

ON SOME FUNDAMENTAL EQUATIONS OF EQUIVARIANT RIEMANNIAN GEOMETRY

ALLEN BACK*, MANFREDO P. DO CARMO AND WU-YI HSIANG*

1. Introduction

First of all, we would like to point out that the terminology “*equivariant Riemannian geometry*” is used here to mean a systematic application of the techniques and ideas of transformation groups to the study of Riemannian geometry rather than in the usual sense of an “equivariant theory” (e.g. equivariant K -theory etc.). In order to clarify the above viewpoint, let us briefly discuss some basic features of transformation groups and Riemannian geometry. In mathematics, the concept of group naturally occurs as *structure-preserving transformations*, namely, the symmetry (or automorphism) group of a given mathematical model. It is quite reasonable to expect that many basic, useful mathematical models are naturally endowed with rich symmetries. Therefore, it is rather natural, and often advantageous, to study the interactions between the structure-preserving transformation group and the structure itself. The classical Galois theory of algebraic equations is a well-known, pioneer example of successful application of the above general philosophy. Symmetries have always been playing an important role in the study of geometry since antiquity. For example, the *axioms of congruence* of Euclidean *geometry* are actually postulates which describe the symmetric properties of the Euclidean model of space. In fact, most of the classical geometric models are, by definition, *homogeneous* under the action of their respective automorphism groups. From the viewpoint of transformation groups, such a geometric model can simply be represented as a *specific homogeneous space* and hence *the geometry is completely determined by its structure-preserving transformation group*. This is exactly the underlying idea of the Erlangen program of F. Klein [35].

The concept of Riemannian space [47] provides a general framework in which one can study various models of space with broad perspective and suitable generality. For example, the classical geometries of Euclidean, spherical and hyperbolic types can then be *unified* as Riemannian spaces of *constant* (sectional) *curvature*. However, such a giant step of generalization inevitably also includes multitudes of abstract arbitrary examples so that most of them are clearly useless and unworthy of study. Therefore, it seems to be one of the special features of Riemannian geometry that selected, nice family of spaces are

Corresponding author: W. Y. Hsiang.

*Supported in part by NSF Grants

more important than general, abstract examples and specific, simple problems are more interesting than general theory. Of course, there are vast numbers of general Riemannian spaces which are rather *asymmetric*. However, it is quite natural that many of the most interesting examples in Riemannian geometry do have large groups of symmetries. It is fair to say that although transformation groups no longer occupy as dominant a role in Riemannian geometry as in the classical geometries, it should still play a useful role in the study of Riemannian geometry.

The purpose of this paper is to present a basic formulation and to derive some fundamental structural equations of equivariant Riemannian geometry which, we hope, will provide some groundwork for the study of interactions between transformation groups and Riemannian geometry.

The publication of this paper has been long delayed, so we have included some discussion of directions of later development.

2. Orbital geometry of a G -Riemannian manifold; a basic setting of equivariant Riemannian geometry

A G -Riemannian manifold, (G, M) , is a Riemannian manifold, M , together with a given *isometric* transformation group, G , which is always assumed to be *closed in the full isometry group of M* . We shall first analyze the geometry of the orbit structure of a given G -Riemannian manifold, (G, M) . Recall that (G, M) is automatically a Lie transformation group and G is automatically compact if M is compact [43]. In case that M is non-compact, then G may be non-compact. However, each isotropy subgroup, G_x , is a closed subgroup of the orthogonal group of T_xM and hence is always compact. Obviously, the slice theorem holds for (G, M) and consequently, the principal orbit type theorem [42] holds for (G, M) even when G is non-compact.

We set once and for all the following notations:

Let G/H be the principal orbit type of (G, M) and H a fixed principal isotropy subgroup. Let \mathfrak{g} and \mathfrak{h} be the Lie algebras of G and H respectively and \mathfrak{p} the orthogonal complement of \mathfrak{h} in \mathfrak{g} with respect to a fixed Ad_G -invariant inner product. Let M_0 be the union of all principal orbits, called the *regular part of M* , and M_S the complement of M_0 in M , called the *singular part of M* . Set $\overline{M}_0 = F(H, M_0)$, the fixed point set of H in M_0 , and \overline{M} the closure of \overline{M}_0 in M . [It is possible that $\overline{M} \neq F(H, M)$.] Let $N(H)$ be the normalizer of H in G and $\overline{G} = N(H)/H$. Then \overline{M} is a \overline{G} -Riemannian manifold which is totally geodesic in M and \overline{M}_0 is the regular part of \overline{M} with \overline{G} acting freely.

Observe that if M is *complete* then the G -Riemannian structure on M is *completely determined by that of its restriction to M_0* . In this section, we shall show that the orbital geometry of (G, M_0) consists of the following three parts:

- (i) the *normal part* of the orbital geometry which can be neatly organized into an *orbital distance metric* (cf. §2.2) on the orbit space M_0/G ,

- (ii) the *tangent part* of the orbital geometry which records the homogeneous Riemannian structure of each fibre of the fibration $G/H \rightarrow M_0 \rightarrow M_0/G$ (cf. Proposition 2, §2.3),
- (iii) a \overline{G} -connection on \overline{M}_0 whose horizontal spaces are exactly those normal spaces to the \overline{G} -orbits in M_0 .

We shall prove that the above three *invariants*, in fact, already consist of a *complete set of invariants for the G -Riemannian structure on M_0 and hence also on M* (cf. Theorem 1, §2.4).

2.1. Invariant smooth functions and the smooth structure on M/G

In this subsection, we recall some basic results of differentiable transformation groups. Here M is only assumed to be a differentiable G -manifold with *all isotropy subgroups compact*. Let $\mathfrak{E}(M)$ be the set of smooth functions on M and $\mathfrak{E}(M)^G$ the set of G -invariant smooth functions on M . It is rather natural to *define* the induced “smooth structure” on the orbit space M/G by setting $\mathfrak{E}(M/G) = \mathfrak{E}(M)^G$, namely, a function $f : M/G \rightarrow \mathbb{R}$ is smooth if and only if its lifting $\tilde{f} : M \rightarrow \mathbb{R}$ is smooth. In the study of such a natural smooth structure on M/G , one has the following basic theorem of J. Schwarz [48]:

Theorem 2.1. (Schwarz) *For any given point $x \in M$, set $\xi = G(x) \in M/G$ and (G_x, V_x) the slice representation at x . Suppose $\{f_1, \dots, f_l\}$ forms a Hilbert basis of the ring of G_x invariant polynomials of V_x , $\mathbb{R}[V_x]^{G_x}$. Then $\{f_1, \dots, f_l\}$ also gives a local imbedding of a suitable neighborhood of ξ in M/G .*

If one combines the above theorem with a theorem of Chevalley-Luna-Richardson [40], it is not difficult to deduce the following:

Theorem 2.2. *The restriction homomorphism:*

$$\mathfrak{E}(M)^G \rightarrow \mathfrak{E}(\overline{M})^{\overline{G}}$$

is always a bijection.

Finally, let us recall some basic facts concerning equivariant maps between *linear G -spaces*:

(1) Suppose V and W are two given *linear G -spaces* (G compact). Let $\mathfrak{E}(V, W)^G$ be the set of all *smooth G -equivariant maps* of V into W . Then $\mathfrak{E}(V, W)^G$ is a finitely generated module over $\mathfrak{E}(V)^G$. Let $\mathbb{R}[V]^G$ be the ring of invariant polynomials of V and $\mathbb{R}[V, W]^G$ the set of G -equivariant polynomial maps of V into W . Then, an $\mathbb{R}[V]^G$ -module basis of $\mathbb{R}[V, W]^G$ is also an $\mathfrak{E}(V)^G$ -module basis of $\mathfrak{E}(V, W)^G$.

(2) Let $f \in \mathbb{R}[V, W]^G$ and $w' \in W^*$, the dual space of W . Then $\tilde{f}(v, w') = \langle f(v), w' \rangle$ is an invariant polynomial of $V \otimes W^*$. Conversely, if $\tilde{f} \in \mathbb{R}[V \otimes W^*]^G$, then $D_{w'} \tilde{f}(v, 0)$

clearly defines a linear functional for each $v \in V$ and hence, by duality, determines a G -equivariant polynomial map of V into W . This is the well-known relationship between *invariants* and *covariants*. It is also useful to recall that a C^∞ -map, θ , between linear spaces V and W satisfying $\theta(\lambda v) = \lambda^k \theta(v)$ for all $\lambda \in \mathbb{R}$ and $v \in V$ must be a polynomial map of degree k .

(3) In general, the above mentioned theorem of Chevalley-Luna-Richardson can not be generalized to equivariant maps. One basic reason obstructing such a generalization lies in the fact that the centralizer of $\overline{G} = N(H)/H$ in the orthogonal group $O(\overline{V})$ and that of G in $O(V)$ may not be bijective.

(4) In the special case $G = S^1 = \{\xi \in C; |\xi| = 1\}$, it is convenient to use complex coordinates. For example, suppose the G -complex-linear spaces V and W are given as follows:

$$V = \{(z_1, \dots, z_n), z_i \in C\}, \xi(z_1, \dots, z_n) = (\xi^{m_1} z_1, \dots, \xi^{m_n} z_n),$$

$$W = C, \xi(z) = \xi^{m_0} z.$$

Then, to each pair of integral n -tuples $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$ satisfying the condition $\sum m_i(\alpha_i - \beta_i) = m_0$, one has a G -equivariant map of V into W defined as follows:

$$f(z_1, \dots, z_n) = z_1^{\alpha_1} \dots z_n^{\alpha_n} \bar{z}_1^{\beta_1} \dots \bar{z}_n^{\beta_n}.$$

It is not difficult to see that maps of the above simple type already forms a module basis of $R[V; W]^G$.

2.2. The normal part of orbital geometry

Let us first consider the regular part, M_0 , which is a smooth fibre bundle over M_0/G with G/H as its typical fibre. Since the restricted metrics on all normal vector spaces of a given orbit $\xi = G(x)$ are mutually isometric under the action of G , it is natural to equip the tangent space of ξ in M_0/G with such a metric so that $M_0 \rightarrow M_0/G$ is a *Riemannian submersion* in the sense of O'Neill [45]. It is not difficult to see that the distance between two points $\xi_i = G(x_i)$, $i = 1, 2$, with respect to such a Riemannian metric on M_0/G is exactly the shortest distance between the two orbits, $G(x_i)$, $i = 1, 2$, in M_0 . We shall call it the *orbital distance metric* of M_0/G , which neatly records the normal part of the orbital geometry of (G, M_0) . According to §2.1, the smooth structure of M_0/G can naturally be described in terms of the invariant functions of M_0 . Therefore, one often needs to compute the orbital distance metric in terms of invariant functions. For this purpose, one has the following proposition of [22].

Proposition 2.1. *Let $\{f_i; 1 \leq i \leq \dim M_0/G\}$ be a system of local coordinate functions in a neighborhood U of ξ . Set*

$$c_{ij} = \langle \nabla f_i, \nabla f_j \rangle_{M_0},$$

where ∇f_i is the gradient vector field of f_i on M_0 and c_{ij} can be expressed as smooth functions in terms of (f_1, \dots, f_d) . Then the matrix (c_{ij}) is non-singular and the orbital distance metric on U is given by

$$ds^2 = \sum_{i,j} c^{ij} df_i df_j, \tag{1}$$

where (c^{ij}) is the inverse matrix of (c_{ij}) .

We refer to [22] for a simple proof of the above useful fact.

Next let us consider the metric structure of the total orbit space, M/G . Let $O(G, M)$ be the set of orbit types of (G, M) equipped with the following partial ordering, namely $G/K_1 \geq G/K_2$ if and only if K_1 is conjugate to a subgroup of K_2 . Then M/G is naturally a stratified set indexed by $O(G, M)$ such that each strata consists of exactly those orbits of the same type. Clearly, there is a natural structure of *metric space* on M/G such that the distance between two points $\xi_1, \xi_2 \in M/G$ is exactly the shortest distance between the two orbits, $\xi_i = G(x_i)$, $i = 1, 2$, in M . Moreover, to each orbit type $G/K \in O(G, M)$, the strata, $(M/G)_K$, consists of a disjoint union of Riemannian manifolds compatible with the restriction of the above metric space structure. Let Y be a connected component of $(M, G)_K$ and Y_1 a *compact region* of Y . Then, there exists a sufficiently small $\varepsilon > 0$ such that

$$N_\varepsilon(Y_1) = \{\xi \in M/G, d(\xi, Y) = d(\xi, Y_1) \leq \varepsilon\}$$

has the structure of a normal fibration $p : N_\varepsilon(Y_1) \rightarrow Y_1$, where $p^{-1}(\eta_1) = \{\xi : d(\xi, Y_1) = d(\xi, \eta_1) \leq \varepsilon\}$ is the *normal cone* of Y_1 at η_1 . Let x_1 be a point on the orbit η_1 with $G_{x_1} = K$ and V_{x_1} the normal vector space of $G(x_1)$ at x_1 . Set $V'_{x_1} \oplus V''_{x_1}$ as the orthogonal decomposition of V_{x_1} with $V'_{x_1} = F(K, V_{x_1})$. Then, it is not difficult to see that V''_{x_1}/K is a local approximation of $p^{-1}(\eta_1)$. We shall call the totality of such structures on M/G the *stratified Riemannian structure of M/G* which amounts to a concise recording of the normal part of the orbital geometry of (G, M) .

