)$. The proof of the classical version in $B_{ap}^2(\mathbb{R})$ due to Bruno and Grande can be found in [17, Theorem 4.1]. This proof easily adapts to the case $B_{ap}^2(\mathbb{R}^d)$.

Theorem 6.1. *Let $\mathcal{F}$ be a family of elements belonging to $B_{ap}^2(\mathbb{R}^d)$, closed and bounded. Then the following statements are equivalent:*

1) *$\mathcal{F}$ is compact in B_{ap}^2-norm;*

2) *$\mathcal{F}$ is B^2_{ap}-equi-continuous, i.e., for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that, if $|h| < \delta$, then*

$$\|\tau_h f - f\|_2 < \epsilon \qquad \forall f \in \mathcal{F} \tag{6.1}$$

and B^2_{ap}-equi-almost-periodic, i.e., for any $\epsilon > 0$, there exists $r = r(\epsilon) > 0$ such that every ball of radius r, contains a common ϵ-almost-period ξ for all $f \in \mathcal{F}$, i.e. for any $\epsilon > 0$ there exists $r = r(\epsilon)$ such that, for all $B(y, r) \subset \mathbb{R}^d$, there exists $\xi \in B(y, r)$ with

$$\|\tau_\xi f - f\|_2 < \epsilon \qquad \forall f \in \mathcal{F}, \tag{6.2}$$

where

$$\|f\|_2 = \left(\lim_{R\to\infty} \fint_{B_R} |f(x)|^2 dx\right)^{\frac{1}{2}}. \tag{6.3}$$

Next, we will use the following lemma, which generalizes the classical and well known averaging lemma of Golse-Perthame-Sentis [27] to almost periodic setting.

Lemma 6.1. *We consider the family of functions $f \in B^2_{ap}(\Lambda; L^2(V))$ satisfying:*

$$\|f\|_{2,2} \le C < \infty \text{ and } \|v \cdot \nabla_y f\|_{2,2} \le C < \infty. \tag{6.4}$$

The family $\mathcal{F} = \left\{\tilde{f} : f \text{ satisfies } (6.4)\right\}$, where

$$\tilde{f} = \int_V f(\cdot, v) d\mu(v), \tag{6.5}$$

is relatively compact in $B^2_{ap}(\Lambda)$.

Let us notice that in view of (4.4), we can reduce the domain of K to

$$D(K) = D(P) = \left\{f \in B^2_{ap}(\Lambda; L^2(V)) : f \text{ satisfies } (6.4)\right\} \tag{6.6}$$

and it is clear that $\|f\|_{D(K)} = \left(\|f\|^2_{2,2} + \|v \cdot \nabla_y f\|^2_{2,2}\right)^{\frac{1}{2}}$ defines a norm on $D(K)$.

Proof of Lemma 6.1. Let $\mathcal{B} = \left\{f \in D(K)/ \, \|f\|_{D(K)} < 1\right\}$ be the open unit ball of $D(K)$. We define

$$\tilde{\mathcal{B}} = \left\{\tilde{f} = \int_V f(\cdot, v) d\mu(v) : f \in \mathcal{B}\right\}.$$

Then $\tilde{\mathcal{B}}$ is obviously a bounded subset of $B^2_{ap}(\Lambda)$. We shall show that the closure $\overline{\tilde{\mathcal{B}}}$ of $\tilde{\mathcal{B}}$ is a closed compact subset of $B^2_{ap}(\Lambda)$. Observe first that $\overline{\tilde{\mathcal{B}}}$ is closed and bounded. Thus, in view of Theorem 6.1, we need to show that $\mathcal{B}$ is B^2_{ap}-equi-continuous and B^2_{ap}-equi-almost-periodic.

i) *B^2_{ap}-equi-continuity.*

Let note that if $f \in \mathcal{B} \subset B^2_{ap}(\Lambda)$, we can pick a sequence $\left(P_n(y, v) = \sum_{j=1}^{m_n} c^n_j(v) e^{i\lambda^n_j \cdot y}\right)_{n\in\mathbb{N}}$ in $\mathcal{P}(\Lambda; L(V)^2)$ *i.e.*, $c^n_j \in L^2(V)$, $\lambda^n_j \in \Lambda$, such that according to Bohr definition we have $f = \lim_{n\mapsto\infty} P_n$ in $L^2(V; B^2_{ap}(\Lambda))$ and obviously $\tilde{f} = \lim_{n\mapsto\infty} \tilde{P}_n$ in $B^2_{ap}(\Lambda)$, where

$$\tilde{P}_n(y) = \sum_{j=1}^{m_n} \tilde{c}^n_j e^{i\lambda^n_j \cdot y} \text{ with } \tilde{c}^n_j = \int_V c^n_j(v) d\mu(v) \in \mathbb{C}.$$

