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Some Alternative Conformable Special Curves via the Conformable Derivative

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Abstract. In this study, some special curves defined on the alternative frame are investigated by means of the conformable derivative. The study consists of three chapters. The first chapter is devoted to the introduction. The second chapter is divided into three subsections. In the first subsection, basic definitions and fundamental concepts related to the alternative frame are presented. The second subsection is concerned with the basic definitions and concepts of the conformable derivative and integral. Finally, the third subsection includes known fundamental definitions and theorems concerning some special curves defined by means of the conformable derivative and integral. In the third chapter, these two structures are combined, forming the original part of the study. In the three-dimensional Euclidean space, some special curves are defined on the alternative frame constructed by means of the conformable derivative and integral, and their characterizations are obtained.

1 Introduction

Differential geometry has become one of the most active research areas of modern mathematics due to its wide range of applications in many scientific disciplines. The foundations of this field were laid in the 17th century with the development of calculus. Scientists such as Pierre de Fermat, René Descartes, and Isaac Newton introduced ideas that made it possible to analyze curves and surfaces analytically. In particular, the concepts of derivative and integral, independently developed by Newton and Leibniz, paved the way for the emergence of differential geometry.

In the 18th century, Leonhard Euler made significant contributions to the study of the curvature of curves and surfaces, thereby establishing the foundations of surface geometry. Euler's work on curvature later enabled Carl Friedrich Gauss to develop the concept known as *Gaussian curvature*. Gauss's seminal work, *Disquisitiones Generales Circa Superficies Curvas*, published in 1827, introduced the concept of intrinsic geometry,

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demonstrating that the geometry of a surface can be described solely by measurements taken on the surface itself. This approach made it possible to study geometry independently of the surface's position in the ambient space [3].

Bernhard Riemann, a student of Gauss, extended differential geometry to higher-dimensional spaces through his famous 1854 lecture entitled *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypotheses Which Lie at the Foundations of Geometry), thereby laying the groundwork for modern Riemannian geometry. This development represented a major revolution not only in mathematics but also in physics, particularly through its fundamental role in the theory of general relativity [10].

With the advent of the 20th century, differential geometry acquired a more abstract structure through its interaction with Lie groups, topology, the calculus of variations, and modern algebraic methods. Today, differential geometry is no longer limited to the classical study of curves and surfaces; it finds applications in a wide range of fields, including Einstein's general theory of relativity, modern theoretical physics, robotics, computer vision, and artificial intelligence.

Within the framework of classical differential geometry, the local geometric properties of space curves are generally described using the Serret–Frenet frame. This frame consists of the tangent t, normal n, and binormal b vectors defined along a curve and characterizes differential properties such as curvature and torsion. However, the Frenet frame is defined only for sufficiently differentiable and regular curves (that is, curves with nonvanishing curvature). This limitation has necessitated the development of alternative frames for certain special curves or under restricted differentiability conditions.

Motivated by this need, the concept of the *alternative frame* has been introduced. Alternative frame systems define an orthonormal frame along a curve in a manner similar to the classical Frenet frame; however, they are constructed based on different differential structures or distinctive geometric principles. Among these frames, one of the most prominent is the $\{n, c, w\}$ frame. In this frame, n represents the classical normal vector of the curve, c denotes a vector obtained from a combination of the tangent and normal vectors, and w is defined as a third vector orthogonal to both. This structure is often used to reflect special directional behaviors of curves or force directions in physical modeling [13]. Studies on alternative frames have increased significantly in recent years [1, 5, 6, 12]. Numerous works have been conducted on the characterization of special curves, the investigation of curve pairs, and their applications in conjunction with conformal and fractional derivative approaches. The greater flexibility and broader range of applications offered by these frames compared to the classical Frenet frame open new perspectives in geometry and physical modeling.

Fractional derivatives and integrals constitute a mathematical field known as *fractional calculus*, which arises from the generalization of classical derivative and integral concepts to rational (and even real or complex) orders. The origins of this field date back to the emergence of calculus itself. The idea of fractional differentiation was first proposed by Gottfried Wilhelm Leibniz in 1695. In a correspondence between Leibniz and the Marquis de l'Hôpital, the question of what a "half-order derivative" might mean was raised, remaining a topic of purely theoretical interest for many years [8]. During the 18th and 19th centuries, prominent mathematicians such as Joseph Fourier, Niels Henrik Abel, Joseph Liouville, and Augustin-Louis Cauchy proposed various definitions to make fractional derivatives and integrals more concrete and formal. In this context, the Riemann–Liouville fractional integral, developed by Liouville and Riemann, became a fundamental building block of fractional calculus [9]. The derivative operator introduced by Caputo in the 19th century has been widely adopted in applied sciences due to its compatibility with physically interpretable initial conditions. This definition has been extensively used in viscoelastic material models, diffusion processes, control theory, biological systems, and even financial mathematics [7].

In this study, In this study, two mathematical structures that have attracted considerable attention in recent years and have been extensively investigated in the field of geometry are examined within a unified framework. By employing the concepts of the conformable derivative and integral, several special curves are redefined on the alternative moving frame in the three-dimensional Euclidean space, and new characterizations of these curves are established. Furthermore, the use of the fractional derivative enables a detailed investigation of the behavior of these curves for different values of the parameter α. The resulting variations are demonstrated through illustrative examples and effectively visualized, highlighting the influence of the conformable approach on the geometric properties of the curves.

2 Preliminaries

2.1 Basic Definitions and Concepts Related to the Alternative Moving Frame

In this subsection, we recall the frame $\{n, c, w\}$, which is derived as an alternative to the Serret–Frenet frame in the three-dimensional Euclidean space, together with some special curve concepts defined by means of this frame.