2.3. The tangent part of orbital geometry

In the regular part M_0 , every orbit is a *homogeneous* Riemannian manifold of G/H -type. The tangent space to a basepoint of G/H may be identified with \mathfrak{p} , and the Riemannian metric on G/H is determined by the metric at the basepoint. Let $\{X_j^*; 1 \leq j \leq m\}$ be a chosen and then fixed orthonormal basis of \mathfrak{p} (with respect to an Ad_G -invariant metric) and X_j the Killing vector field of the one-parameter group of isometrics $\text{Exp}(tX_j^*)$. Set

$$a_{ij}(x) = \langle X_i(x), X_j(x) \rangle_{M_0}, \quad x \in \overline{M_0}. \tag{2}$$

Then the matrix valued function $A(x) = (a_{ij}(x))$, $x \in M_0$, provides a systematic way of recording the intrinsic geometry of all principal orbits. $A(x)$ satisfies the following covariance property with respect to the \overline{G} -action on $\overline{M_0}$:

Proposition 2.2. *The matrix $A(x)$, $x \in \overline{M}_0$, is H -invariant with respect to $Ad_H \mathfrak{p}$ and is \overline{G} -covariant, namely,*

$$A(nx) = B(n) \cdot A(x) \cdot B(n)^t, \quad n \in N(H), \tag{3}$$

where $B(n)$ is the matrix of $Ad(n)|_{\mathfrak{p}}$ with respect to the basis $\{X_j^*; 1 \leq j \leq m\}$.

Proof. $A(x)$ is by definition H -invariant. One need only check the covariance property (3). Observe that

$$X_i(nx) = L_n(Ad(n^{-1})X_i^*)(x) = L_n(B(n^{-1})X_i^*)(x)$$

and the homogeneous metric on each orbit is invariant under the left translation. Hence

$$\begin{aligned} A(nx) &= (\langle B(n^{-1})X_i(x), B(n^{-1})X_j(x) \rangle) \\ &= B(n^{-1})^t \cdot A(x) \cdot B(n^{-1}) \\ &= B(n)A(x)B(n)^t. \end{aligned} \tag{4}$$

Note that the number of copies and multiplicity of each irreducible component of the isotropy representation determine how many independent functions are contained in the matrix A . By virtue of the Schur lemma, the representation spaces of non-equivalent components are orthogonal, and a component of multiplicity n is described by $\binom{n}{2}$ real functions.

2.4. \overline{G} -connection and a uniqueness theorem

In the setting of Riemannian geometry, the orbital submersion $G/H \rightarrow M_0 \rightarrow M_0/G$ has a natural \overline{G} -connection defined on its associated principal bundle $\overline{G} \rightarrow \overline{M}_0 \rightarrow \overline{M}_0/\overline{G} = M_0/G$, namely, the horizontal spaces are exactly those normal spaces to the \overline{G} -orbits in \overline{M}_0 (which are also the normal spaces to the G -orbits in M_0). Let θ be the connection form gotten by orthogonal projection and Y_1, Y_2 arbitrary horizontal vector fields on \overline{M}_0 . Then the curvature form of the above \overline{G} -connection is given by

$$d\theta(Y_1, Y_2) = -\frac{1}{2}\theta([Y_1, Y_2]). \tag{5}$$

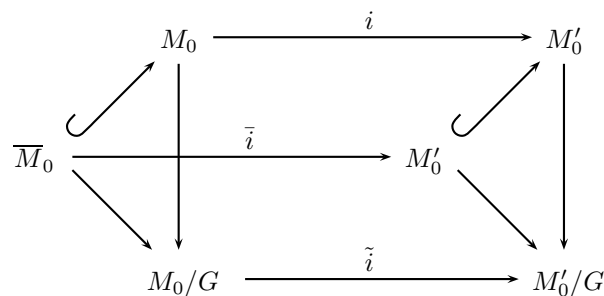
In the setting of O’Neill [45], the above curvature form also gives the fundamental tensor of the orbital submersion.

Combining the above three geometric invariants of the orbital geometry of a given G -Riemannian manifold M , one has the following simple but basic theorem of uniqueness:

Uniqueness Theorem *Suppose M and M' are two complete G -Riemannian manifolds with the same principal orbit type G/H . If there exists a \overline{G} -bundle map $\bar{i} : \overline{M}_0 \rightarrow \overline{M}'_0$ satisfying*

- (i) the induced map $\tilde{i} : \overline{M}_0/\overline{G} \rightarrow \overline{M}'_0/\overline{G}$ is an isometry with respect to their orbital distance metrics,
 - (ii) \tilde{i} preserves the \overline{G} -connections,
 - (iii) $A'(\tilde{i}x) = A(x)$ for all $x \in \overline{M}_0$,
- then \tilde{i} can be uniquely extended to a G -equivariant isometry of $M \rightarrow M'$.

Proof. Let $i : M_0 \rightarrow M'_0$ be the associated G/H -bundle map of \tilde{i} , where G/H is a G -space with \overline{G} as its automorphism group. Then, one has the following commutative diagram of maps:



where i is, by construction, G -equivariant. In order to check that the above G -equivariant map is an isometry, one needs only to check it at an arbitrary point $x \in \overline{M}_0$. Let $M_x, M'_{x'}$ ($x' = \tilde{i}(x)$) be the tangent spaces of M, M' at x, x' respectively, and let

$$M_x = T_x + N_x, \quad M'_{x'} = T'_{x'} + N'_{x'}$$

be the decompositions into the tangent and normal subspaces of the orbits $G(x)$ and $G(x')$ respectively. It follows from the assumptions (i)–(iii) that di_x maps T_x and N_x isometrically onto $T'_{x'}$ and $N'_{x'}$, respectively. Hence, di_x is an isometry of M_x onto $M'_{x'}$. This proves that i is an isometry of M_0 onto M'_0 . Finally, it follows from the completeness of M and M' and the open dense set property of M_0 and M'_0 that the above isometry $i : M_0 \rightarrow M'_0$ can be uniquely extended to an isometry of M onto M' [36].

2.5. A basic setting of equivariant Riemannian geometry

The above uniqueness theorem proves that the geometry of a given G -Riemannian manifold (G, M) is completely determined by the following three invariants, namely,

- (i) the orbital distance metric on M_0/G ,
- (ii) a \overline{G} -covariant, matrix-valued function $A(x), x \in \overline{M}_0$ which records the homogeneous metric of each G -orbit of M_0
- (iii) a \overline{G} -connection in the associated principal bundle $\overline{G} \rightarrow \overline{M}_0 \rightarrow M_0/G$, which is exactly the *twisting invariant* of the *orbital submersion* $M_0 \rightarrow M_0/G$.

Therefore, a natural setting of equivariant Riemannian geometry is to reduce the study of equivariant geometric problems to corresponding problems solely in terms of the above three basic invariants. For example, one should be able to compute the various curvatures of (G, M) solely in terms of the above three invariants (cf. §3). In case $(G, W) \subset (G, M)$ is an invariant submanifold, then the second fundamental form of $W \subset M$ should again be computable in terms of the second fundamental form of $W/G \subset M/G$ together with the second and third invariants. (cf. §4). Roughly speaking, if a given geometric problem is G -equivariant, then the geometry in the tangential directions of the G -orbits are reigned in by the given G -action. Hence, by properly understanding the interaction of the given geometric problem and the G -action, it should be possible to condense the given geometric problem down to an associated problem at the level of the orbit space. For example, suppose the given geometric problem is expressible in terms of a certain system of differential equations. Then, there should be a reduced system of differential equations, defined at the level of the orbit space, if the geometric problem happens to be G -equivariant.

In our approach, it is natural to understand many aspects of the behavior of the geometry along singular orbits as a limit of behavior in the regular part M_0 . However the covariance relation in Proposition 2.2. (as well as the usual covariance properties of connection forms in principal bundles) pose precise conditions on the basic invariants near the singular orbits. These conditions may be understood by finding bases for appropriate spaces of equivariant polynomials. Alternatively, when the invariant theory is too involved, there is a natural infinitesimal method for identifying the boundary conditions along the singular orbits. This is described in appendix I.

3. Fundamental equations of orbital submersion

For a given complete G -Riemannian manifold, (G, M) , the restriction of the orbital projection to the regular part is a *Riemannian submersion*: $M_0 \rightarrow M_0/G$. We shall call it the orbital submersion of (G, M) . Following the terminology of O'Neill [45], we shall call the tangent (resp. normal) spaces to the G -orbits the vertical (resp. horizontal) spaces and denote the orthogonal projection onto the vertical (resp. horizontal) spaces by \mathcal{V} (resp. \mathcal{H}). Recall from [45] that the second fundamental form of the fibers, T , is the tensor defined by

$$T_X Y = \mathcal{V} \nabla_{\mathcal{V} X} \mathcal{H} Y + \mathcal{H} \nabla_{\mathcal{V} X} \mathcal{V} Y \quad (6)$$

and the fundamental tensor of the submersion, A , is defined by

$$A_X Y = \mathcal{H} \nabla_{\mathcal{H} X} \mathcal{V} Y + \mathcal{V} \nabla_{\mathcal{H} X} \mathcal{H} Y. \quad (7)$$

In [45], a set of fundamental equations of a Riemannian submersion was derived by O'Neill expressing the Riemann tensor of the total space in terms of that of its base space and each fibre together with the *above two tensors and their covariant derivatives*.

In the special case of orbital submersion of (G, M) , the uniqueness theorem of §2 asserts that the three basic invariants (cf. §2) already constitute a complete set of invariants of (G, M) . Therefore, it is rather natural to seek a set of equations which will

enable us to compute the above two tensors, T and A , and their covariant derivatives solely in terms of the three basic invariants. Thus, they will also enable us to compute the various curvatures of M solely in terms of the three basic invariants introduced in §2. Technically, the result of this section is a natural synthesis of the work of O’Neill [45] and that of Nomizu [44] and Wang [50].

3.1. On the choice of a special frame field

In the study of a specific geometric problem, one of the typical, crucial steps is the choice of a convenient coordinate system which is particularly suitable for the given geometric situation. In the case of orbital submersion, that amounts to the choice of a special frame field so that the tensor fields T and A together with their covariant derivatives can be effectively computed in terms of the three basic invariants of §2. In the special situation of orbital submersion, $M_0 \rightarrow M_0/G$, it is easy to see that the vertical spaces can be spanned by *Killing vector fields* and the horizontal spaces can be spanned by invariant vector fields. Moreover, since the geometric problem is clearly G -equivariant and \overline{M}_0 is a totally geodesic submanifold which intersects every G -orbit, one needs only to carry out the computations of those local invariants at the points $x \in \overline{M}_0$. Therefore, in the computation of this section, we shall mainly use the following special frame field defined over a \overline{G} -invariant neighborhood, U , of x in \overline{M}_0 , namely,

$$\left\{ \begin{array}{l} \text{vertical part:} \quad \text{the } m \text{ killing vector fields, } \{X_i, 1 \leq i \leq m\} \text{ (cf. §2.3),} \\ \text{horizontal part:} \quad \text{the } d \text{ orthonormal invariant vector fields, } \{Y_\alpha, 1 \leq \alpha \leq d = \\ \quad \text{dim } M_0/G\}, \text{ where } \{Y_\alpha\} \text{ is the unique horizontal lifting of an ar-} \\ \quad \text{bitrary, but fixed, orthonormal frame field, } \{Y_\alpha^*\} \text{ of } U/G. \end{array} \right.$$

It is easy to verify that both T_E and A_E are *skew symmetric* with respect to $\langle \cdot, \cdot \rangle$ and they interchange the vertical and horizontal subspaces. Hence, the tensor T (resp. A) is completely determined by the following set of values.

$$\{\langle T_{X_i} X_j, Y_\alpha \rangle; 1 \leq i, j \leq m, 1 \leq \alpha < d\},$$

(resp. $\{\langle A_{Y_\alpha} Y_\beta, X_i \rangle; 1 \leq \alpha, \beta \leq d, 1 \leq i \leq m\}$).

Proposition 3.1. *In terms of the above special frame field, the tensors T and A of the orbital submersion, $M_0 \rightarrow M_0/G$, are completely determined by the following simple formulae:*

$$\left\{ \begin{array}{l} \text{(i)} \quad A_{Y_\alpha} Y_\beta = \frac{1}{2} \mathcal{V}[Y_\alpha, Y_\beta] \\ \text{(ii)} \quad \langle T_{X_i} X_j, Y_\alpha \rangle = -\frac{1}{2} Y_\alpha \langle X_i, X_j \rangle = -\frac{1}{2} Y_\alpha a_{ij}(x). \end{array} \right. \tag{8}$$

Proof. (i) is exactly the Lemma 2 of [45].

Since X_i is a Killing vector field and Y_α is an invariant vector field, one has $[X_i, Y_\alpha] = 0$ and hence $\nabla_{X_i} Y_\alpha = \nabla_{Y_\alpha} X_i$

$$\begin{aligned} Y_\alpha \langle X_i, X_j \rangle &= \langle \nabla_{Y_\alpha} X_i, X_j \rangle + \langle X_i, \nabla_{Y_\alpha} X_j \rangle \\ &= \langle \nabla_{X_i} Y_\alpha, X_j \rangle + \langle X_i, \nabla_{X_j} Y_\alpha \rangle \\ &= -\langle \nabla_{X_i} X_j, Y_\alpha \rangle - \langle \nabla_{X_j} X_i, Y_\alpha \rangle \\ &= -2\langle T_{X_i} X_j, Y_\alpha \rangle. \end{aligned}$$

Remark. (i) shows that the tensor A is essentially just the \overline{G} -connection of $\overline{M}_0 \rightarrow \overline{M}_0/\overline{G}$. (ii) neatly expresses the components of T in terms of the normal derivatives of the tangential part of the invariants, namely, $(a_{ij}(x))$. The result is also expressible as the horizontal Lie derivative of the metric evaluated in the directions (X_i, X_j) .