By so doing, we can suppose without lost of generality that

$$f(y,v)=\sum_{j=1}^{\infty}c_j(v)e^{i\lambda_j^n\cdot y} \text{ with } c_j(v)\in L^2(V) \text{ and } \quad \lambda_j\in\Lambda \;\; \forall \;\; j\in\mathbb{N}.$$

Now if $\epsilon>0$, for $h\in\mathbb{R}^d$, we set for a fixed $m\in\mathbb{N}$

$$\tilde{P}_m(y)=\sum_{j=1}^{m}\tilde{c}_je^{i\lambda_j\cdot y} \text{ with } \tilde{c}_j=\int_V c_j(v)d\mu(v)\in\mathbb{C} \;\; \forall \;\; j=1,2,...,m, \tag{6.7}$$

and thanks to (3.6) we have (with $\tau_h\tilde{P_m}(y)=\tilde{P_m}(y+h)$)

$$\begin{aligned}\left\|\tau_h\tilde{P_m}-\tilde{P_m}\right\|_2^2&=\lim_{R\to\infty}⨍_{B_R}\left|\sum_{j=1}^{m}\tilde{c_j}e^{i\lambda_j\cdot y}\left(e^{i\lambda_j\cdot h}-1\right)\right|^2dy,\\&=\lim_{R\to\infty}⨍_{B_R}\left|\sum_{j=1}^{m}2i\tilde{c_j}e^{i\lambda_j\cdot(y+\frac{h}{2})}\sin\left(\frac{\lambda_j\cdot h}{2}\right)\right|^2dy,\\&=\left(\sum_{j=1}^{m}\sum_{k=1}^{m}4\tilde{c_j}\bar{\tilde{c_k}}\sin\left(\frac{\lambda_j\cdot h}{2}\right)\sin\left(\frac{\lambda_k\cdot h}{2}\right)\right)\left(\lim_{R\to\infty}⨍_{B_R}e^{i(\lambda_j-\lambda_k)\cdot(y+\frac{h}{2})}dy\right);\end{aligned}$$

Since $\lim_{R\to\infty}⨍_{B_R}e^{i(\lambda_j-\lambda_k)\cdot y}dy=\delta_k^j$, where δ_k^j are Christoffel's coefficients and $\bar{c}$ is the conjugate complex of c in $\mathbb{C}$, we are leads to

$$\left\|\tau_h\tilde{P_m}-\tilde{P_m}\right\|_2^2=\sum_{j=1}^{m}4\left|\tilde{c_j}\right|^2\sin^2\left(\frac{\lambda_j\cdot h}{2}\right). \tag{6.8}$$

But we have for $j=1,...,m$,

$$\begin{aligned}\left|\tilde{c_j}\right|^2&=\left|\int_V c_j(v)d\mu(v)\right|^2,\\&=\left|\int_V c_j(v)\left(1_{E^c_{\alpha,\lambda_j}}+1_{E_{\alpha,\lambda_j}}\right)d\mu(v)\right|^2,\\&\leq 2\mu\left(E^c_{\alpha,\lambda_j}\right)\int_V\left|c_j(v)\right|^2d\mu(v)+2\mu\left(V\right)\int_V\left|c_j(v)\right|^2 1_{E_{\alpha,\lambda_j}}d\mu(v),\end{aligned} \tag{6.9}$$

where E_{α,λ_j} and E^c_{α,λ_j} are defined in (4.11). Next, in (6.8), thanks to (4.10) we have

$$1\geq\left|\sin\left(\frac{\lambda_j\cdot h}{2}\right)\right|\leq\frac{|\lambda_j\cdot h|}{2}\leq\frac{|h|}{2}\left|\lambda_j\right|\leq\frac{|h|}{2}\frac{|v\cdot\lambda_j|}{\alpha}, \quad \forall v\in E_{\alpha,\lambda_j}. \tag{6.10}$$