In the three-dimensional Euclidean space, every unit-speed curve has at least four continuous derivatives, and by means of these derivatives, three orthogonal unit vector

fields are obtained along the curve. These vector fields are the tangent, principal normal, and binormal vector fields, denoted by t, n, and b, respectively. By using these vector fields, another orthonormal moving frame, called the alternative frame $\{n, c, w\}$, can be defined.

Definition 1. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed regular curve. An alternative frame $\{n, c, w\}$, which is different from the Serret–Frenet frame, is defined by

$$\begin{aligned} n(s) &= n(s), \\ c(s) &= \frac{n'(s)}{\|n'(s)\|}, \\ w(s) &= n \times c, \end{aligned}$$

where the vectors are called the principal normal vector, the derivative of the principal normal vector, and the Darboux vector, respectively. That is,

$$\begin{aligned} n(s) &= n(s), \\ c(s) &= \frac{-\kappa(s)\, t(s) + \tau(s)\, b(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}, \\ w(s) &= \frac{\tau(s)\, t(s) + \kappa(s)\, b(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}, \end{aligned}$$

or equivalently,

$$\begin{aligned} t(s) &= \frac{-\kappa(s)\, c(s) + \tau(s)\, w(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}, \\ n(s) &= n(s), \\ b(s) &= \frac{\tau(s)\, c(s) + \kappa(s)\, w(s)}{\sqrt{\kappa^2(s) + \tau^2(s)}}. \end{aligned}$$

Moreover, the relations $n \times c = w$, $c \times w = n$, and $w \times n = c$ hold among the elements of the frame. The derivative formulas of the alternative moving frame can be expressed in matrix form as

$$\begin{pmatrix} n'(s) \\ c'(s) \\ w'(s) \end{pmatrix} = \begin{pmatrix} 0 & f(s) & 0 \\ -f(s) & 0 & g(s) \\ 0 & -g(s) & 0 \end{pmatrix} \begin{pmatrix} n(s) \\ c(s) \\ w(s) \end{pmatrix}.$$

Here,

$$f(s) = \sqrt{\kappa^2(s) + \tau^2(s)} \quad \text{and} \quad g(s) = \frac{\kappa^2}{\kappa^2 + \tau^2} \left(\frac{\tau}{\kappa}\right)'$$

are the curvatures of the alternative moving frame [13].

Theorem 2.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed curve in the three-dimensional Euclidean space defined with respect to the alternative moving frame $\{n, c, w, f, g\}$. Then, the ratio*

$$\frac{g(s)}{f(s)} = \sigma$$

is constant if and only if the curve α is a slant helix [13].

Definition 2. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed curve defined in the three-dimensional Euclidean space with respect to the alternative moving frame $\{n, c, w, f, g\}$. If the vector field c of the curve α makes a constant angle $\phi \neq \frac{\pi}{2}$ with a fixed direction vector u along the curve, then α is called a *c-slant helix*. That is, along the curve α,

$$\langle c, u \rangle = \cos\phi, \quad \phi = \text{constant} \neq \frac{\pi}{2}. \tag{2.1}$$

(For more detailed information, see, [13]).

2.2 Local Conformable Derivatives and Integrals

In this subsection, the basic definitions and fundamental theorems of conformable fractional derivatives and integrals are presented. Throughout this section and the subsequent ones, the symbol α denotes the order of the conformable fractional derivative. Therefore, in order to avoid confusion, the curves taken from the space will be denoted by $x = x(s)$ in the sequel.

Let $f : [0, \infty) \to \mathbb{R}$ be an arbitrary function and let $t > 0$. Then, the classical derivative of f at the point t is defined as

$$\frac{df}{dt} = \lim_{\epsilon \to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon}.$$

Starting from this classical definition of the derivative, the question "Can one take a half-order derivative?" naturally leads to the concept of fractional derivatives.

For this purpose, let us first denote by D_α the fractional derivative operator of order α. For $\alpha = 1$, the operator D_1 satisfies the following properties:

1. $D_1(af + bg) = aD_1(f) + bD_1(g), \quad \forall\ a, b \in \mathbb{R},$
2. $D_1(t^p) = pt^{p-1}, \quad \forall\ p \in \mathbb{R},$
3. $D_1(fg) = fD_1(g) + gD_1(f),$
4. $D_1\left(\frac{f}{g}\right) = \frac{gD_1(f) - fD_1(g)}{g^2},$
5. $D_1(\lambda) = 0, \quad$ for any constant function $f(t) = \lambda.$

Now, we present the simplest and most useful definition of the fractional derivative for $\alpha \in (0,1)$. It should be noted that this definition can be generalized for any α. However, the case $\alpha \in (0,1)$ is the most important one, and once it is established, the remaining cases become simpler [11].

Definition 3. Let $f : [0,\infty) \to \mathbb{R}$ be a given function. For $\alpha \in (0,1)$ and $t > 0$, the conformable fractional derivative of order α of the function f is defined by

$$D_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon}. \tag{2.2}$$

If this limit exists, then the function f is said to be α-differentiable at the point t, and this derivative is sometimes denoted by $f^{(\alpha)}(t)$. In particular, if f is α-differentiable on the interval $(0,a)$ and the limit $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then it is defined as

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Moreover, this approach yields results consistent with the classical Riemann–Liouville and Caputo fractional derivatives, especially for polynomial functions (up to a constant multiplicative factor) [2].