Proposition 3.2. *In terms of the special frame field and the tensors T, A (determined by (i), (ii) of Proposition 3.1.), one has the following complete description of the covariant differentiation on M_0 , namely,*

$$\left\{ \begin{array}{l} \text{(i) } \nabla_{X_i} X_j = T_{X_i} X_j + \widehat{\nabla}_{X_i} X_j, \\ \text{(ii) } \nabla_{X_i} Y_\alpha = A_{Y_\alpha} X_i + T_{X_i} Y_\alpha, \\ \text{(iii) } \nabla_{Y_\alpha} X_i = A_{Y_\alpha} X_i + T_{X_i} Y_\alpha, \\ \text{(iv) } \nabla_{Y_\alpha} Y_\beta = (\widetilde{\nabla}_{Y_\alpha} Y_\beta^*) + A_{Y_\alpha} Y_\beta, \end{array} \right. \tag{9}$$

where $\widehat{\nabla}$ is the connection on the orbit $G(x)$ and $\widetilde{\nabla}$ is the connection on the orbit space M_0/G .

Proof. Most of these are noted in O’Neill [45] as easy consequences of the definition of T and A . The others just follow from $[X_i, Y_\alpha] = 0$. For example,

$$\nabla_{Y_\alpha} X_i = A_{Y_\alpha} X_i + \mathcal{V} \nabla_{Y_\alpha} X_i$$

and $[X_i, Y_\alpha] = 0$ implies

$$\langle \nabla_{Y_\alpha} X_i, X_j \rangle = \langle \nabla_{X_i} Y_\alpha, X_j \rangle = \langle T_{X_i} Y_\alpha, X_j \rangle.$$

This proves (iii) because $T_{X_i} Y_\alpha$ is, by definition, vertical.

3.2. The covariant derivatives of T and A

In the fundamental equations of Riemannian submersion [45], one needs the covariant derivatives of T and A . The following “algebraic” relationships which hold for any Riemannian submersion were proved or indicated in [45].

Let V_1, V_2 and V_3 (resp. Z_1, Z_2 and Z_3) be arbitrary vertical (resp. horizontal) vector fields of the total space of a Riemannian submersion. Then the following “algebraic” identities hold in general, namely,

$$\left\{ \begin{array}{l} \text{(a)} \quad (\nabla_{V_1} A)_{V_2} = -A_{T_{V_1} V_2}, \quad (\nabla_{Z_1} T)_{Z_2} = -T_{A_{Z_1} Z_2}, \\ \text{(b)} \quad (\nabla_{Z_1} A)_{V_1} = -A_{A_{Z_1} V_1}, \quad (\nabla_{V_1} T)_{Z_1} = -T_{T_{V_1} Z_1}, \\ \text{(c)} \quad \langle (\nabla_{V_1} A)_{Z_1} V_2, V_3 \rangle = \langle T_{V_1} V_2, A_{Z_1} V_3 \rangle - \langle T_{V_1} V_3, A_{Z_1} V_2 \rangle, \\ \quad \langle (\nabla_{Z_1} T)_{V_1} Z_2, Z_3 \rangle = \langle A_{Z_1} Z_2, T_{V_1} Z_3 \rangle - \langle A_{Z_1} Z_3, T_{V_1} Z_2 \rangle, \\ \text{(d)} \quad \langle (\nabla_{V_1} A)_{Z_1} Z_2, Z_3 \rangle = \langle T_{V_1} Z_2, A_{Z_1} Z_3 \rangle - \langle T_{V_1} Z_3, A_{Z_1} Z_2 \rangle, \\ \quad \langle (\nabla_{Z_1} T)_{V_1} V_2, V_3 \rangle = \langle A_{Z_1} V_2, T_{V_1} V_3 \rangle - \langle A_{Z_1} V_3, T_{V_1} V_2 \rangle, \\ \text{(e)} \quad \langle (\nabla_{Z_1} A)_{Z_2} V_1, V_2 \rangle = \langle A_{Z_1} V_1, A_{Z_2} V_2 \rangle - \langle A_{Z_1} V_2, A_{Z_2} V_1 \rangle, \\ \quad \langle (\nabla_{V_1} T)_{V_2} Z_1, Z_2 \rangle = \langle T_{V_1} Z_1, T_{V_2} Z_2 \rangle - \langle T_{V_1} Z_2, T_{V_2} Z_1 \rangle, \\ \text{(f)} \quad \langle (\nabla_{Z_1} A)_{Z_2} Z_3, Z_4 \rangle = \langle A_{Z_1} Z_3, A_{Z_2} Z_4 \rangle - \langle A_{Z_1} Z_4, A_{Z_2} Z_3 \rangle, \\ \quad \langle (\nabla_{V_1} T)_{V_2} V_3, V_4 \rangle = \langle T_{V_1} V_3, T_{V_2} V_4 \rangle - \langle T_{V_1} V_4, T_{V_2} V_3 \rangle, \\ \text{(g)} \quad \mathcal{C} \langle (\nabla_{Z_1} A)_{Z_2} Z_3, V_1 \rangle = \mathcal{C} \langle A_{Z_2} Z_3, T_{V_1} Z_1 \rangle \\ \quad \mathcal{C} \langle (\nabla_{V_1} T)_{V_2} V_3, Z_1 \rangle = \mathcal{C} \langle T_{V_2} V_3, A_{Z_1} V_1 \rangle \end{array} \right. \quad (10)$$

where \mathcal{C} denotes the cyclic sum of three terms.

Notice the duality between each pair of formulae for A and T under reversal of horizontal and vertical vector fields. As was pointed out in [45], the above formulae enable one to reduce to just four types of “non-algebraic” covariant derivatives of T and A for a general Riemannian submersion, namely,

$$\begin{aligned} &\langle (\nabla_{Z_1} A)_{Z_2} Z_3, V_1 \rangle, \quad \langle (\nabla_{V_1} A)_{Z_1} Z_2, V_2 \rangle, \\ &\langle (\nabla_{V_1} T)_{V_2} V_3, Z_1 \rangle, \quad \langle (\nabla_{Z_1} T)_{V_1} V_2, Z_2 \rangle. \end{aligned}$$

However, in the special case of orbital submersion, one can again compute the above four types of covariant derivatives in terms of the basic invariants of §2.

Proposition 3.3. *In terms of the special frame field, one has the following formula for the computation of the above four types of covariant derivatives of T and A , namely,*

$$\left\{ \begin{array}{l} \text{(a)} \quad \langle (\nabla_{X_i} A)_{Y_\alpha} Y_\beta, X_j \rangle = X_i \langle A_{Y_\alpha} Y_\beta, X_j \rangle - \langle A_{Y_\alpha} X_i, A_{Y_\beta} X_j \rangle \\ \quad \quad \quad + \langle A_{Y_\beta} X_i, A_{Y_\alpha} X_j \rangle - \langle A_{Y_\alpha} Y_\beta, \widehat{\nabla}_{X_i} X_j \rangle \\ \text{(a')} \quad \langle (\nabla_{Y_\alpha} T)_{X_i} X_j, Y_\beta \rangle = Y_\alpha \langle T_{X_i} X_j, Y_\beta \rangle + \langle T_{X_i} Y_\alpha, T_{X_j} Y_\beta \rangle \\ \quad \quad \quad + \langle T_{X_j} Y_\alpha, T_{X_i} Y_\beta \rangle - \langle T_{X_i} X_j, \widetilde{\nabla}_{Y_\alpha} Y_\beta^* \rangle \\ \text{(b)} \quad \langle (\nabla_{Y_\alpha} A)_{Y_\beta} Y_\gamma, X_i \rangle = Y_\alpha \langle A_{Y_\beta} Y_\gamma, X_i \rangle - \langle A_{Y_\gamma} X_i, \widetilde{\nabla}_{Y_\alpha} Y_\beta^* \rangle \\ \quad \quad \quad + \langle A_{Y_\beta} X_i, \widetilde{\nabla}_{Y_\alpha} Y_\gamma^* \rangle - \langle A_{Y_\beta} Y_\gamma, T_{X_i} Y_\alpha \rangle \\ \text{(b')} \quad \langle (\nabla_{X_i} T)_{X_j} X_k, Y_\alpha \rangle = X_i \langle T_{X_j} X_k, Y_\alpha \rangle + \langle T_{X_k} Y_\alpha, \widehat{\nabla}_{X_i} X_j \rangle \\ \quad \quad \quad + \langle T_{X_j} Y_\alpha, \widehat{\nabla}_{X_i} X_k \rangle - \langle T_{X_j} X_k, A_{Y_\alpha} X_i \rangle \end{array} \right. \quad (11)$$

Proof. The validity of the above set of formulae heavily relies on the special choice of frame field, especially the fact that $[X_i, Y_\alpha] = 0$ and hence $\nabla_{X_i} Y_\alpha = \nabla_{Y_\alpha} X_i$. Since the above formula (a), (a') and (b), (b') are essentially dual to each other [the only *sign difference* comes from the fact that A is skew symmetric for horizontal vector fields but T is symmetric for vertical vector fields], we shall only prove (a) and (b) in the following:

$$\begin{aligned}
\langle (\nabla_{X_i} A)_{Y_\alpha} Y_\beta, X_j \rangle &= X_i \langle A_{Y_\alpha} Y_\beta, X_j \rangle - \langle A_{\nabla_{X_i} Y_\alpha} Y_\beta, X_j \rangle \\
&\quad - \langle A_{Y_\alpha} (\nabla_{X_i} Y_\beta), X_j \rangle - \langle A_{Y_\alpha} Y_\beta, \nabla_{X_i} X_j \rangle \\
&= X_i \langle A_{Y_\alpha} Y_\beta, X_j \rangle - \langle A_{A_{Y_\alpha} X_i} Y_\beta, X_j \rangle \\
&\quad + \langle \nabla_{X_i} Y_\beta, A_{Y_\alpha} X_j \rangle - \langle A_{Y_\alpha} Y_\beta, \widehat{\nabla}_{X_i} X_j \rangle \\
&= X_i \langle A_{Y_\alpha} Y_\beta, X_j \rangle - \langle A_{Y_\alpha} X_i, A_{Y_\beta} X_j \rangle \\
&\quad + \langle A_{Y_\beta} X_i, A_{Y_\alpha} X_j \rangle - \langle A_{Y_\alpha} Y_\beta, \widehat{\nabla}_{X_i} X_j \rangle \\
\langle (\nabla_{Y_\alpha} A)_{Y_\beta} Y_\gamma, X_i \rangle &= Y_\alpha \langle A_{Y_\beta} Y_\gamma, X_i \rangle - \langle A_{\nabla_{Y_\alpha} Y_\beta} Y_\gamma, X_i \rangle \\
&\quad - \langle A_{Y_\beta} (\nabla_{Y_\alpha} Y_\gamma), X_i \rangle - \langle A_{Y_\beta} Y_\gamma, \nabla_{Y_\alpha} X_i \rangle \\
&= Y_\alpha \langle A_{Y_\beta} Y_\gamma, X_i \rangle - \langle A_{Y_\gamma} X_i, \widetilde{\nabla}_{Y_\alpha} Y_\beta^* \rangle \\
&\quad + \langle A_{Y_\beta} X_i, \widetilde{\nabla}_{Y_\alpha} Y_\gamma^* \rangle - \langle A_{Y_\beta} Y_\gamma, T_{X_i} Y_\alpha \rangle.
\end{aligned}$$

3.3. Formulae for the Riemannian curvature tensor of an orbital submersion

In the very special case of a homogeneous Riemannian metric on G/H , the homogeneous Riemannian manifold is completely determined by the pair of Lie groups (G, H) and an Ad_H -invariant inner product on the complementary subspace, \mathfrak{p} , of \mathfrak{h} in \mathfrak{g} . The result of Nomizu-Wang [44, 50] shows how to compute the Levi-Civita connection and hence also the *Riemannian curvature tensor* of the above homogeneous Riemannian manifold in terms of the invariant inner product on \mathfrak{p} and the Lie structure of $(\mathfrak{g}, \mathfrak{h})$. We state their result as the following proposition:

Proposition 3.4. (Nomizu-Wang) *Let X, Y, Z be arbitrary Killing vector fields on a given Riemannian manifold M and*

$$U(X, Y) = \nabla_X Y - \frac{1}{2}[X, Y].$$

Then $U(X, Y) = U(Y, X)$ and satisfies the following identity:

$$2\langle U(X, Y), Z \rangle = -\{\langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle\}$$

Proof. It is well-known that $D_X = L_X - \nabla_X$ is a tensor of type $(1, 1)$ and is skew symmetric if and only if X is a Killing vector field. It follows from the general identity

$$\nabla_X Y - \nabla_Y X - [X, Y] \equiv 0$$

that $U(X, Y) = U(Y, X)$. Since $U(X, Y) = \frac{1}{2}[X, Y] - D_X Y$ it follows from the skew symmetry of $D_X Y$ that

$$\langle U(X, Y), Z \rangle + \langle Y, U(X, Z) \rangle = \frac{1}{2} \{ \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle \} \tag{12}$$

From this and from the two identities resulting by cyclic permutation of X, Y, Z , one obtains the following identity by using the symmetric property of U

$$\langle U(X, Y), Z \rangle = -\frac{1}{2} \{ \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle \} \tag{13}$$

Remark. The above formula (13) provides effective machinery to compute $\widehat{\nabla}_{X_i} X_j$ and hence also $\langle \widehat{R}(X_i X_j) X_k, X_l \rangle$ in terms of the Lie algebra data of $(\mathfrak{g}, \mathfrak{h})$ and the inner product data $a_{ij}(x) = \langle X_i(x), X_j(x) \rangle, x \in \overline{M}_0$. However in carrying out such explicit computation in terms of the bracket operation of \mathfrak{g} , one must be careful to note that for

$$X_i \leftrightarrow X_i^* \in \mathfrak{g}, X_j \leftrightarrow X_j^* \in \mathfrak{g}$$

then $[X_i, X_j]$ (bracket as Killing vector fields) $\leftrightarrow -[X_i^*, X_j^*]$ (bracket as elements of \mathfrak{g}).