Inserting (6.10) and (6.9) in (6.8) and in view of assumption (4.10), we can compute it as follows:

$$\begin{aligned}\left\|\tau_h\tilde{P}_m-\tilde{P}_m\right\|_2^2\leq 8\sum_{j=1}^{m}\mu\left(E^c_{\alpha,\lambda_j}\right)\sin^2\left(\frac{\lambda_j\cdot h}{2}\right)\int_V\left|c_j(v)\right|^2d\mu(v),\\+8\mu\left(V\right)\sum_{j=1}^{m}\int_V\sin^2\left(\frac{\lambda_j\cdot h}{2}\right)\left|c_j(v)\right|^2 1_{E_{\alpha,\lambda_j}}d\mu(v),\end{aligned}$$

$$\leq 8\kappa\alpha\sum_{j=1}^{m}\int_V |c_j(v)|^2\,d\mu(v),$$

$$+\,8\mu\,(V)\sum_{j=1}^{m}\int_V \sin^2\left(\frac{\lambda_j\cdot h}{2}\right)|c_j(v)|^2\,1_{E_{\alpha,\lambda_j}}\,d\mu(v), \tag{6.11}$$

$$\leq 8\kappa\alpha\,\|P_m\|_{2,2}^2 + 8\mu\,(V)\sum_{j=1}^{m}\frac{|h|^2}{4}\int_V |\lambda_j|^2\,|c_j(v)|^2\,1_{E_{\alpha,\lambda_j}}\,d\mu(v),$$

$$\leq 8\kappa\alpha\,\|P_m\|_{2,2}^2 + 2|h|^2\mu\,(V)\sum_{j=1}^{m}\int_V \frac{|v\cdot\lambda_j|^2}{\alpha^2}\,|c_j(v)|^2\,d\mu(v),$$

$$= 8\kappa\alpha\,\|P_m\|_{2,2}^2 + 2\mu(V)\frac{|h|^2}{\alpha^2}\,\|v\cdot\nabla_y P_m\|_{2,2}^2\,.$$

Now, taking $\alpha = |h|^{\frac{2}{3}}$ and $K' = \max\left(8\kappa, 2\mu(V)\right)$, we are led to

$$\left\|\tau_h\tilde{P}_m - \tilde{P}_m\right\|_2^2 \leq K'|h|^{\frac{2}{3}}\left(\|P_m\|_{2,2}^2 + \|v\cdot\nabla_y P_m\|_{2,2}^2\right).$$

So that letting $m\mapsto\infty$ we have

$$\left\|\tau_h\tilde{f} - \tilde{f}\right\|_2^2 \leq K'|h|^{\frac{2}{3}}\left(\|f\|_{2,2}^2 + \|v\cdot\nabla_y f\|_{2,2}^2\right) \leq K'|h|^{\frac{2}{3}},$$

since $f\in\mathcal{B}$, that is, $\|f\|_{D(K)}^2 = \|f\|_{2,2}^2 + \|v\cdot\nabla_y f\|_{2,2}^2 < 1$. Hence, the B^2_{ap}-equi-continuity is obtained in view of Theorem 6.1 for

$$\delta = \left(\frac{\epsilon}{\sqrt{2K'}}\right)^3. \tag{6.12}$$

ii) B^2_{ap}*-equi-almost periodicity.* At this stage, for the sake of simplicity we are going to use $d=3$. By so doing, β-condition (3.9) is fulfilled for $\beta=3$ as notice in Subsection 3.2. But the result can be generalized for any $d>3$.

Now let $f\in\mathcal{B}$. As above, we can suppose without lost of generality that

$$f(y,v) = \sum_{j=1}^{\infty} c_j(v)e^{i\lambda_j\cdot y} \text{ with } c_j(v)\in L^2(V;\mathbb{C}) \text{ and } \lambda_j\in\Lambda \quad \forall\ j.$$

For $\epsilon>0$ and $h\in\mathbb{R}^d$, we consider once more for a fixed $m\in\mathbb{N}$

$$\tilde{P}_m(y) = \sum_{j=1}^{m}\tilde{c}_j e^{i\lambda_j\cdot y} \text{ with } \tilde{c}_j = \int_V c_j(v)d\mu(v)\in\mathbb{C} \quad \forall\ j=1,2,...,m.$$