Theorem 2.2. *If the function $f : [0,\infty) \to \mathbb{R}$ is α-differentiable at the point $t_0 > 0$, where $\alpha \in (0,1)$, then the function f is continuous at t_0 [2].*

Theorem 2.3. *Let $\alpha \in (0,1)$, and let the functions f and g be α-differentiable at the point $t > 0$. Then the following properties hold [2]:*

1. $D_\alpha(af + bg) = aD_\alpha(f) + bD_\alpha(g)$, *for all* $a, b \in \mathbb{R}$,
2. $D_\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$,
3. $D_\alpha(\lambda) = 0$, *for any constant function* $f(t) = \lambda$,
4. $D_\alpha(fg) = fD_\alpha(g) + gD_\alpha(f)$,
5. $D_\alpha\left(\frac{f}{g}\right) = \frac{gD_\alpha(f) - fD_\alpha(g)}{g^2}$,
6. *If f is differentiable, then*

$$D_\alpha(f)(t) = t^{1-\alpha}\frac{d}{dt}f(t).$$

Definition 4. Let $\alpha \in (0,1)$ and $c \geq 0$. The conformable fractional integral of order α of a function f is defined by

$$I_\alpha^c(f)(t) = I_1^c\left(t^{\alpha-1}f\right) = \int_c^t \frac{f(x)}{x^{1-\alpha}}\,dx,$$

which is expressed in terms of the generalized Riemann integral [11].

2.3 Some Special Curves Defined by Means of the Conformable Derivative

In this section, definitions and theorems related to some special curves that were previously defined by the classical derivative and later recharacterized using the conformable fractional derivative are presented.

Definition 5. Let $x = x(s)$ be a unit-speed conformable curve in the three-dimensional Euclidean space, and let s denote the arc-length parameter of this curve. Moreover, let $\mathbf{t} = D_\alpha(x)(s)\, s^{1-\alpha}$ be the unit tangent vector, $\mathbf{n} = \dfrac{D_\alpha(\mathbf{t})(s)}{\|D_\alpha(\mathbf{t})(s)\|}$ be the principal normal vector, and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ be the binormal vector of the curve. The triple $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is called the conformable Frenet frame of the curve $x(s)$. Then, the conformable Frenet formulas are given by

$$\begin{pmatrix} D_\alpha \mathbf{t}(s) \\ D_\alpha \mathbf{n}(s) \\ D_\alpha \mathbf{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_\alpha(s) & 0 \\ -\kappa_\alpha(s) & 0 & \tau_\alpha(s) \\ 0 & -\tau_\alpha(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix}. \tag{2.3}$$

Here, $\kappa_\alpha(s) = \|D_\alpha \mathbf{t}(s)\|$ denotes the curvature of the curve $x(s)$, and $\tau_\alpha(s) = \langle D_\alpha \mathbf{n}(s), \mathbf{b}(s)\rangle$ denotes its torsion [4].

Proposition 2.1. *Let $x = x(s)$ be a unit-speed conformable curve in $\mathbb{E}^3$. Then the conformable curvature and torsion satisfy*

$$\kappa_\alpha(s) = s^{1-\alpha}\kappa(s), \qquad \tau_\alpha(s) = s^{1-\alpha}\tau(s),$$

where $\kappa(s)$ and $\tau(s)$ denote the classical curvature and torsion, respectively [4].

Conclusion 1. Let $x = x(s)$ be a unit-speed conformable curve in the three-dimensional Euclidean space, and let s denote its arc length. From the definitions of the Serret–Frenet vectors, it follows that when x is unit-speed, the conformable derivative does not affect the Frenet vectors themselves, that is, there is no change in the vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$. However, considering equations (4) and (5), it is observed that the curvature and torsion of the curve x change under the conformable fractional derivative [4].

Definition 6. Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed conformable curve in the three-dimensional Euclidean space. If the unit tangent vector of the curve x, defined with respect to the conformable frame, makes a constant angle with a fixed direction $\mathbf{u}$, then the curve x is called a *conformable helix* [4].

Theorem 2.4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed conformable curve in the three-dimensional Euclidean space. If the curve x is a conformable helix with respect to the conformable frame, then*

$$\frac{\kappa_\alpha}{\tau_\alpha} = constant$$

must hold [4].

Definition 7. Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed conformable curve in the three-dimensional Euclidean space. If the unit principal normal vector of the curve x, defined with respect to the conformable frame, makes a constant angle with a fixed direction $\mathbf{u}$, then the curve x is called a *conformable slant helix* [4].

Theorem 2.5. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit-speed conformable curve in the three-dimensional Euclidean space. If the curve x is a conformable slant helix with respect to the conformable frame, then*

$$D_\alpha\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)(s)\frac{1}{\kappa_\alpha(s)} = constant$$

holds [4].

Example 1. Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a regular conformable curve with arbitrary speed, parametrized as

$$x(s) = \left(-\frac{3}{50}\int s^{\alpha-1}\sin s\, ds,\ \frac{3}{50}\int s^{\alpha-1}\cos s\, ds,\ \frac{1}{100}\int s^{\alpha-1}\, ds\right).$$

For the values $\alpha \to 1,\ 0.9,\ 0.7,\ 0.5,\ 0.3,\ 0.1$, the curve $x(s)$ is illustrated in Figure 1, [4].

3 Some Alternative Conformable Special Curves via the Conformable Derivative

In this section, first, the alternative conformable frame $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$ will be defined by means of the conformable derivative. Then, on this defined frame, the definitions of some alternative conformable curves and curve pairs, together with their various characterizations, will be presented. In order to avoid any possible confusion, it is emphasized once again that the frames obtained by the classical derivative are denoted by $\{t(s), n(s), b(s)\}$ and $\{n(s), c(s), w(s)\}$, whereas the frame elements obtained by means of the conformable derivative are denoted by $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ and $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$.