In the general case of a G -Riemannian (G, M) with an intransitive G -action, the geometric structure is an orbital submersion, $M_0 \rightarrow M_0/G$, which is completely determined by the three basic invariants of §2 (cf. Uniqueness Theorem, §2). Therefore, it is rather natural to apply the formalism of O’Neill on Riemannian submersion to generalize the above result to derive the following formulae for the Riemannian curvature tensor of (G, M) .

Theorem 3.3. *In terms of the special frame field, the various components of the Riemannian curvature tensor of M can be efficiently computed at points of \overline{M}_0 by the following set of formulae (using the Kobayashi-Nomizu sign convention):*

$$\left\{ \begin{array}{l} \text{(a) } \langle R(X_i, X_j) X_k, X_l \rangle = \langle \widehat{R}(X_i, X_j) X_k, X_l \rangle \\ \qquad \qquad \qquad - \langle T_{X_j} X_k, T_{X_i} X_l \rangle + \langle T_{X_i} X_k, T_{X_j} X_l \rangle \\ \text{(b) } \langle R(Y_\alpha, Y_\beta) Y_\gamma, Y_\delta \rangle = \langle \widetilde{R}(Y_\alpha^* Y_\beta^*) Y_\gamma^* Y_\delta^* \rangle + 2 \langle A_{Y_\alpha} Y_\beta, A_{Y_\gamma} Y_\delta \rangle \\ \qquad \qquad \qquad - \langle A_{Y_\beta} Y_\gamma, A_{Y_\alpha} Y_\delta \rangle - \langle A_{Y_\gamma} Y_\alpha, A_{Y_\beta} Y_\delta \rangle \\ \text{(c) } \langle R(Y_\alpha, Y_\beta) Y_\gamma, X_i \rangle = -Y_\gamma \langle A_{Y_\alpha} Y_\beta, X_i \rangle + \langle A_{Y_\beta} X_i, \widetilde{\nabla}_{Y_\gamma^*} Y_\alpha^* \rangle \\ \qquad \qquad \qquad - \langle A_{Y_\alpha} X_i, \widetilde{\nabla}_{Y_\gamma^*} Y_\beta^* \rangle + \langle A_{Y_\beta} Y_\gamma, T_{X_i} Y_\alpha \rangle \\ \qquad \qquad \qquad + \langle A_{Y_\gamma} Y_\alpha, T_{X_i} Y_\beta \rangle \\ \text{(d) } \langle R(Y_\alpha, X_i) Y_\beta, X_j \rangle = -Y_\alpha \langle T_{X_i} X_j, Y_\beta \rangle - \langle T_{X_i} Y_\beta, T_{X_j} Y_\alpha \rangle \\ \qquad \qquad \qquad + \langle T_{X_i} X_j, \widetilde{\nabla}_{Y_\alpha^*} Y_\beta^* \rangle - X_i \langle A_{Y_\alpha} Y_\beta, X_j \rangle \\ \qquad \qquad \qquad - \langle A_{Y_\beta} X_i, A_{Y_\alpha} X_j \rangle + \langle A_{Y_\alpha} Y_\beta, \widehat{\nabla}_{X_i} X_j \rangle \\ \text{(e) } \langle R(X_i, X_j) X_k, Y_\alpha \rangle = \langle T_{X_i} X_k, A_{Y_\alpha} X_j \rangle - \langle T_{X_j} X_k, A_{Y_\alpha} X_i \rangle \\ \qquad \qquad \qquad + \frac{1}{2} [- \langle [Y_\alpha, U(X_i, X_k)], X_j \rangle + \langle [Y_\alpha, U(X_j, X_k)], X_i \rangle] \end{array} \right. \tag{14}$$

Proof. (a) and (b) are just the formula {0} and {4} in §4 of [45]. Furthermore O'Neill [45] proved

$$\begin{aligned} \langle R(Y_\alpha, Y_\beta)Y_\gamma, X_i \rangle &= \langle A_{Y_\beta}Y_\gamma, T_{X_i}Y_\alpha \rangle + \langle A_{Y_\gamma}Y_\alpha, T_{X_i}Y_\beta \rangle \\ &\quad - \langle A_{Y_\alpha}Y_\beta, T_{X_i}Y_\alpha \rangle - \langle (\nabla_{Y_\gamma}A)_{Y_\alpha}Y_\beta, X_i \rangle \end{aligned} \quad \{3\}$$

$$\begin{aligned} \langle R(Y_\alpha, X_i)Y_\beta, X_j \rangle &= -\langle (\nabla_{Y_\alpha}T)_{X_i}X_j, Y_\beta \rangle - \langle (\nabla_{X_i}A)_{Y_\alpha}Y_\beta, X_j \rangle \\ &\quad + \langle T_{X_i}Y_\alpha, T_{X_j}Y_\beta \rangle - \langle A_{Y_\alpha}X_i, A_{Y_\beta}X_j \rangle. \end{aligned} \quad \{2\}$$

Using (11-b) for the 4-th term of {3} and (11-a'), (11-a) for the first, second terms of {2}, it is straightforward to obtain the above formulae (c) and (d). Finally, let us prove (e) as follows: We start with {1} of O'Neill [45].

$$\langle R(X_i, X_j)X_k, Y_\alpha \rangle = \langle (\nabla_{X_i}T)_{X_j}X_k, Y_\alpha \rangle - \langle (\nabla_{X_j}T)_{X_i}X_k, Y_\alpha \rangle \quad \{1\}$$

By (11-b') of proposition 3.3., one has

$$\begin{aligned} \langle R(X_i, X_j)X_k, Y_\alpha \rangle &= X_i \langle T_{X_j}X_k, Y_\alpha \rangle + \langle T_{X_k}Y_\alpha, \widehat{\nabla}_{X_i}X_j \rangle + \langle T_{X_j}Y_\alpha, \widehat{\nabla}_{X_i}X_k \rangle \\ &\quad - \langle T_{X_j}X_k, A_{Y_\alpha}X_i \rangle - X_j \langle T_{X_i}X_k, Y_\alpha \rangle - \langle T_{X_k}Y_\alpha, \widehat{\nabla}_{X_j}X_i \rangle \\ &\quad - \langle T_{X_i}Y_\alpha, \widehat{\nabla}_{X_j}X_k \rangle + \langle T_{X_i}X_k, A_{Y_\alpha}X_j \rangle \end{aligned}$$

Using (8-ii) of proposition 3.1., the skew symmetric property of T_Y and the fact that $[X_i, Y_\alpha] = [X_j, Y_\alpha] = [X_k, Y_\alpha] = 0$, one reduces the above formula into

$$\begin{aligned} \langle R(X_i, X_j)X_k, Y_\alpha \rangle &= \langle T_{X_i}X_k, A_{Y_\alpha}X_j \rangle - \langle T_{X_j}X_k, A_{Y_\alpha}X_i \rangle \\ &\quad + \frac{1}{2}Y_\alpha \{ X_j \langle X_i, X_k \rangle - \langle \widehat{\nabla}_{X_j}X_i, X_k \rangle - \langle X_i, \widehat{\nabla}_{X_j}X_k \rangle \\ &\quad - X_i \langle X_j, X_k \rangle + \langle \widehat{\nabla}_{X_i}X_j, X_k \rangle + \langle X_j, \nabla_{X_i}X_k \rangle \} \\ &\quad + \frac{1}{2} \langle X_i, [Y_\alpha, \widehat{\nabla}_{X_j}X_k] \rangle - \frac{1}{2} \langle X_j, [Y_\alpha, \widehat{\nabla}_{X_i}X_k] \rangle \\ &= \langle T_{X_i}X_k, A_{Y_\alpha}X_j \rangle - \langle T_{X_j}X_k, A_{Y_\alpha}X_i \rangle \\ &\quad + \frac{1}{2} [-\langle [Y_\alpha, U(X_i, X_k)]X_j \rangle + \langle [Y_\alpha, U(X_j, X_k)], X_i \rangle] \end{aligned}$$

Remarks

- (i) We follow the sign convention of [36] for the definition of Riemannian curvature tensor, which happens to differ by a sign with that of O'Neill [45].
- (ii) (14-a) is simply the Gauss equation of each orbit and (14-e) is the Codazzi equation of each orbit.
- (iii) From the above set of formulae for the Riemannian curvature tensor, it is then rather straightforward to write down formulae for other type of curvature tensors.

- (iv) The above set of formulae simplify considerably in the special case of generalized rotational manifold, i.e. $\dim M_0/G = 1$; we shall discuss this special case and its applications in §5.

4. The second fundamental form of an equivariant isometric immersion

In the study of differential geometry of submanifolds, one is particularly interested in those nice submanifolds whose *second fundamental forms* satisfy some simple analytic conditions, e.g., minimal submanifolds, hypersurfaces of constant mean curvature, W -hypersurfaces etc. Analytically, such nice submanifolds are characterized by certain specific differential equations. Suppose the submanifold M happens to be invariant under a given isometry group G of the ambient space N . Then there should be a corresponding *reduced differential equation which characterizes the “submanifold” M/G in the orbit space N/G* . In this section, we shall apply the results of §3 to establish a set of simple formulae which express *the second fundamental form of M in N* in terms of the second fundamental form of M/G in N/G and the two fundamental tensors T and A of the orbital submersions, thus providing a unified way of settling the above type of reduction problem.

Let $i : (G, M) \rightarrow (G, N)$ be an equivariant, isometric immersion of G -Riemannian manifolds. One may assume without loss of generality that the principal orbit type of (G, M) is the same as that of (G, N) . By restricting the immersion, i , to the regular part, M_0 , of M , one has the following commutative diagrams:

$$\begin{array}{ccc}
 M_0 & \xrightarrow{i} & N_0 \\
 \downarrow \pi & & \downarrow \pi' \\
 M_0/G & \xrightarrow{\tilde{i}} & N_0/G
 \end{array}
 \qquad
 \begin{array}{ccc}
 \overline{M}_0 & \xrightarrow{i} & \overline{N}_0 \\
 \downarrow \bar{\pi} & & \downarrow \bar{\pi}' \\
 M_0/G & \xrightarrow{\tilde{i}} & N_0/G
 \end{array}$$

Let II and \widetilde{II} be respectively the second fundamental forms of $i : M_0 \rightarrow N_0$ and $\tilde{i} : M_0/G \rightarrow N_0/G$ (with respect to the orbital distance metric). We shall adopt the following system of notations for this section, namely,

- (i) Let A, T (resp. A', T') be the two fundamental tensors of the orbital submersions $\pi : M_0 \rightarrow M_0/G$ (resp. $\pi' : N_0 \rightarrow N_0/G$),
- (ii) $\nabla, \nabla', \widetilde{\nabla}$ and $\widetilde{\nabla}'$ are respectively the covariant derivations in $M_0, N_0, M_0/G$ and N_0/G ,
- (iii) τ, ν and $\tilde{\tau}, \tilde{\nu}$ are respectively the orthogonal projections onto the tangential or the normal subspaces of M_0 and M_0/G .

In terms of the above system of notations, the second fundamental forms II and \widetilde{II} can be defined as the following tensors, namely,

$$\begin{cases} II_X Y = \nu \nabla'_{\tau X}(\tau Y) + \tau \nabla'_{\tau X}(\nu Y), \\ \widetilde{II}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nu} \widetilde{\nabla}'_{\widetilde{\tau} \widetilde{X}}(\widetilde{\tau} \widetilde{Y}) + \widetilde{\tau} \widetilde{\nabla}'_{\widetilde{\tau} \widetilde{X}}(\widetilde{\nu} \widetilde{Y}), \end{cases} \tag{15}$$

where X, Y (resp. $\widetilde{X}, \widetilde{Y}$) are vector fields defined on M_0 (resp. M_0/G). At each point, $\widetilde{II}_{\widetilde{X}}$ (resp. II_X) is a skew symmetric linear operator and it reverses the tangential and normal subspaces. Therefore, one needs only to compute the value of $II_X Y$ for tangential vector fields.