Then, proceeding as for the B^2_{ap}-equi-continuity, we obtain as in (6.11) the following inequality

$$\left\|\tau_h\tilde{P}_m - \tilde{P}_m\right\|_2^2 \leq 8\kappa\alpha\sum_{j=1}^{m}\int_V |c_j(v)|^2\,d\mu(v)$$

$$+\,8\mu\,(V)\sum_{j=1}^{m}\int_V \sin^2\left(\frac{\lambda_j\cdot h}{2}\right)|c_j(v)|^2\,1_{E_{\alpha,\lambda_j}}\,d\mu(v). \tag{6.13}$$

But as for $v \in E_{\alpha,\lambda_j}$ we have $|v\cdot\lambda_j| > \alpha|\lambda_j|$, i.e. $\frac{1}{|v\cdot\lambda_j|} < \frac{1}{\alpha|\lambda_j|}$, it follows from (6.13) that

$$\begin{aligned}\left\|\tau_h\tilde{P}_m - \tilde{P}_m\right\|_2^2 \leq & 8\kappa\alpha\sum_{j=1}^{m}\int_V |c_j(v)|^2\,d\mu(v) \\ & + 8\mu(V)\sum_{j=1}^{m}\int_V \frac{\sin^2\left(\frac{\lambda_j\cdot h}{2}\right)}{|v\cdot\lambda_j|^2}|v\cdot\lambda_j|^2\,|c_j(v)|^2\,1_{E_{\alpha,\lambda_j}}\,d\mu(v), \\ \leq & 8\kappa\alpha\sum_{j=1}^{m}\int_V |c_j(v)|^2\,d\mu(v) \\ & + 8\mu(V)\sum_{j=1}^{m}\frac{\sin^2\left(\frac{\lambda_j\cdot h}{2}\right)}{\alpha^2|\lambda_j|^2}\int_V |v\cdot\lambda_j|^2\,|c_j(v)|^2\,1_{E_{\alpha,\lambda_j}}\,d\mu(v).\end{aligned} \tag{6.14}$$

Now, consider only the last term at right-hand-side of (6.14), we have by Holder inequality

$$\begin{aligned}&\sum_{j=1}^{m}\frac{\sin^2\left(\frac{\lambda_j\cdot h}{2}\right)}{\alpha^2|\lambda_j|^2}\int_V |v\cdot\lambda_j|^2\,|c_j(v)|^2\,1_{E_{\alpha,\lambda_j}}\,d\mu(v) \\ \leq & \frac{1}{\alpha^2}\left[\sum_{j=1}^{m}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}\right]^{\frac{1}{2}}\left[\sum_{j=1}^{m}\left(\int_V |v\cdot\lambda_j|^2\,|c_j(v)|^2\,d\mu(v)\right)^2\right]^{\frac{1}{2}},\end{aligned} \tag{6.15}$$

and since $\sum_{j=1}^{m} a_j^r \leq \left(\sum_{j=1}^{m} a_j\right)^r$ for $r \geq 1$ and $a_j \geq 0$, in view of (6.15), (6.14) leads to

$$\begin{aligned}\left\|\tau_h\tilde{P}_m - \tilde{P}_m\right\|_2^2 \leq & 8\kappa\alpha\sum_{j=1}^{m}\int_V |c_j(v)|^2\,d\mu(v) \\ & + \frac{8\mu(V)}{\alpha^2}\left[\sum_{j=1}^{m}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}\right]^{\frac{1}{2}}\left[\sum_{j=1}^{m}\int_V |v\cdot\lambda_j|^2\,|c_j(v)|^2\,d\mu(v)\right].\end{aligned} \tag{6.16}$$

We then set $C = \max(8\kappa, 8\mu(V))$. Since $\|f\|_{2,2}^2 = \sum_{j=1}^{\infty}\int_V |c_j(v)|^2\,d\mu(v)$ and $\|v\cdot\nabla_y f\|_{2,2}^2 = \sum_{j=1}^{\infty}\int_V |v\cdot\lambda_j|^2\,|c_j(v)|^2\,d\mu(v)$, letting $m\to\infty$, we obtain

$$\begin{aligned}\left\|\tau_h\tilde{f} - \tilde{f}\right\|_2^2 \leq & C\left(\alpha\sum_{j=1}^{\infty}\int_V |c_j(v)|^2\,d\mu(v)\right. \\ & \left. + \frac{1}{\alpha^2}\left[\sum_{j=1}^{\infty}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}\right]^{\frac{1}{2}}\left[\sum_{j=1}^{\infty}\int_V |v\cdot\lambda_j|^2\,|c_j(v)|^2\,d\mu(v)\right]\right) \\ = & C\left(\alpha\|f\|_{2,2}^2 + \frac{1}{\alpha^2}\left[\sum_{j=1}^{\infty}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}\right]^{\frac{1}{2}}\|v\cdot\nabla_y f\|_{2,2}^2\right).\end{aligned} \tag{6.17}$$