Let $x(s)$ be a unit speed curve, and recall that the conformable Frenet frame is defined by

$$\mathbf{t}(s) = \frac{D_\alpha x(s)}{\|D_\alpha x(s)\|}, \quad \mathbf{n}(s) = \frac{D_\alpha \mathbf{t}(s)}{\|D_\alpha \mathbf{t}(s)\|}, \quad \mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s).$$

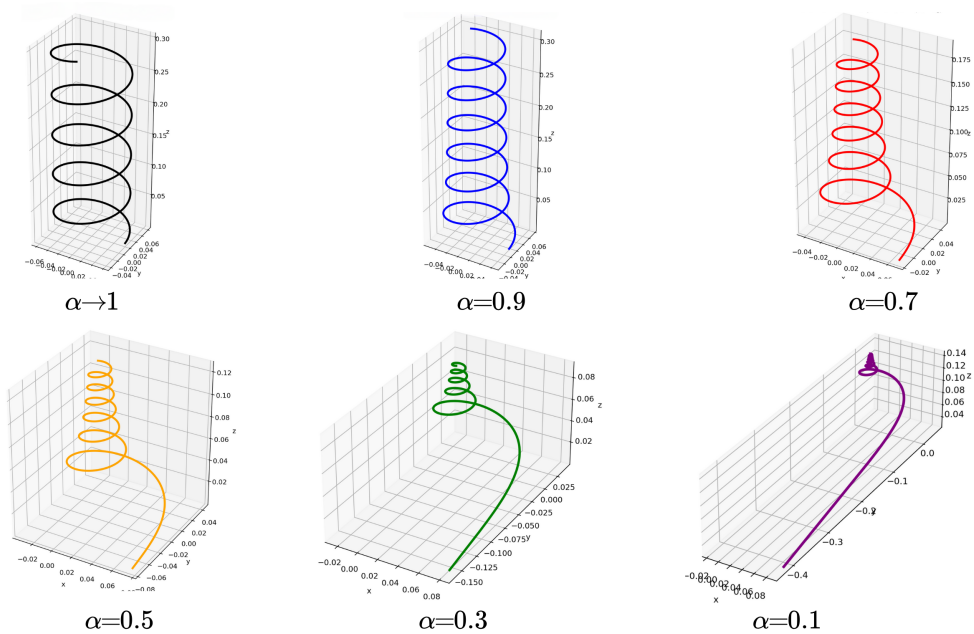


Figure 1: The curve $x(s)$ for different values of α.

In order to show how the alternative conformable frame elements are obtained, we now examine the conformable Frenet frame elements in detail. Let

$$D_\alpha \mathbf{t}(s) = a_1 \mathbf{t}(s) + b_1 \mathbf{n}(s) + c_1 \mathbf{b}(s).$$

Taking the inner product of both sides successively with $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, we obtain

$$\langle D_\alpha \mathbf{t}(s), \mathbf{t}(s) \rangle = a_1 \langle \mathbf{t}(s), \mathbf{t}(s) \rangle + b_1 \langle \mathbf{n}(s), \mathbf{t}(s) \rangle + c_1 \langle \mathbf{b}(s), \mathbf{t}(s) \rangle,$$

which implies

$$\langle s^{1-\alpha} \kappa \mathbf{n}, \mathbf{t} \rangle = a_1, \quad \Rightarrow \quad a_1 = 0.$$

Similarly,

$$\langle D_\alpha \mathbf{t}(s), \mathbf{n}(s) \rangle = a_1 \langle \mathbf{t}(s), \mathbf{n}(s) \rangle + b_1 \langle \mathbf{n}(s), \mathbf{n}(s) \rangle + c_1 \langle \mathbf{b}(s), \mathbf{n}(s) \rangle,$$

hence

$$\langle s^{1-\alpha} \kappa \mathbf{n}, \mathbf{n} \rangle = b_1, \quad \Rightarrow \quad b_1 = \kappa_\alpha.$$

Finally,

$$\langle D_\alpha \mathbf{t}(s), \mathbf{b}(s) \rangle = a_1 \langle \mathbf{t}(s), \mathbf{b}(s) \rangle + b_1 \langle \mathbf{n}(s), \mathbf{b}(s) \rangle + c_1 \langle \mathbf{b}(s), \mathbf{b}(s) \rangle,$$

which yields

$$\langle s^{1-\alpha} \kappa \mathbf{n}, \mathbf{b} \rangle = c_1, \quad \Rightarrow \quad c_1 = 0.$$

Applying the same procedure to

$$D_\alpha \mathbf{n}(s) = a_2\mathbf{t}(s) + b_2\mathbf{n}(s) + c_2\mathbf{b}(s), \quad D_\alpha \mathbf{b}(s) = a_3\mathbf{t}(s) + b_3\mathbf{n}(s) + c_3\mathbf{b}(s),$$

the derivative change matrix of the Serret-Frenet frame obtained by means of the conformable derivative is given by

$$\begin{bmatrix} D_\alpha \mathbf{t}(s) \\ D_\alpha \mathbf{n}(s) \\ D_\alpha \mathbf{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{bmatrix}.$$

Before proceeding to the definition of the alternative conformable frame elements $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$, we now present the definitions of the spherical indicatrix curves obtained by means of the conformable derivative, which will be used in the theorems.