Theorem 4.4. *Let $\{X_i\}$ and $\{Y_\alpha\}$ be the special frame field of §3.1 and let the normal space of M_0 at $x \in \overline{M_0}$ be identified with that of M_0/G at \tilde{x} under $d\pi'$. Then, one has the following equations:*

$$\begin{cases} \text{(a)} & A \text{ is the pullback of } A', \text{ i.e., } A_{Y_\alpha} Y_\beta = A'_{Y_\alpha} Y_\beta \\ \text{(b)} & II_{X_i} X_j = T'_{X_i} X_j - T_{X_i} X_j \\ \text{(c)} & II_{Y_\alpha} Y_\beta = \widetilde{II}_{\widetilde{Y}_\alpha} \widetilde{Y}_\beta \\ \text{(d)} & II_{X_i} Y_\alpha = A'_{Y_\alpha} X_i - A_{Y_\alpha} X_i = II_{Y_\alpha} X_i \end{cases} \tag{16}$$

Proof. (a) follows immediately from the interpretation that

$$A_{Y_\alpha} Y_\beta = \frac{1}{2} \mathcal{V}[Y_\alpha, Y_\beta]$$

(b), (c) and (d) follows from (15) and (9)-(i), (ii), (iii), (iv) of Proposition 3.2. by straightforward computations. For example,

$$\begin{aligned} II_{X_i} X_j &= \nu \nabla'_{X_i} X_j = \nabla'_{X_i} X_j - \nabla_{X_i} X_j \\ &= (T'_{X_i} X_j + \widehat{\nabla}_{X_i} X_j) - (T_{X_i} X_j + \widehat{\nabla}_{X_i} X_j) \\ &= T'_{X_i} X_j - T_{X_i} X_j \\ II_{X_i} Y_\alpha &= \nu \nabla'_{X_i} Y_\alpha = \nabla'_{X_i} Y_\alpha - \nabla_{X_i} Y_\alpha \\ &= (A'_{Y_\alpha} X_i + T'_{X_i} Y_\alpha) - (A_{Y_\alpha} X_i + T_{X_i} Y_\alpha) \end{aligned}$$

and the following computation shows that $T'_{X_i} Y_\alpha = T_{X_i} Y_\alpha$. By definition, both $T'_{X_i} Y_\alpha$ and $T_{X_i} Y_\alpha$ are tangential to the orbits and hence, one can check their equality by computing their inner products with X_j , namely,

$$\begin{aligned} \langle T'_{X_i} Y_\alpha, X_j \rangle &= -\langle Y_\alpha, T'_{X_i} X_j \rangle = \frac{1}{2} Y_\alpha \langle X_i, X_j \rangle, \\ \langle T_{X_i} Y_\alpha, X_j \rangle &= -\langle Y_\alpha, T_{X_i} X_j \rangle = \frac{1}{2} Y_\alpha \langle X_i, X_j \rangle. \end{aligned}$$

Among various local invariants of differential geometry of submanifolds, the simplest and also one of the most important one is certainly the *mean curvature*, namely,

$$H = \operatorname{tr} II \quad (\text{with respect to an orthonormal frame}).$$

The following is a useful formula for the mean curvature of a G -invariant submanifold $(G, M) \subset (G, N)$.

Proposition 4.1. *Let $v(x) = \det(A_{ij}(x))^{\frac{1}{2}}$, $x \in M_0$, be the volume function of the orbit $G(x)$. Then*

$$\operatorname{tr} II = \operatorname{tr} \widetilde{II} - \operatorname{grad}(\ln v(x)) \quad (\text{or } H = \widetilde{H} - \operatorname{grad}(\ln v(x))) \quad (17)$$

where H and \widetilde{H} are the (principal) mean curvature vector fields of $M_0 \subset N_0$ and $M_0/G \subset N_0/G$ respectively.

Proof. Let \widetilde{Z} be an arbitrary normal vector field of M_0/G in N_0/G and Z be the unique invariant, normal vector field of M_0 in N_0 such that $\pi'(Z) = \widetilde{Z}$. Since the expressions of both sides are, by definition, normal vector fields, one needs only to check the following equality

$$\langle \operatorname{tr} II, Z \rangle = \langle \operatorname{tr} \widetilde{II}, \widetilde{Z} \rangle - \langle \operatorname{grad}(\ln v), Z \rangle$$

to hold for such an arbitrary Z . Straightforward computation using (16)-(b), (c), (d) and (8)-(ii) shows that

$$\begin{aligned} \langle \operatorname{tr} II, Z \rangle &= \langle \operatorname{tr} \widetilde{II}, \widetilde{Z} \rangle - \frac{1}{2} \sum_{i,j} a^{ij} (Z a_{ij}) \\ &= \langle \operatorname{tr} \widetilde{II}, \widetilde{Z} \rangle - \frac{1}{2} Z(\ln \det(a_{ij})) \\ &= \langle \operatorname{tr} \widetilde{II}, \widetilde{Z} \rangle - \langle \operatorname{grad}(\ln v), Z \rangle \end{aligned}$$

Remarks.

- (i) In the special case of G -invariant minimal submanifolds, G compact, one has the equation

$$\operatorname{tr} \widetilde{II} - \operatorname{grad}(\ln v(x)) = 0 \quad (17_0)$$

which is clearly equivalent to the equation obtained by the reduction procedure of Hsiang-Lawson [31]. However, the above formula (17) has two advantages over the reduction procedure of [31], namely, (a), (17) still holds even if G is non-compact (e.g., this makes it directly applicable to the case studied in [52]), because one uses the volume function instead of the total volume of orbits in (17); (b), (17) gives the mean curvature vector field directly, while the procedure of [31] will give the mean curvature field with a distortion (such distortion is inconvenient in the study of invariant hypersurface of constant mean curvature, etc.).

- (ii) Technically, theorem 4.4. is a straightforward consequence of the proposition 3.1. and proposition 3.2. which make the covariant derivatives easily computable in terms of the orbital geometric data. Therefore, it is not difficult to derive similar formulae for other fundamental tensors associated to equivariant submersions or mapping etc., as long as they are neatly defined in terms of covariant derivations.

5. Preliminary examples of problems and applications I: Generalized rotational (almost homogeneous) manifolds

The basic setting of §2 together with the fundamental structure equations of §3 and §4 provide a general machinery for studying geometric problems which are *equivariant* under the action of an *intransitive* transformation group. Following [31], we shall call the dimension of M/G the *cohomogeneity* of (G, M) , which is clearly a kind of geometric degree of the intransitivity of the G -action on M . Intuitively speaking, the existence of a structural-preserving transformation group provides a possibility of organizing *the geometry of tangential directions to the orbits* into some set of Lie group theoretical invariants, which are usually algebraic in nature. Therefore, it is natural to expect that a given problem of differential geometry with an intransitive symmetry group can be reduced to an analytical problem only involving the normal directions of the orbits, preferably to a problem that can be “pushed down” to the orbit space.

Among all Riemannian manifolds with *intransitive* symmetry groups, those G -Riemannian manifold of *cohomogeneity one* are certainly the simplest and the most accessible ones from the viewpoint of equivariant differential geometry. We propose to call them *generalized rotational manifolds*, or almost homogeneous manifolds. We would like to point out here that the class of generalized rotational manifolds, in fact, constitutes a rather rich family of Riemannian manifolds which are, so far, almost entirely unexplored. It is our opinion that the understanding of quite a few basic problems of global differential geometry will be considerably improved by *testing* them on the family of generalized rotational manifolds. Technically, as one can see from the simplicity of the fundamental equations of this special case (cf §5.1 of this section), the geometry of generalized rotational manifolds provides an interesting but rather manageable interplay of Lie group theory and ODE (or rather calculus). Of course, problems and applications discussed in this section are only some of the simplest, preliminary examples of the geometry of generalized rotational manifolds.

5.1. Structural equations of generalized rotational manifolds

In the especially simple case of a generalized rotational G -manifold, M_0/G is, by assumption, an open interval. Therefore, the tensor A is automatically zero and there exists an arc Γ in \overline{M} which forms a fundamental domain of (G, M) and is perpendicular to every G -orbit. We shall parameterize the orbit space by the arc-length of $\Gamma \simeq M/G$, say s , and denote the restriction of $a_{ij}(x)$ to Γ by $a_{ij}(s)$. We shall denote by Y the unique lifting of $\frac{d}{ds}$ to a unit normal vector field to the G -orbits.

Theorem 5.5. *Let M be a G -Riemannian manifold of generalized rotational type, Γ a fixed, fundamental normal arc and $\{X_1, \dots, X_m, Y\}$ the special frame field of §3.1 (restricted to Γ). Then the fundamental equations of §3 can be simplified as follows:*

- (i) $\nabla_{X_i} X_j = -\frac{1}{2}a'_{ij}Y + U(X_i, X_j) + \frac{1}{2}[X_i, X_j]$
 $\nabla_{X_i} Y = \sum_j (\frac{1}{2} \sum_k a'_{ik} a^{kj}) X_j = \nabla_Y X_i$
 $\nabla_Y Y = 0$
- (ii) $\langle R(X_i, X_j)X_k, X_l \rangle = \langle \widehat{R}(X_i, X_j)X_k, X_l \rangle + \frac{1}{4}[a'_{ik}a'_{jl} - a'_{jk}a'_{il}]$
 $\langle R(Y, X_i)X_j, Y \rangle = -\frac{1}{2}a''_i + \frac{1}{4} \sum_{k,l} a'_{ik} a'_{jl} a^{kl}$
 $\langle R(X_i, X_j)X_k, Y \rangle = \frac{1}{2}[\langle [Y, U(X_j, X_k)], X_i \rangle - \langle [Y, U(X_i, X_k)], X_j \rangle]$
- (iii) $\text{Ric}(X_j, X_k) = \langle R(Y, X_j)X_k, Y \rangle + \sum_{i,l} a^{il} \langle R(X_i, X_j)X_k, X_l \rangle$
 $= \widehat{\text{Ric}}(X_j, X_k) - \frac{1}{2}a''_{jk} + \frac{1}{2} \sum_{i,l} a'_{ik} a'_{jl} a^{il} - \frac{1}{2}a'_{jk}(\log v)'$
where $v = \det(A_{ij})^{1/2}$.

Proof. These are straightforward from Propositions 3.2. and 3.4. and Theorem 3.3.. The function v enters because of the well known identity $\text{Tr}(A^{-1}A') = (\log \det A)'$.

5.2. Reflectionally symmetric spaces and rotational manifolds of classical type

Let M be a given Riemannian manifold, $I(M)$ the (full) isometry group of M and $I(M, p)$ the subgroup of $I(M)$ fixing the point $p \in M$. We say that M is *reflectionally symmetric* with respect to a given direction $X_p \in T_p M$ if there exists an isometric involution $\sigma(X_p) \in I(M, p)$ which reverses X_p and fixes all the perpendicular directions of X_p in $T_p(M)$. M is said to be reflectionally symmetric at p if it is reflectionally symmetric with respect to every direction at p (i.e., all $X_p \in T_p M$) and M is simply said to be a reflectionally symmetric space if it is reflectionally symmetric at *every point* $p \in M$.

Classically, the Euclidean, spherical and non-Euclidean (i.e., hyperbolic) geometries are the three types of geometries which are commonly characterized by the usual *axioms of congruence*. From the general setting of Riemannian manifolds, it is quite easy to see that the usual axioms of congruences are logically equivalent to the property of *reflectional symmetry* (with respect to every direction). Locally, it is also clear that M is reflectionally symmetric at p if and only if $I(M, p) \simeq O(n)$, $n = \dim M$. We shall call a Riemannian manifold, M^n , a rotational manifold of classical type if $I(M^n) \supset O(n)$. It is easy to show that the principal orbit type of $(O(n), M^n)$ is either S^{n-1} or RP^{n-1} . Moreover, if the principal orbit type is RP^{n-1} , then M is diffeomorphic to $RP^{n-1} \times (M/G)$ and hence $\pi_1(M^n) \simeq \mathbb{Z}_2$.

Proposition 5.1. *Let M^n be a simply connected, complete Riemannian manifold which is reflectionally symmetric at p , i.e., $I(M^n, p) \simeq O(n)$. Then*

- (i) *there exists a unique function, $f(r)$, such that $\sum(p, r) = \{x \in M^n; d(p, x) = r\}$ is isometric to $S^{n-1}(f(r))$, the $(n - 1)$ -sphere of radius $f(r)$, and M^n is uniquely characterized by this function up to $O(n)$ -isometry. (We shall call it the characteristic function of orbit sizes of M^n .)*
- (ii) *M^n is a reflectional symmetric space if and only if M^n is of constant sectional curvature, say $K = K(M^n)$, and the above function is given as follows, namely*

$$f(r) = \begin{cases} r & \text{if } K = 0 \text{ (Euclidean),} \\ \frac{1}{\sqrt{K}} \sin \sqrt{K}r & \text{if } K > 0 \text{ (spherical),} \\ \frac{1}{\sqrt{-K}} \sinh \sqrt{-K}r & \text{if } K < 0 \text{ (hyperbolic).} \end{cases} \tag{18}$$

Proof. By the assumption $I(M^n, p) \simeq O(n)$, it is obvious that $O(n)$ acts transitively over the subset $\sum(p, r) = \{x \in M^n; d(p, x) = r\}$. There is no $O(n)$ -orbit of RP^{n-1} -type, for otherwise, M^n will be of the diffeomorphic type of RP^n and hence contradict the assumption $\pi_1(M^n) = 1$. Therefore, there exists a unique function $f(r)$ such that $\sum(p, r)$ is isometric to $S^{n-1}(f(r))$. On the other hand, it follows immediately from the uniqueness theorem (cf. §2) that M^n is completely determined by the above function, $f(r)$, up to $O(n)$ -isometry.