As in view of the β-condition (3.9) we have $S=\sum_{j=1}^{\infty}\frac{1}{|\lambda_j|^4}<\infty$, then $\exists N=N(\epsilon)\in\mathbb{N}$ such that

$$\sum_{j=N+1}^{\infty}\frac{1}{|\lambda_j|^4}<\frac{\epsilon^{12}}{2C^6}.$$

Setting

$$\delta=\delta_\epsilon=\left(\frac{\epsilon^{12}}{2SC^6}\right)^{\frac{1}{4}},$$

we note that if $h\in\mathbb{R}^d$ is the solution of the following Kronecker system of inequalities

$$|\lambda_j\cdot h|<\delta \mod [2\pi],\ j=1,2,....,N, \tag{6.18}$$

then, as

$$\left|\sin\left(\frac{\lambda_j\cdot h}{2}\right)\right|\le\frac{|\lambda_j\cdot h|}{2}\le|\lambda_j\cdot h|,$$

such h will also be solution of

$$\left|\sin\left(\frac{\lambda_j\cdot h}{2}\right)\right|<\delta \text{ for } j=1,2,...,N. \tag{6.19}$$

Thus for such h, we will have

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}&=\sum_{j=1}^{N}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}+\sum_{j=N+1}^{\infty}\frac{\sin^4\left(\frac{\lambda_j\cdot h}{2}\right)}{|\lambda_j|^4}, \qquad (6.20)\\&\le\delta^4\sum_{j=1}^{N}\frac{1}{|\lambda_j|^4}+\sum_{j=N+1}^{\infty}\frac{1}{|\lambda_j|^4},\\&\le\left(\left(\frac{\epsilon^{12}}{2SC^6}\right)^{\frac{1}{4}}\right)^4\times S+\frac{\epsilon^{12}}{2C^6},\\&=\frac{\epsilon^{12}}{2C^6}+\frac{\epsilon^{12}}{2C^6}=\frac{\epsilon^{12}}{C^6}.\end{aligned}$$

Coming back to inequality (6.17) and inserting (6.20) we obtain

$$\left\|\tau_h\tilde{f}-\tilde{f}\right\|_2^2\le C\left(\alpha\|f\|_{2,2}^2+\frac{1}{\alpha^2}\frac{\epsilon^6}{C^3}\|v\cdot\nabla_y f\|_{2,2}^2\right). \tag{6.21}$$

So that if $\alpha=\frac{\epsilon^2}{C}$ and as $\left(\|f\|_{2,2}^2+\|v\cdot\nabla_y f\|_{2,2}^2\right)<1$, (6.21) gives us

$$\left\|\tau_h\tilde{f}-\tilde{f}\right\|_2^2\le\epsilon^2\left(\|f\|_{2,2}^2+\|v\cdot\nabla_y f\|_{2,2}^2\right)\le\epsilon^2.$$

That is

$$\left\|\tau_h\tilde{f}-\tilde{f}\right\|_2\le\epsilon. \tag{6.22}$$

As it is well known by Kronecker's theorem [21, 30, 35], the solution of the system (6.18) is a relatively dense subset of $\mathbb{R}^d$. Hence, for a corresponding $\epsilon>0$, it follows from (6.22) that there exists a relatively dense subset of $\mathbb{R}^d$ of ϵ-common almost period in the sense of $B^2_{ap}(\Lambda)$ for all $\tilde{f}\in\mathcal{B}$. This shows that $\tilde{\mathcal{B}}$ is B^2_{ap}-equi-almost periodic and ends the proof of the lemma. □

We are now able to prove the compactness of operator K stated in Proposition 6.1.