Definition 8. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in the three-dimensional Euclidean space. Let the Frenet frame elements of the conformable tangent indicatrix x_t of the curve x be denoted by $\{\mathbf{t}_t, \mathbf{n}_t, \mathbf{b}_t, \kappa_{x_t}, \tau_{x_t}\}$. Then,

$$\mathbf{t}_t = \mathbf{n},$$

$$\mathbf{n}_t = \frac{\kappa_\alpha}{f_\alpha}\left(-\mathbf{t} + \frac{\tau_\alpha}{\kappa_\alpha}\,\mathbf{b}\right),$$

$$\mathbf{b}_t = \frac{\kappa_\alpha}{f_\alpha}\left(\frac{\tau_\alpha}{\kappa_\alpha}\,\mathbf{t} + \mathbf{b}\right),$$

$$s_t = \int \kappa_\alpha\, ds, \qquad \kappa_{x_t} = \frac{f_\alpha}{\kappa_\alpha}, \qquad \tau_{x_t} = \frac{g_\alpha}{\kappa_\alpha}.$$

The Frenet derivative change matrix is given by

$$\begin{pmatrix} \mathbf{t}_t' \\ \mathbf{n}_t' \\ \mathbf{b}_t' \end{pmatrix} = \begin{pmatrix} 0 & -\kappa_{x_t} & 0 \\ \kappa_{x_t} & 0 & -\tau_{x_t} \\ 0 & \tau_{x_t} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}_t \\ \mathbf{n}_t \\ \mathbf{b}_t \end{pmatrix}.$$

Definition 9. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in the three-dimensional Euclidean space. Let the Frenet frame elements of the conformable normal indicatrix x_n of the curve x be denoted by $\{\mathbf{t}_n, \mathbf{n}_n, \mathbf{b}_n, \kappa_{x_n}, \tau_{x_n}\}$. Then,

$$\mathbf{t}_n = \frac{\kappa_\alpha}{f_\alpha}\left(-\mathbf{t} + \frac{\tau_\alpha}{\kappa_\alpha}\,\mathbf{b}\right),$$

$$\mathbf{n}_n = \frac{\sigma_\alpha}{\sqrt{1+\sigma_\alpha^2}}\left(\frac{\kappa_\alpha}{f_\alpha}\left(\frac{\tau_\alpha}{\kappa_\alpha}\,\mathbf{t} + \mathbf{b}\right) - \frac{1}{\sigma_\alpha}\,\mathbf{n}\right),$$

$$\mathbf{b}_n = \frac{1}{\sqrt{1+\sigma_\alpha^2}}\left(\frac{\kappa_\alpha}{f_\alpha}\left(\frac{\tau_\alpha}{\kappa_\alpha}\,\mathbf{t}+\mathbf{b}\right)+\sigma_\alpha\mathbf{n}\right),$$

$$s_n = \int f_\alpha\, ds, \qquad \kappa_{x_n} = \sqrt{1+\sigma_\alpha^2}, \qquad \tau_{x_n} = \frac{\Gamma_\alpha}{\sqrt{1+\sigma_\alpha^2}},$$

$$\Gamma_\alpha = \frac{\sigma_\alpha'}{f_\alpha\,(1+\sigma_\alpha^2)^{3/2}}.$$

The Frenet derivative change matrix is given by

$$\begin{pmatrix}\mathbf{t}_n'\\ \mathbf{n}_n'\\ \mathbf{b}_n'\end{pmatrix} = \begin{pmatrix}0 & -\kappa_{x_n} & 0\\ \kappa_{x_n} & 0 & -\tau_{x_n}\\ 0 & \tau_{x_n} & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}_n\\ \mathbf{n}_n\\ \mathbf{b}_n\end{pmatrix}.$$

Definition 10. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in the three-dimensional Euclidean space. Let the Frenet frame elements of the conformable binormal indicatrix x_b of the curve x be denoted by $\{\mathbf{t}_b, \mathbf{n}_b, \mathbf{b}_b, \kappa_{x_b}, \tau_{x_b}\}$. Then,

$$\mathbf{t}_b = -\mathbf{n},$$

$$\mathbf{n}_b = \frac{\kappa_\alpha}{f_\alpha}\left(\mathbf{t}-\frac{\tau_\alpha}{\kappa_\alpha}\,\mathbf{b}\right),$$

$$\mathbf{b}_b = \frac{\kappa_\alpha}{f_\alpha}\left(\frac{\tau_\alpha}{f_\alpha}\,\mathbf{t}+\mathbf{b}\right),$$

$$s_b = \int \tau_\alpha\, ds, \qquad \kappa_{x_b} = \frac{f_\alpha}{\tau_\alpha}, \qquad \tau_{x_b} = -\frac{g_\alpha}{\tau_\alpha}.$$

The Frenet derivative matrix representation is given by

$$\begin{pmatrix}\mathbf{t}_b'\\ \mathbf{n}_b'\\ \mathbf{b}_b'\end{pmatrix} = \begin{pmatrix}0 & -\kappa_{x_b} & 0\\ -\kappa_{x_b} & 0 & -\tau_{x_b}\\ 0 & \tau_{x_b} & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}_b\\ \mathbf{n}_b\\ \mathbf{b}_b\end{pmatrix}.$$

Now, using the conformable Serret-Frenet frame, we construct the conformable alternative moving frame.

Definition 11. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve in the three-dimensional Euclidean space. Define

$$\mathbf{n}(s) = \frac{D_\alpha\mathbf{t}(s)}{\|D_\alpha\mathbf{t}(s)\|}, \qquad \mathbf{c}(s) = \frac{D_\alpha\mathbf{n}(s)}{\|D_\alpha\mathbf{n}(s)\|}, \qquad \mathbf{w}(s) = \mathbf{n}(s)\times\mathbf{c}(s).$$

Here, since

$$D_\alpha\mathbf{n}(s) = -\kappa_\alpha\mathbf{t}(s)+\tau_\alpha\mathbf{b}(s) \quad \text{and} \quad \|D_\alpha\mathbf{n}(s)\| = \sqrt{\kappa_\alpha^2+\tau_\alpha^2} = f_\alpha,$$

it can be written as

$$\mathbf{n}(s) = \mathbf{n}(s), \qquad \mathbf{c}(s) = -\frac{\kappa_\alpha}{f_\alpha}\mathbf{t}(s) + \frac{\tau_\alpha}{f_\alpha}\mathbf{b}(s), \qquad \mathbf{w}(s) = \frac{\tau_\alpha}{f_\alpha}\mathbf{t}(s) + \frac{\kappa_\alpha}{f_\alpha}\mathbf{b}(s).$$