Since $O(n)$ acts transitively over the space of orthonormal two-frames at p , it is clear that the sectional curvatures of M^n at p is independent of the choice of sectional-planes. If M^n is a reflectional symmetric space, then it is easy to show that M^n is automatically homogeneous. Hence, the sectional curvatures are also independent of the point $p \in M^n$, namely, M^n is of constant sectional curvature, say $K = K(M^n)$.

Conversely, suppose $(O(n), M^n)$, is simply connected, complete and of constant sectional curvature equal to K . Then it follows directly from (ii) of theorem 5.5. that the function $f(r)$ satisfies the following ODE.

$$f'' + Kf = 0 \tag{19}$$

and the obvious initial condition $f(0) = 0, f'(0) = 1$. Hence

$$f(r) = \begin{cases} r & \text{if } K = 0, \\ \frac{1}{\sqrt{K}} \sin \sqrt{K}r & \text{if } K > 0 \\ \frac{1}{\sqrt{-K}} \sinh \sqrt{-K}r & \text{if } K < 0 \end{cases}$$

which uniquely determines M^n as the Euclidean, spherical or hyperbolic space respectively.

Remark. To each positive C^2 -function $f(r), 0 \leq r \leq b$ (resp. $0 \leq r \leq \infty$) satisfying the boundary conditions, namely,

$$f(0) = f(b) = 0, \quad f'(0) = f'(b) = 1$$

(resp. $f(0) = 0, f'(0) = 1$ but $f(r)$ does not $\rightarrow 0$ as $r \rightarrow \infty$) there exists a unique simply connected, complete $O(n)$ -Riemannian manifold, say $M^n(f)$ such that the above given function, $f(r)$, is exactly its characteristic function of orbit sizes (cf Proposition 5.1).

Proposition 5.2. (Unified laws of sine and cosine) *Let $M^n(f)$ be the above $O(n)$ -Riemannian manifold with $f(r)$ as its characteristic function of orbit sizes and $\triangle OAB$ be a (geodesic) triangle in $M^n(f)$ with the vertex O fixed under $O(n)$. Then*

$$\begin{cases} (i) & f(\overline{OA}) \cdot \sin A = f(\overline{OB}) \cdot \sin B \\ (ii) & \overline{AB} = \int_{\overline{AB}}^{\overline{OB}} \frac{f(r)dr}{\sqrt{f^2(r) - f^2(\overline{OA}) \cdot \sin^2 A}} \end{cases} \tag{20}$$

where $\overline{OA}, \overline{OB}, \overline{AB}$ are respectively the lengths of the three sides of $\triangle OAB$.

Corollary. *In the special case that $M^n(f)$ is of constant sectional curvature K , then $f(r)$ is given by (18) and the above formulae (20) reduce to the well-known laws of sine and cosine in the Euclidean, spherical and hyperbolic geometries respectively.*

Proof. By assumption, $I(M^n(f), 0) \simeq O(n) \simeq O(T_0M^n(f))$. Let $O(n - 2)$ be the subgroup of $O(T_0M^n(f))$ fixing both the directions \overrightarrow{OA} and \overrightarrow{OB} . Then, it is easy to see that the fixed point set, $F(O(n - 2), M^n(f)) \simeq M^2(f)$, which is a totally geodesic submanifold of $M^n(f)$ containing $\triangle OAB$. Therefore one may reduce the proof of (20)-(i), (ii) to the special case of $n = 2$.

For $M^2(f)$, it is convenient to parameterize by polar coordinates (r, θ) . Then, $ds^2 = dr^2 + f^2(r)d\theta^2$, and straightforward computation will show that geodesics are characterized by the following ODE:

$$f(r) \frac{d\alpha}{ds} \cos \alpha + f'(r) \sin \alpha = 0 \quad (\cos \alpha \neq 0) \tag{21}$$

where α is the angle between $\frac{\partial}{\partial r}$ and the tangential direction. Hence, geodesics are characterized (unless $\alpha = \pi/2$) by the following first integral of (21), namely,

$$f(r) \cdot \sin \alpha = \text{constant}, \tag{22}$$

[i.e., $f(r(A)) \sin \alpha(A) = f(r(B)) \sin \alpha(B)$ for two arbitrary points A, B on a geodesic curve].

It follows from (22) that

$$\begin{aligned} \frac{dr}{ds} &= \cos \alpha = \frac{\sqrt{f^2(r) - c^2}}{f(r)}, \quad c = f(\overline{OA}) \cdot \sin \alpha(A) \\ \overline{AB} &= \int_A^B ds = \int_{r(A)}^{r(B)} \frac{f(r)dr}{\sqrt{f^2(r) - c^2}} \end{aligned}$$

5.3. Two point homogeneous spaces and rotational manifolds of spherical type

The Euclidean, spherical and hyperbolic geometries are exactly those Riemannian spaces which satisfy the two *axioms of congruences*: namely, (i) two geodesic intervals \overline{AB} and $\overline{A'B'}$ are congruent if they have the same length, and (ii) two geodesic triangles $\triangle ABC$ and $\triangle A'B'C'$ are congruent if they have respective equalities of one angle and its two adjacent sides. Naturally, it is interesting to study the family of slightly more general Riemannian spaces satisfying only the first axiom of congruence. Such Riemannian spaces are called two point homogeneous spaces. They were classified by H.C. Wang [51], and it turns out that they are exactly those symmetric spaces of rank one. From the viewpoint of transformation groups, the first axiom of congruences amounts to asserting that the space M is homogeneous and $I(M, p)$ acts transitively on the unit sphere of $T_p M$. [The second axiom is equivalent to the assertion that $I(M, p)$ acts transitively on the orthonormal two frames of $T_p M$]. A rotational manifold (G, M^n) is said to be of *spherical type* if the principal G -orbits are of the diffeomorphic type of S^{n-1} . Recall that *homogeneous spaces*, G/H , of *spherical type* were classified by Montgomery-Samelson [41] and A. Borel [9]. From the viewpoint of Riemannian geometry, there are the following four types of spherical, homogeneous Riemannian manifolds:

- (i) Real type: $O(n)/O(n-1)$ (or $SO(n)/SO(n-1)$) and the special cases $G_2/SU(3)$, $Spin(7)/G_2$.
- (ii) Complex type: $U(n)/U(n-1)$ (or $SU(n)/SU(n-1)$).
- (iii) Quaternion type: $Sp(n) \times Sp(1)/Sp(n-1) \times Sp(1)$, $n \geq 2$.
[partial quaternion type: $Sp(n) \times A/Sp(n-1) \times A$, $A \subset Sp(1)$]
- (iv) Cayley type: $S^{15} = Spin(9)/Spin(7)$.

Correspondingly, the isotropy representations of H on the tangent space of G/H at the base point are respectively the following:

- (i) an irreducible real representation,
- (ii) $(U(n-1), \mathbb{C}^{n-1})_{\mathbb{R}} + 1$ (trivial representation),
- (iii) $(Sp(n-1), \mathbb{H}^{n-1})_{\mathbb{R}} + (SO(3), \mathbb{R}^3)$,
- (iv) $(Spin(7), \mathbb{R}^8) + (SO(7), \mathbb{R}^7)$.

Therefore, the geometry of a rotational manifold of spherical type is uniquely determined by *two characteristic functions of orbit sizes*, except the case of real type which has only one such function and the case of partial quaternion type with reducible $\rho_3|_A$ which has more than two such functions.

Since rank one symmetric spaces are geometrically characterized by a single congruence - axiom of intervals it is rather natural to study the following basic problems:

Problem 5.1. *What are the minimal conditions for the congruence of two geodisical triangles $\triangle ABC$ and $\triangle A'B'C'$ in a given rank one symmetric space $M = G/K$?*

Problem 5.2. *What are the laws of trigonometry in M . The first problem can be reformulated for an arbitrary homogeneous Riemannian manifold $M = G/H$ and in terms of Lie-theoretical invariants, namely,*

Problem 5.1.' *Let $M = G/H$ be a given, homogeneous Riemannian manifold and (H, \mathcal{P}) be the isotropy representation. What is a complete set of geometric (resp. algebraic) invariants of (H, \mathcal{P}) ? What is a complete set of geometric (resp. algebraic) invariants of $(H, \mathcal{P} \oplus \mathcal{P})$?*

In view of the unified treatment of the laws of sine and cosine of the classical geometries in the framework of rotational manifolds (cf. Proposition 5.2., §5.2), it is rather natural to reformulate Problem 5.2. in terms of the setting of the rotational manifolds as follows:

Let (G, M) be a generalized rotational manifold with G/H as the principal orbit type. Let l be the dimension of H -invariant quadratic forms of (H, \mathcal{P}) and r the arc length parameter of M/G . Then the geometry of M is completely determined by l characteristic functions, say $\{f_1(r), \dots, f_l(r)\}$, of orbit sizes. Let $\gamma = \{\gamma(s)\}$ be a geodesic curve in M parameterized by its arc length s , and $r(s) = \pi(\gamma(s))$, $\mathfrak{D}(r) = \pi^{-1}(r)$, $\pi : M \rightarrow M/G$. Let $\underline{\pi}(\gamma, s)$ be the component of $\dot{\gamma}(s)$ in the tangential direction of $\mathfrak{D}(r(s))$ and $\hat{\pi}(\gamma, s)$ be its image in \mathcal{P}/H , namely

$$\begin{array}{ccc} \underline{\pi}(\gamma, s) & \xrightarrow{\hspace{10em}} & \hat{\pi}(\gamma, s) \\ \uparrow \text{fl} & & \uparrow \text{fl} \\ T(\mathfrak{D}(r(s))) \simeq T(G/H) & \rightarrow & T(G/H)/G \simeq \mathcal{P}/H \end{array}$$

It is clear that, to each geodesic curve γ , $\hat{\pi}(\gamma, s)$ defines a graph in the space of $(M_0/G) \times (\mathcal{P}/H)$ and moreover, two geodesic curves γ_1 and γ_2 are G -equivalent if and only if $\hat{\pi}(\gamma_1, s)$ and $\hat{\pi}(\gamma_2, s)$ are the same graph.

Problem 5.2.' *How to describe the above graphs, $\hat{\pi}(\gamma)$ in terms of the characteristic functions, $\{f_1(r), \dots, f_l(r)\}$ of orbit sizes of (G, M) ?*

In order to study the above Problem 5.2.', the following theorem of Clairaut-Noether is conveniently useful:

Theorem 5.6. (Clairaut, Noether [4]) *Let γ be a geodesic curve and X a Killing vector field on M . Then*

$$\langle X, \dot{\gamma}(x) \rangle = \text{constant} \tag{23}$$

along γ , where $\dot{\gamma}(s)$ is the unit tangent vector of γ .

Proof. $\frac{d}{ds}\langle X, \dot{\gamma}(s) \rangle = \langle \nabla_{\dot{\gamma}} X, \dot{\gamma} \rangle + \langle X, \nabla_{\dot{\gamma}} \dot{\gamma} \rangle$, where $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ because γ is a geodesic curve.

Let $\phi(t)$ be the 1-parameter subgroup of isometries generated by X and $\gamma_t = \phi(t) \cdot \gamma$, $\dot{\gamma}_t = \frac{\partial}{\partial s} \gamma(t, s)$. It is easy to see that

$$2\langle \nabla_X \dot{\gamma}_t, \dot{\gamma}_t \rangle = X \langle \dot{\gamma}_t, \dot{\gamma}_t \rangle = 0 \quad \text{and} \quad [X, \dot{\gamma}_t] = 0$$

Hence

$$\langle \nabla_{\dot{\gamma}_t} X, \dot{\gamma}_t \rangle = \langle \nabla_X \dot{\gamma}_t, \dot{\gamma}_t \rangle = 0, \quad \text{and} \quad \frac{d}{ds} \langle X, \dot{\gamma} \rangle = 0$$

We refer to [29] for a systematic treatment of the trigonometries on symmetric spaces of rank one.

5.4. A problem of H. Hopf and the problem of generalized rotational manifolds of strictly positive sectional curvatures

The following is a well-known problem posed by H. Hopf.

Problem 5.3. *Does the manifold $S^2 \times S^2$ admit a Riemannian metric with strictly positive sectional curvatures?*

It is rather natural to generalize the above problem to a setting of symmetric spaces, namely

Problem 5.3'. *Let M be of the diffeomorphic type of a simply connected, compact, symmetric space of rank 2. Does M admit a Riemannian metric with strictly positive sectional curvatures?*

Remarks.