Proof of Proposition 6.1. i) Let first notice that for $f \in D(K)$, we have

$$\|f\|_{2,2} \leq \left(\|f\|_{2,2}^2 + \|v \cdot \nabla_y f\|_{2,2}^2\right)^{\frac{1}{2}} = \|f\|_{D(K)} . \tag{6.23}$$

This implies that the injection of $D(K)$ in $B^2_{ap}(\Lambda; L^2(V))$ is continuous. On the other hand, we can easily check that

$$\|Kf\|_{2,2} \leq \sup_{y\in\mathbb{R}^d}\left(\|\sigma(y,\cdot,\cdot)\|_{2,2}\right)\|f\|_{2,2} . \tag{6.24}$$

Therefore K is a continuous operator in $B^2_{ap}(\Lambda; L^2(V)) \equiv L^2(V; B^2_{ap}(\Lambda))$, satisfying:

$$\|K\|_{\mathcal{L}(D(K),L^2(V;B^2_{ap}))} \leq \|K\|_{\mathcal{L}(L^2(V;B^2_{ap}))} \leq \sup_{y\in\mathbb{R}^d}\left(\|\sigma(y,\cdot,\cdot)\|_{2,2}^2\right). \tag{6.25}$$

Where $\mathcal{L}(D(K), L^2(V; B^2_{ap}))$ (resp. $\mathcal{L}(L^2(V; B^2_{ap})))$ is the set of linear operator from $D(K)$ (resp. $L^2(V; B^2_{ap}(\Lambda)))$ to $L^2(V; B^2_{ap}(\Lambda))$. Now, as $\sigma \in B^2_{ap}(\Lambda, L^2(V \times V))$, then by Bohr definition of almost periodic function it can be approximated uniformly by linear combination (with coefficients in $L^2(V) \times L^2(V)$) of elements of $\mathcal{P}(\Lambda; L^2(V) \times L^2(V))$ and consequently

$$\sigma = \lim_{n\to\infty} k_n \text{ in } B^2_{ap}(\Lambda, L^2(V^2)), \text{ with } k_n(y,v,w) = \sum_{k=1}^{m_n} g_k^n(v)\phi_k^n(w)e^{i\lambda_k^n\cdot y}, \tag{6.26}$$

where g_k^n, $\phi_k^n \in L^2(V)$, are regular enough for the multiplication mapping $f \mapsto g_k^n f$ to be continuous in $D(K)$. Then, the collision operators K_n, defined by

$$K_n f(y,v) = \int_V k_n(y,v,w)f(y,w)d\mu(w), \tag{6.27}$$

are compact from $D(K)$ into $B^2_{ap}(\Lambda; L^2(V))$, since the mappings

$$\begin{array}{ccccccc} D(K) & \longrightarrow & D(K) & \longrightarrow & B^2_{ap}(\Lambda) & \longrightarrow & B^2_{ap}(\Lambda; L^2(V)) \\ & \times\phi_k^n & & \widetilde{\cdot}\times e^{i\lambda_k^n\cdot y} & & \times g_k^n & \\ f & \longmapsto & \phi_k^n f & \longmapsto & \widetilde{\phi_k^n f} e^{i\lambda_k^n\cdot y} = \int_V \phi_k^n f e^{i\lambda_k^n\cdot y}d\mu & \longmapsto & g_k^n\widetilde{\phi_k^n f}e^{i\lambda_k^n\cdot y} \end{array}$$

are compact as composed of bounded mappings and the compact mapping $\phi_k^n f \longmapsto \widetilde{\phi_k^n f}e^{i\lambda_k^n\cdot y}$. But the K_n are also convergent in norm to the operator K which is therefore compact from $D(K)$ into $B^2_{ap}\left(\Lambda; L^2(V)\right)$.

ii) For the proof of this statement, we refer the reader to [23, Section 5.3. and Section 5.4.]. $\square$

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Patrick ATIOFACK FOUEGAP Department of Mathematics and Computer Science, University of Dschang, P.O. Box 67, Dschang, Cameroon

E-mail: minlefack@gmail.com

David DONGO Department of Mathematics and Computer Science, University of Dschang, P.O. Box 67, Dschang, Cameroon

E-mail: dongodavid@yahoo.fr

Jean Louis WOUKENG Department of Mathematics and Computer Science, University of Dschang, P.O. Box 67, Dschang, Cameroon

E-mail: jwoukeng@yahoo.fr