Now, let

$$D_\alpha\mathbf{n}(s) = a_1\mathbf{n}(s) + b_1\mathbf{c}(s) + c_1\mathbf{w}(s).$$

Taking the inner product of both sides successively with $\mathbf{n}(s)$, $\mathbf{c}(s)$, and $\mathbf{w}(s)$, we obtain

$$\langle D_\alpha\mathbf{n}(s), \mathbf{n}(s)\rangle = a_1\langle \mathbf{n}(s), \mathbf{n}(s)\rangle + b_1\langle \mathbf{c}(s), \mathbf{n}(s)\rangle + c_1\langle \mathbf{w}(s), \mathbf{n}(s)\rangle,$$

$$\Rightarrow \langle -\kappa_\alpha\mathbf{t} + \tau_\alpha\mathbf{b}, \mathbf{n}\rangle = a_1, \qquad \Rightarrow a_1 = 0.$$

$$\langle D_\alpha\mathbf{n}(s), \mathbf{c}(s)\rangle = a_1\langle \mathbf{n}(s), \mathbf{c}(s)\rangle + b_1\langle \mathbf{c}(s), \mathbf{c}(s)\rangle + c_1\langle \mathbf{w}(s), \mathbf{c}(s)\rangle,$$

$$\Rightarrow \left\langle -\kappa_\alpha\mathbf{t} + \tau_\alpha\mathbf{b}, -\frac{\kappa_\alpha}{f_\alpha}\mathbf{t} + \frac{\tau_\alpha}{f_\alpha}\mathbf{b}\right\rangle = \frac{\kappa_\alpha^2 + \tau_\alpha^2}{f_\alpha}\, b_1,$$

$$\Rightarrow b_1 = f_\alpha.$$

$$\langle D_\alpha\mathbf{n}(s), \mathbf{w}(s)\rangle = a_1\langle \mathbf{n}(s), \mathbf{w}(s)\rangle + b_1\langle \mathbf{c}(s), \mathbf{w}(s)\rangle + c_1\langle \mathbf{w}(s), \mathbf{w}(s)\rangle,$$

$$\Rightarrow \left\langle -\kappa_\alpha\mathbf{t} + \tau_\alpha\mathbf{b}, \frac{\tau_\alpha}{f_\alpha}\mathbf{t} + \frac{\kappa_\alpha}{f_\alpha}\mathbf{b}\right\rangle = c_1, \qquad \Rightarrow c_1 = 0.$$

As a result of these operations, the first row of the matrix representation is obtained. Similarly, the other row elements are found, and the following matrix equation can be written as

$$\begin{pmatrix} D_\alpha\mathbf{n}(s) \\ D_\alpha\mathbf{c}(s) \\ D_\alpha\mathbf{w}(s) \end{pmatrix} = \begin{pmatrix} 0 & -f_\alpha & 0 \\ f_\alpha & 0 & -g_\alpha \\ 0 & g_\alpha & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n}(s) \\ \mathbf{c}(s) \\ \mathbf{w}(s) \end{pmatrix}. \tag{3.1}$$

Now, we define some special curves parametrized by the parameter s by means of the alternative conformable frame.

Definition 12. Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve defined in the three-dimensional Euclidean space and lying on the alternative conformable frame $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$. If the vector field $\mathbf{c}(s)$ of the curve x makes a constant angle $\theta \neq \frac{\pi}{2}$ with a fixed direction $\mathbf{u}$, then this curve is called an alternative conformable slant helix, or equivalently, a conformable c-slant helix. This situation is expressed by the following condition:

$$\langle \mathbf{c}(s), \mathbf{u}\rangle = \cos\theta, \qquad \theta \neq \frac{\pi}{2}. \tag{3.2}$$

Theorem 3.1. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve defined in the three-dimensional Euclidean space and lying on the alternative conformable frame $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$. If the curve x is an alternative conformable slant helix, then the axis $\mathbf{u}$ of the curve x is given as follows:*

$$\mathbf{u} = \left\{ \frac{g_\alpha}{(f_\alpha^2 + g_\alpha^2)} \left(D_\alpha \left(\frac{g_\alpha}{f_\alpha} \right) \right) f_\alpha^2 \, \mathbf{n}(s) + \mathbf{c}(s) + \frac{f_\alpha^2 + g_\alpha^2}{D_\alpha \left(\frac{g_\alpha}{f_\alpha} \right) f_\alpha} \, \mathbf{w}(s) \right\} \cos\theta.$$

Here, θ is a constant angle, and $f_\alpha = \sqrt{\kappa_\alpha^2 + \tau_\alpha^2}$ and $g_\alpha = \sigma_\alpha f_\alpha$.