- (i) The set of simply connected, compact, symmetric spaces consists of a family of nice Riemannian manifolds which are natural generalizations of the space of spherical geometry. However, their sectional curvatures are, in general, only non-negative and are strictly positive for all plane sections only when the rank of M is one.
- (ii) If $M = G/K$ is of rank two, then the set of 2-planes in $T(G/K)$ with zero sectional curvature consists of only a single G -orbit. Therefore, it is rather natural to ask whether it is possible to equip M with some non-homogeneous Riemannian metrics whose sectional curvatures are *strictly positive*.
- (iii) Technically, the study of sectional curvatures of a *general* Riemannian structure on a *given* high dimensional manifold is a rather unmanageable task. A simple-minded approach to test the above problem within a technically still manageable range is to study the following modified problems:

Problem 5.3." *Let M be of the diffeomorphic type of a simply connected, compact, symmetric space of rank 2. Does M admit a Riemannian structure of low cohomogeneity (resp. generalized rotational type) with strictly positive sectional curvatures?*

Problem 5.4. *Classify the diffeomorphism types of all those manifolds which admit some Riemannian structures of generalized rotational type with strictly positive sectional curvatures.*

5.5. Generalized rotational Riemannian manifolds of the diffeomorphism type of Kervaire sphere

In the study of differentiable transformation groups on exotic spheres, one finds that the exotic spheres of Kervaire type are the most symmetric among all exotic spheres of the same dimension and they are also the only kind of exotic spheres which admit compact group actions of cohomogeneity one [20]. Geometrically, a Riemannian metric is a kind of super-structure that builds on top of a differentiable structure. Therefore, it is rather natural to have a comparison study between *the geometries* of various *Riemannian structures on exotic spheres* and that of the standard sphere. More specifically, one may have a comparison study between the geometries of those almost homogeneous Riemannian structures on a Kervaire sphere and that of the standard sphere. We shall discuss various specific problems on the geometries of almost homogeneous metrics on a Kervaire sphere in a succeeding paper.

6. Preliminary examples of problems and applications II: Generalized rotational hypersurfaces

Roughly speaking, in intrinsic geometry, one studies the existence and uniqueness of certain types of geometric models or the logical relationship among various properties of space models; in extrinsic geometry, one studies the existence and uniqueness of certain types of subspaces N in a given space M or the logical relationships among various properties of the pair (M, N) . Therefore, in the study of extrinsic geometry, one *begins* with the *selection of the ambient space* where all the subspaces of the investigation shall live. Of course, it is quite natural that one is mainly interested in the case that the ambient space is nice and simple, e.g., Euclidean spaces, spherical or hyperbolic spaces or symmetric spaces, etc. In fact, most of the interesting examples of ambient spaces are highly symmetric. Hence, it is not surprising that the techniques of equivariant differential geometry turns out to be quite useful in the study of global differential geometry of submanifolds in such highly symmetric ambient spaces. We shall only briefly discuss a few simple basic examples of such applications in this section.

6.1. Generalized rotational hypersurfaces in a given highly symmetric Riemannian manifold

Let M be a given highly symmetric Riemannian manifold and N be a hypersurface in M , $G = I(M, N)$. N is called a *generalized rotational hypersurface* of M if $\dim N/G = 1$. Similar to the role played by the generalized rotational manifolds in the study of intrinsic differential geometry, the generalized rotational hypersurfaces constitutes a rich family of submanifolds (of codimension 1) which are readily accessible to the technique of equivariant differential geometry. Therefore, in the study of many basic problems of global differential geometry of submanifolds such as the spherical Bernstein problem, the problem of soap bubbles etc., it is rather natural to first test them in the realm of generalized rotational hypersurfaces.

For a given highly symmetric Riemannian manifold M (e.g., Euclidean space, spheres, etc.), generalized rotational hypersurfaces in M are, by definition, invariant hypersurfaces of a certain isometric transformation group of *cohomogeneity 2*. Hence, in order to test some given geometric problems in the realm of generalized rotational hypersurfaces of M , the first technical preparation is to classify isometric transformation groups, (G, M) , of cohomogeneity 2, and then, to compute the orbital geometry of such a transformation group (G, M) . Based on the above classification and computation it is then not difficult to apply the fundamental equations of §4 to write down the reduced differential equation of a given problem at the level of the orbit space, M/G . In fact, only at this stage, is it then possible to have a realistic assessment of the geometric feasibility and the degree of difficulty of the analytical task of the equivariant geometric problem.

6.2. Generalizations of a theorem of Delaunay

In the study of global differential geometry of submanifolds of a given Riemannian space M , the basic objects of interest are usually those *complete* submanifolds satisfying some simple local conditions, e.g., the minimal submanifolds, the hypersurfaces of constant mean curvature or of constant scalar curvature in E^n , S^n or H^n . Technically speaking, the local conditions are usually given by some differential equations and one studies the existence, the uniqueness and the geometric behavior of “global solutions” of such geometric differential equations. However, non-trivial global solutions of such geometric differential equations are, in general, rather difficult to construct. For examples, when Lagrange first derived the equation of minimal surfaces in E^3 , the only global solution he knew was the trivial one, namely, a plane; and it was not until 1969 [8] that a non-trivial, entire solution of the minimal equation was finally constructed by Bombieri-De Giorgi-Giusti, thus settling the problem of Bernstein. Historically, many examples of non-trivial global solutions were firstly discovered by suitably exploiting the rich symmetries of the ambient spaces. The following theorem of Delaunay is a typical example of imposing a suitable symmetry condition to facilitate the construction of some examples.

Delaunay Theorem [17]. Let N^2 be a rotational surface of constant mean curvature in E^3 . Then the generating curve of N^2 is the locus of a foci by rolling a suitable conic along the axis of rotation.

We would like to point out that there are various natural directions to generalize the above beautiful result. The following are some obvious but rather interesting directions that one might try to explore:

- (i) One may generalize the ambient space, E^3 , to other highly symmetric spaces such as high dimensional Euclidean spaces, E^n , or S^n , H^n , symmetric spaces etc.
- (ii) The rotation group $(O(2), E^3)$ can be generalized to isometric transformation groups (G, M) of cohomogeneity 2.
- (iii) The uniform local condition of constant mean curvature can be generalized to a suitable W -condition, namely, a uniform functional relationship, say $\phi(k_1, \dots, k_{n-1}) = 0$, among the principal curvatures of a given hypersurface.

We refer to [26, 33, 30, 28] for some of the simplest generalizations of Delaunay theorem in the simple case of invariant hypersurfaces of $(O(n-1), E^n)$ or $(O(n-1), M^n(c))$ where $M^n(c)$ is the simply connected Riemannian n -space of constant sectional curvature c .

6.3. The spherical Bernstein problem and the problem of generalized equator in compact symmetric spaces

One of the simplest and also the most intensively studied non-linear, geometric PDE is the minimal equation. It is of the following form in the simplest case of minimal graph in \mathbb{R}^n , namely,

$$\sum_{k=1}^{n-1} D_k \frac{D_k u}{\sqrt{1 + |Du|^2}} = 0 \quad (24)$$

The classical Bernstein problem asks whether an *entire* solution, $u \in C^2(\mathbb{R}^{n-1})$, of (24) is necessarily a *linear* function. It was proved to be affirmative for the dimensions $n \leq 8$ and settled to be negative for $n \geq 9$ by the successive efforts of Bernstein [7]. De Giorgi [16], Almgren [3], Simons [49] and Bombieri-De Giorgi-Giusti [8]. Following that, Chern [14] proposed the following spherical Bernstein problem:

Problem 5.5. Let \sum^{n-1} be a minimal imbedding of the differential $(n-1)$ -sphere into $S^n(1)$. Is it necessarily an equator?

The beginning case of $n = 3$ was proved by Almgren [3] and Calabi [10] before the above problem was proposed. In two recent papers of the third author [27], *infinitely many distinct* (i.e., non-congruent) examples of *minimal imbedding* of S^{n-1} into $S^n(1)$ were constructed for the dimensions $n = 4, 5, 6, 7, 8, 10, 12$ and 14 by studying minimal hypersurfaces of generalized rotational type in $S^n(1)$. Therefore, it seems natural turn around and propose the following problems:

Problem 5.5.' *Are there infinitely many, distinct, minimal imbeddings of S^{n-1} into $S^n(1)$ for all $n \geq 4$?*

[It was conjectured in [27] that the answer of Problem 5.5.' should be affirmative for all $n \geq 4$].

Problem 5.5." *Are there non-equatorial, minimal, imbeddings of S^{n-1} into $S^n(1)$, for sufficiently large dimensions n , whose cones are stable?*

Since simply-connected, compact, symmetric spaces are natural generalizations of the spherical spaces, it is interesting to extend the study of spherical geometry to a broader spectrum of geometry of compact symmetric spaces. For example, it is rather natural to generalize Problem 5.5. as follows:

Problem 5.6. *Let M^n be a given simply-connected, compact, symmetric space, $n \geq 4$. Are there infinitely many distinct minimal imbeddings of S^{n-1} into M^n ?*

In fact, in the cases that $rk(M^n) \geq 2$, even the existence of a single minimal imbedding of S^{n-1} into M^n is already a non-trivial problem. The answer of problem 5.6. was proved to be affirmative at least in the following cases, namely,

- (i) $S^2 \times S^2$, $S^3 \times S^3$, $SU(3)/SO(3)$, $SU(3)$, (Wu-teh Hsiang and Wu-yi Hsiang [23]),
- (ii) CP^k , $k \geq 2$, (Wu-teh Hsiang – Wu-yi Hsiang – Per Tomter [24]).

Again, the above results were proved by studying minimal hypersurfaces of generalized rotational type. One might consider those minimal imbeddings of S^{n-1} into M^n as generalizations of the equator hypersurface in M^n . It was conjectured that the answer of Problem 5.6. should be affirmative for all simply-connected compact symmetric spaces of dimensions ≥ 4 .

6.4. The problem of soap bubbles

It follows from the variational principle that a free soap bubble *in equilibrium* must be a closed surface of *constant mean curvature*. Hence, it is natural to define *soap bubbles* in a given Riemannian space, M , to be those *closed hypersurfaces of constant mean curvature*. They are simple, basic objects of interest in the study of global differential geometry of submanifolds of M . The study of soap bubbles in E^n has a long and interesting history [cf. 18]. On the uniqueness side of soap bubbles in E^n , there are the following theorems which characterize the round sphere as the only soap bubble satisfying some *additional conditions*.

- (i) Liebmann (1900, [39]): a strictly convex soap bubble in E^3 must be a round sphere.
- (ii) H. Hopf (1951, [18]): A soap bubble of genus zero in E^3 must be a round sphere.
- (iii) A.D. Alexandrov (1958, [1]): An imbedded soap bubble in E^n must be a round sphere.

New examples of spherical soap bubbles of generalized rotational types were constructed by Hsiang-Teng-Yu [32] and Hsiang [25], again by the method of equivariant differential geometry.

6.5. Concluding remarks

(i) The new results mentioned in 6.2, 6.3, and 6.4 are only preliminary examples of applications of equivariant method. They are also initial indications of the richness of the seemingly rather special family of generalized rotational hypersurfaces in E^n , S^n , H^n or symmetric spaces. In fact, the basic geometric properties of generalized rotational hypersurfaces as well as their significance in the general context of global geometry of submanifolds are largely unexplored.

(ii) Of course, the construction of non-trivial, basic objects such as minimal spheres or soap bubbles by exploiting the rich symmetries of the ambient space is only the first step. The next logical step will, then, be to understand the geometric properties of such new examples so that one can also characterize them in some similar ways as one did for the round spheres. For examples, one may ask:

Problem 5.7. *Is an $O(n-2)$ -invariant soap bubble in E^n automatically $O(2) \times O(n-2)$ -invariant?*

Problem 5.7.' *Is an $O(n-2)$ -equivariant minimal imbedding of S^{n-1} into $S^n(1)$ necessarily also $O(2) \times O(n-2)$ -equivariant?*

Problem 5.8. *Is a minimal imbedding of S^3 into $S^4(1)$ necessarily $O(2) \times O(2)$ -equivariant?*

(iii) The theorem of Delaunay and the theorem of A.D. Alexandrov are two typical results of equivariant differential geometry of submanifolds. Roughly speaking, one first exploits the rich symmetries of the ambient space to *discover* basic objects of interest, and then, one examines the geometric properties of the new objects in order to characterize them by proving symmetry theorems of Alexandrov type.

Appendix I. Infinitesimal Computations and Existence

In section §2.5 we showed that the three basic invariants of orbital distance metric, \overline{G} connection, and matrix valued covariant function uniquely determine a Riemannian G -manifold. This appendix has two purposes. First we wish to point out that the question (even along singular orbits) of whether such a set of invariants actually come from a metric may be reduced to a problem in the slice. Secondly, we call attention to the natural infinitesimal method for understanding jets of equivariant metrics at points with a non-principal isotropy group.

Our basic tool is the slice theorem describing a tubular neighborhood of any orbit $G(x)$. The slice S at x is the intersection of an open neighborhood of 0 in $T_x M$ with

the set of normal vectors to the orbit $G(x)$. Then using the exponential map, a tubular neighborhood of $G(X)$ is identified (up to equivariant diffeomorphism) with the bundle $G \times_K S \rightarrow G/K$ where K is the isotropy group at x . The action of K on S is orthogonal and is called the slice representation at x . The projection $G \times_K S \rightarrow G/K$ is just a local version of the orbit map. Note that principal orbits are exactly the places where the slice representation is trivial.

If A is a smooth K -manifold and Y is a smooth G -manifold with K a subgroup of G , it is immediate that any smooth K -equivariant map $\theta : A \rightarrow Y$ has a smooth extension to a G -map $\tilde{\theta} : G \times_K A \rightarrow Y$. Using this, it is easy to see:

Proposition I.1. *Let S be the slice at a point x with isotropy group K . Then any K -invariant smooth inner product defined on $T_y M \forall y \in S$ has a (unique) smooth extension to a G -invariant Riemannian metric on a neighborhood of $G(x)$.*

So to check the smoothness of the metric determined by our three basic geometric invariants, we need only check smoothness at points of a slice. As a consequence, in the special case when M has only one orbit type (i.e., $M_S = \emptyset$), we get the following simple existence theorem.