Proof. Assume that the curve x is an alternative conformable slant helix in the three-dimensional Euclidean space. Then, in order to obtain the vector $\mathbf{u}$, the following expression can be written:

$$\mathbf{u} = a_1 \mathbf{n}(s) + b_1 \mathbf{c}(s) + c_1 \mathbf{w}(s).$$

Moreover, taking the conformable derivative with respect to the parameter s of both sides of the equality

$$\langle \mathbf{c}(s), \mathbf{u} \rangle = \cos\theta$$

yields the following equalities:

$$\begin{aligned} \langle D_\alpha \mathbf{c}(s), \mathbf{u} \rangle + \langle \mathbf{c}(s), D_\alpha \mathbf{u} \rangle &= 0, \\ \langle -f_\alpha \mathbf{n}(s) + g_\alpha \mathbf{w}(s), \mathbf{u} \rangle + \langle \mathbf{c}(s), D_\alpha \mathbf{u} \rangle &= 0, \\ -f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle + g_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle &= 0. \end{aligned} \tag{3.3}$$

Now, taking the conformable derivative with respect to the parameter s of both sides of Eq. (3.3), we obtain

$$\begin{aligned} -D_\alpha f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle - f_\alpha D_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle + D_\alpha g_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle + g_\alpha D_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle = 0, \\ - D_\alpha f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle - f_\alpha \big(\langle D_\alpha \mathbf{n}(s), \mathbf{u} \rangle + \langle \mathbf{n}(s), D_\alpha \mathbf{u} \rangle \big) \\ + D_\alpha g_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle + g_\alpha \big(\langle D_\alpha \mathbf{w}(s), \mathbf{u} \rangle + \langle \mathbf{w}(s), D_\alpha \mathbf{u} \rangle \big) = 0. \end{aligned}$$

Moreover, substituting

$$D_\alpha \mathbf{n}(s) = f_\alpha \mathbf{c}(s), \qquad D_\alpha \mathbf{w}(s) = -g_\alpha \mathbf{c}(s)$$

into the last equation and simplifying, we obtain

$$\begin{aligned} -D_\alpha f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle - f_\alpha^2 \langle \mathbf{c}(s), \mathbf{u} \rangle + D_\alpha g_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle - g_\alpha^2 \langle \mathbf{c}(s), \mathbf{u} \rangle = 0, \\ -D_\alpha f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle + D_\alpha g_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle - (f_\alpha^2 + g_\alpha^2) \cos\theta = 0, \end{aligned}$$

$$-D_\alpha f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle + D_\alpha g_\alpha \langle \mathbf{w}(s), \mathbf{u} \rangle = (f_\alpha^2 + g_\alpha^2) \cos\theta. \tag{3.4}$$

Here, from Eq. (3.3), we know that

$$\langle \mathbf{w}(s), \mathbf{u} \rangle = \frac{f_\alpha}{g_\alpha} \langle \mathbf{n}(s), \mathbf{u} \rangle.$$

Substituting this expression into Eq. (3.4) and performing the necessary algebraic manipulations, we obtain

$$-D_\alpha f_\alpha \langle \mathbf{n}(s), \mathbf{u} \rangle + D_\alpha g_\alpha \frac{f_\alpha}{g_\alpha} \langle \mathbf{n}(s), \mathbf{u} \rangle = (f_\alpha^2 + g_\alpha^2) \cos\theta,$$

$$\langle \mathbf{n}(s), \mathbf{u} \rangle \frac{-D_\alpha f_\alpha g_\alpha + D_\alpha g_\alpha f_\alpha}{g_\alpha} = (f_\alpha^2 + g_\alpha^2) \cos\theta,$$

$$\langle \mathbf{n}(s), \mathbf{u} \rangle D_\alpha \left(\frac{g_\alpha}{f_\alpha} \right) \frac{f_\alpha^2}{g_\alpha} = (f_\alpha^2 + g_\alpha^2) \cos\theta,$$

$$\langle \mathbf{n}(s), \mathbf{u} \rangle = \frac{g_\alpha (f_\alpha^2 + g_\alpha^2)}{D_\alpha \left(\frac{g_\alpha}{f_\alpha} \right) f_\alpha^2} \cos\theta.$$

Substituting this result into Eq. (3.3), we obtain

$$\langle \mathbf{w}(s), \mathbf{u} \rangle = \frac{f_\alpha^2 + g_\alpha^2}{D_\alpha \left(\frac{g_\alpha}{f_\alpha} \right) f_\alpha} \cos\theta.$$

Finally, substituting these inner product values into the axis equation, the desired expression is obtained. This completes the proof. □

Corollary 3.2. *Let $x : I \subset \mathbb{R} \to E^3$ be a unit speed conformable curve lying on the alternative conformable frame $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$ in the three-dimensional Euclidean space. Then the curve x is an alternative conformable slant helix if and only if*

$$\frac{(f_\alpha^2 + g_\alpha^2)^{3/2}}{D_\alpha \left(\frac{g_\alpha}{f_\alpha} \right) f_\alpha^2} = \tan\theta = \textit{constant}. \tag{3.5}$$

Here, θ denotes the constant angle between the vector field $\mathbf{c}(s)$ and a fixed direction $\mathbf{u}$.

Proof. Assume that the curve x is an alternative conformable slant helix in the three-dimensional Euclidean space. From the definition of an alternative conformable slant helix, we know that the axis $\mathbf{u}$ of the curve is constant and of unit length. Using this information together with the expression of $\mathbf{u}$, Eq. (3.5) is obtained.

Conversely, assume that Eq. (3.5) holds. In this case, from the vector field

$$\mathbf{u} = \left\{ \frac{g_\alpha(f_\alpha^2+g_\alpha^2)}{D_\alpha\left(\frac{g_\alpha}{f_\alpha}\right)f_\alpha^2}\,\mathbf{n}(s) + \mathbf{c}(s) + \frac{(f_\alpha^2+g_\alpha^2)}{D_\alpha\left(\frac{g_\alpha}{f_\alpha}\right)f_\alpha}\,\mathbf{w}(s) \right\}\cos\theta,$$

it is seen that $\langle \mathbf{c}(s), \mathbf{u}\rangle = \cos\theta$. Taking the conformable derivative of this equality with respect to the parameter s and using the alternative conformable frame elements, we obtain

$$-f_\alpha\langle \mathbf{n}(s), \mathbf{u}\rangle + g_\alpha\langle \mathbf{w}(s), \mathbf{u}\rangle = s^{1-\alpha}(-\sin\theta\, D_\alpha\theta).$$

Since the equalities $\langle \mathbf{n}(s), \mathbf{u}\rangle$ and $\langle \mathbf{w}(s), \mathbf{u}\rangle$ are known from the previous theorem, it is clear that $-\sin\theta\, D_\alpha\theta = 0$. That is, θ is a constant function. Therefore, x is an alternative conformable slant helix. Hence, the proof is complete. □

Theorem 3.3. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed conformable curve lying on the alternative conformable frame $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$ in the three-dimensional Euclidean space. Then, the conformable tangent indicatrix $\mathbf{x_t}$ of the curve x is a conformable slant helix if and only if the curve x is an alternative conformable slant helix.*

Proof. Assume that the tangent indicatrix of the curve x is a conformable slant helix. In this case, the equalities

$$\mathbf{t_t} = \mathbf{n} \qquad \text{and} \qquad \mathbf{n_t} = \mathbf{c}$$

can be written. Since $\mathbf{x_t}$ is a conformable slant helix, there exists a fixed direction $\mathbf{u}$ such that

$$\langle \mathbf{n_t}, \mathbf{u}\rangle = \text{constant}.$$

From the above equality, it is seen that

$$\langle \mathbf{c}, \mathbf{u}\rangle = \text{constant}.$$

Therefore, the curve x is an alternative conformable slant helix.

Conversely, assume that the curve x is an alternative conformable slant helix. Accordingly, it is clear that the tangent indicatrix of the curve x is a conformable slant helix. This completes the proof. □

Example 2. A function $x : I \subset \mathbb{R} \to \mathbb{E}^3$ is a regular alternative conformable slant helix curve with arbitrary speed, and it is parametrized as follows:

$$x(s) = \left(\int s^{\alpha-1}\left(\tfrac{\sqrt{2}}{2}\sin s\sin(\sqrt{2}s) + \cos s\cos(\sqrt{2}s)\right)\mathrm{d}s,\ \int s^{\alpha-1}\left(\tfrac{\sqrt{2}}{2}\sin s\cos(\sqrt{2}s) - \cos s\sin(\sqrt{2}s)\right)\mathrm{d}s,\ \tfrac{\sqrt{2}}{2}\int s^{\alpha-1}\sin s\,\mathrm{d}s\right).$$

For different values of α, the alternative conformable slant helix curve is given in Figure 2.

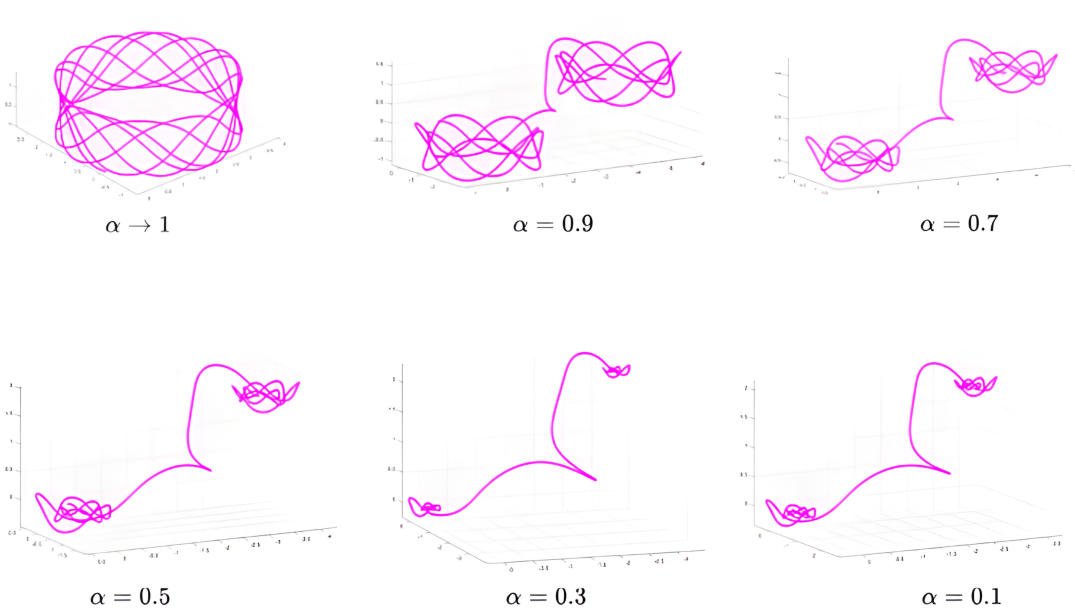


Figure 2: Alternative conformable slant helix.

Theorem 3.4. *Let $x : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed conformable curve lying on the alternative conformable frame $\{\mathbf{n}(s), \mathbf{c}(s), \mathbf{w}(s)\}$ in the three-dimensional Euclidean space. Then, the conformable binormal indicatrix $\mathbf{x_b}$ of the curve x is a conformable slant helix if and only if the curve x is an alternative conformable slant helix.*

Proof. Assume that the binormal indicatrix of the curve x is a conformable slant helix. In this case, the equalities

$$\mathbf{t_b} = -\mathbf{n} \qquad \text{and} \qquad \mathbf{n_b} = -\mathbf{c}$$

can be written. Since $\mathbf{x_b}$ is a conformable slant helix, there exists a fixed direction $\mathbf{u}$ such that

$$\langle \mathbf{n_b}, \mathbf{u} \rangle = \text{constant}.$$

From the above equality, it is seen that

$$\langle \mathbf{c}, \mathbf{u} \rangle = \text{constant}.$$

Therefore, the curve x is an alternative conformable slant helix.

Conversely, assume that the curve x is an alternative conformable slant helix. Accordingly, it is clear that the binormal indicatrix of the curve x is a conformable slant helix. This completes the proof. □

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