Proposition I.2. *Suppose M is a given G -manifold with all orbits of the same type G/H . Then an arbitrary set of geometric invariants of types (i), (ii) and (iii) can be realized by a suitable G -Riemannian structure on M . In fact, any smooth choice of Ad - H invariant inner product on \mathfrak{p} at points of S has a smooth extension to an $N(H)$ covariant function h of type (iii) defined on the open set $\overline{G}(S)$ of \overline{M} .*

Of course the above proposition always applies to the regular part M of any G -Riemannian manifold M . At points of M_S , the actual application of Proposition I.1. depends on the understanding of smooth K -equivariant maps between linear representations.

Now suppose $x \in M_S$ has connected isotropy group K and slice S . Let $x \in M$, so $K \supset H$. By choosing an orthonormal basis $\{Q_i : i = 1, \dots, k\}$ of S , we can define linear coordinates u_i on S by $w = \sum u_i Q_i$ for $w \in S$. We shall also use Q_i to denote the vector fields $\frac{\partial}{\partial u^i}$ on S . The union of this set of vector fields with the set of Killing fields X_i^* induced by elements $X_i \in \mathfrak{p}$ span TM along S . If T is an element of the Lie algebra \mathfrak{k} of K , then the linear actions of K on S and \mathfrak{g} give us

$$[T^*, Q_i] = \sum \alpha_i^j(T) Q_j$$

$$[T, X_k] = \sum \beta_k^j(T) X_j + Z_k(T)$$

where the $\alpha_i^j(T)$ and $\beta_k^j(T)$ are constants and $Z_k(T) \in \mathfrak{h}$. Let w_a denote either of the vector fields X_i^* or Q_i , and $g_{ij} = \langle w_i, w_j \rangle$. Then the above formulae may be summarized by

$$[T^*, w_a] = \sum \gamma_a^b(T) w_b$$

(Note $[T, X_k]^* = -[T^*, X_k^*]$ and $Z_k^*(T) = 0$.)

At the origin x of S ($u_i = 0$), $T^* = 0$. But then

$$\begin{aligned} 0 &= T^*Q_i\langle w_a, w_b \rangle \\ &= Q_iT^*\langle w_a, w_b \rangle + [T^*, Q_i]\langle w_a, w_b \rangle \\ &= Q_i(\langle [T^*, w_a], w_b \rangle + \langle w_a, [T^*, w_b] \rangle) + [T^*, Q_i]\langle w_a, w_b \rangle \quad \text{at } x \end{aligned}$$

or

$$\sum_m \gamma_a^m(T)Q_i g_{mb} + \gamma_b^m(T)Q_i g_{am} + \sum_j \alpha_i^j(T)Q_j g_{ab} = 0 \quad \text{at } x$$

For elements T of \mathfrak{k} not in \mathfrak{h} , these are in general nontrivial conditions on the first derivative, of the g_{ab} at the origin, and consequently conditions on the derivatives of our geometric invariants (i), (ii) and (iii).

Using the usual multi-index notation $I = (i_1, \dots, i_r)$, let $Q_I = Q_{i_1}, Q_{i_2}, \dots, Q_{i_r}$ denote the associated r 'th order differential operator. Then we obtain similar higher order conditions

$$0 = \sum_m \gamma_a^m(T)D_I g_{mb} + \gamma_b^m(T)D_I g_{am} + \sum_{p=1}^r \sum_{q=1}^k \alpha_{i_p}^q(T)D_J g_{ab} \quad \text{at } x \quad (*)$$

where J is the multi-index of length r obtained from I by replacing i_p by q . These conditions are sufficient in the following sense:

Theorem I.1. *If K is connected, then any polynomial (analytic) functions in the u_i satisfying the conditions (*) for all elements T of \mathfrak{k} determine, a G -invariant (smooth) Riemannian metric in a neighborhood of x .*

Proof. By proposition I.2., we need only show that the functions g_{ab} on S determine a K -invariant quadratic form. Since K is connected, this will be the case if and only if

$$T^*\langle w_a, w_b \rangle = \langle [T^*, w_a], w_b \rangle + \langle w_a, [T^*, w_b] \rangle$$

or

$$T^*g_{ab} = \sum_p \gamma_a^p(T)g_{pb} + \gamma_b^p(T)g_{pa} \quad \forall T \in \mathfrak{k}.$$

The condition $[T^*, Q_i] = \sum_j \alpha_i^j(T)Q_j$ implies that $T^*(u^i) = \sum_j \alpha_j^i(T)u^j$. Thus for analytic functions, the invariance condition on the g_{ab} separates out into a homogeneous condition on each degree. Using

$$D_I T^* = T^*D_I - \sum_{p=1}^r \sum_{q=1}^k \alpha_{i_p}^q(T)D_J$$

and realizing that $D_I g_{ab}$ at 0 is just a constant times the Taylor coefficient of $x^{i_1} x^{i_2} \dots x^{i_k}$ in g_{ab} , we see that the conditions (*) are just the requirement that the Taylor series of T^*g_{ab} and $\sum_p \gamma_a^p(T)g_{pb} + \sum_b \gamma_b^p(T)g_{pa}$ agree.

Of course the condition (*) in the theorem need only be checked on elements of a basis of \mathfrak{k} . And if $T \in \mathfrak{h}$, then they follow automatically from the \overline{G} covariance of the function h . When K is not connected, this theorem may be used to construct smooth extensions invariant under the connected component of K , and extra discrete conditions may be added if necessary.

References

- [1] A. D. Alexandrov, *Uniqueness theorems for surfaces in the large V*, Vestnik Leningrad University **13** (1958), No. 19, 5–8.
- [2] A.D. Alexandrov, *A characteristic property of spheres*, Annali di mat pura et appl. **58** (1962), 303–315.
- [3] F.J. Almgren Jr., *Some interior regularity theorem for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. **85** (1966), 277–292.
- [4] V.I. Arnol'd, *Mathematical methods of classical mechanics*, GTM, **60**, Springer-Verlag, 1989.
- [5] A. Back and W.Y. Hsiang, *Equivariant geometry and Kervaire spheres*, Trans. Amer. Math. Soc. **304** (1987), no. 1, 207–227.
- [6] J.L. Barbosa and M. do Carom, *Hopf's conjecture for stable immersed surfaces*, An. Acad. Brasil. Ciênc. **55** (1983), no. 1, 15–17.
- [7] S. Bernstein, *Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles du type elliptique*, Com. Inst. Sci. Math. Mech. Univ. Kharkov **15** (1915–17), 38–45.
- [8] E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.
- [9] A. Borel, *Le plan projectif de octaves et les sphères comme espaces homogènes*, Comptes Rendue de l'Académie des Sciences, Paris **230** (1960), 1378–1383.
- [10] E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Diff. Geometry **1** (1967), 111–126.
- [11] É. Cartan, *Sur une classe remarquable despaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264 and **55** (1927), 114–134.
- [12] É. Cartan, *La géométrie des groupes de transformations*, J. Math. Pures Appl. **6** (1927), 1–119.
- [13] S.Y. Cheng and S.T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Annalen **225** (1977), 195–204.
- [14] S.S. Chern, *Differential geometry, its past and its future*, Actes, Cong. Intern. Math. (1970), Tome 1, 41–53.
- [15] G. Darboux, *Lecons sur la théorie générale des surfaces*, Tome 3, Ch. VII-X, 1894 Gauthier-Villars, Paris.
- [16] E. De Giorgi, *Una estensione del teoreme di Bernstein*, Ann. Sc. Norm Sup. Pisa **19** (1965), 79–85.
- [17] C. Delaunay, *Sur la surface de revolution dont la courbure mayenne est constante*, J. Math. Pures & Appl. **6** (1841), 309–320.
- [18] H. Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Math., Nachr. **4** (1951), 232–249.

- [19] H. Hopf, *Zur Differential geometrie geschlossener Flächen in Euklidischen Raum*. Conv. Int. Geometria Diff. Italia 1953, 45–54, Ed. Cremonese, Roma, (1954).
- [20] W.C. Hsiang and W.Y. Hsiang, *On compact subgroups of diffeomorphism groups of Kervaire spheres*, Ann. of Math., **85** (1967), 359–369.
- [21] W.T. Hsiang and W.Y. Hsiang, *Examples of codimension one closed minimal submanifolds in some symmetric spaces I, On the existence of codimension-one minimal spheres in compact symmetric spaces of rank 2. II.*, J. of Diff. Geometry **15** (1980), no. 4, 543–551 (1981), J. Differential Geom. **17** (1982), no. 4, 583–594 (1983).
- [22] W.T. Hsiang and W.Y. Hsiang, *On the construction of constant mean curvature imbeddings of exotic and/or knotted spheres into the unit sphere*, Invent. Math. **82** (1985), no. 3, 423–445.
- [23] W.T. Hsiang and W.Y. Hsiang, *An infinite family of minimal imbeddings of S^{2n-1} into $S^n(1) \times S^n(1)$, $n = 2, 3$* , (mimeo at Univ. of Calif., Berkeley, CA, (1982)).
- [24] W.T. Hsiang, W.Y. Hsiang and P. Tomter, *On the construction of infinitely many, mutually non-congruent, examples of minimal imbeddings S^{2n-1} into CP^n , $n \geq 2$* , Bull. Amer. Math. Soc. (N.S.) **8** (1983), no. 3, 463–465.
- [25] W.Y. Hsiang, *Generalized rotational hypersurfaces of constant mean curvature in the Euclidean spaces, I*, J. of Diff. Geometry **17** (1982), 337–356.
- [26] W.Y. Hsiang, *On generalization of theorems of A.D. Alexandrov and C. Delaunay on hypersurfaces of constant mean curvature*, Duke Math., J. **49** (1982), 485–496.
- [27] W.Y. Hsiang, *Minimal cones and spherical Bernstein problem I, II*, Ann. of Math. (2) **118** (1983), no. 1, 61–73, Invent. Math. **74** (1983), no. 3, 351–369, (Announced in Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 2, 377–379).
- [28] W.Y. Hsiang, *On rotational W -hypersurfaces in spaces of constant curvature and generalized laws of sine and cosine*, Bull. Inst. Math. Acad. Sinica **11** (1983), no. 3, 349–373.
- [29] W.Y. Hsiang, *On the laws of trigonometries of two-point homogeneous spaces*, Ann. Global Anal. Geom. **7**, (1989), 29–45.
- [30] W.Y. Hsiang, *On W -hypersurfaces of generalized rotational type in E^{n+1} , I.* (to appear in Math. Ann.)
- [31] W.Y. Hsiang and H.B. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. of Diff. Geometry **5** (1971), 1–38.
- [32] W.Y. Hsiang, Z.H. Teng and W. Yu, *New examples of constant mean curvature immersions of $(2k - 1)$ -spheres into euclidean $2k$ -space*, Ann. of Math. (2) **117** (1983), no. 3, 609–625.
- [33] W.Y. Hsiang and W. Yu, *A generalization of a theorem of Delaunay*, J. of Diff. Geometry **16** (1981), 161–177.
- [34] W. Killing, *Ueber die Clifford-Klein'schen Raumformen (German)*, Math. Ann. **39** (1991), no. 2, 257–278.
- [35] F. Klein, *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Math. Ann. **43** (1893), 63–100.
- [36] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol II, Interscience, (1969), New York.
- [37] H.B. Lawson, Jr., *The equivariant Plateau problem and interior regularity*, Trans. AMS **173** (1972), 231–249.
- [38] S. Lie and F. Engel, *Theorie der Transformations gruppen*, **3** volumes, B.G. Tenbner, Leipzig, 1888–1893.

- [39] H. Liebmann, *Über die Verbiegung der geschlossenen Flächen positiver Krümmung*, Math. Ann. **53** (1900), 81–112.
- [40] Luna and Richardson, *A Generalization of the Chevalley Restriction Theorem*, Duke Math. J. **46** (1979), 487–496.
- [41] D. Montgomery and H. Samelson, *Transformation groups of spheres*, Ann. of Math. (2) **44**, (1943), 454–470.
- [42] D. Montgomery and C.T. Yang, *The existence of a slice*, Ann. of Math., **65** (1957), 108–116.
- [43] S. Myers and N. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math., **40** (1939), 400–416.
- [44] K. Nomizu, *Invariant affine connections on homogeneous spaces*, Amer. J. Math. **76** (1954), 33–65.
- [45] B. O'Neill, *The fundamental equations of submersion*, Michigan Math. J., **13** (1966), 459–469.
- [46] R. S. Palais, *Global formulation of the Lie theory of transformation groups*, Mem. AMS #22, (1957).
- [47] B. Riemann, *Über die Hypothesen, welche der Geometrie zugrunde liegen*, Ges. Werke.
- [48] G. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68.
- [49] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. **88** (1968), 62–105.
- [50] H.C. Wang, *On invariant connections over a principal fibre bundle*, Nagoya Math. J. **13** (1958), 1–19.
- [51] H.C. Wang, *Two point homogeneous spaces*, Ann. of Math. **55** (1952), 177–191.
- [52] S.P. Wang and S.W. Wei, *Bernstein conjecture in hyperbolic geometry*, Seminar on minimal submanifolds, edited by E. Bombieri, 339–358, Ann. of Math. Stud., **103**, Princeton Univ. Press, Princeton, NJ, 1983.

Department of Mathematics, Cornell University, NY 14853, U.S.A.

IMPA, rue Luiz de Camoes 68, Rio de Janeiro, Brazil, 20060RJ.

Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A.