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p-HARMONIC MORPHISMS, COHOMOLOGY CLASSES AND SUBMERSIONS

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Abstract. By studying cohomology classes that are related with p-harmonic morphisms, we extend some previous results of the first author on Riemannian submersions with minimal fibers to n-harmonic morphisms which are submersions.

1. Introduction

In the study of topology on a compact Riemannian manifold M, it is well-known that nontrivial fundamental groups $\pi_1(M)$, homology classes, and cohomology classes can be represented by stable closed geodesics, stable minimal currents, and harmonic forms on M, by Cartan's Theorem ([3]), Federer-Fleming's Theorem ([7]), and Hodge Theorem ([8]), respectively. In an analogous spirit, it is shown in [13] that homotopy classes can be represented by p-harmonic maps (For definition and examples of p-harmonic maps, see e.g. [14]):

Theorem A. *If N is a compact Riemannian manifold, then for any positive integer i, each class in $\pi_i(N)$ can be represented by a $C^{1,\alpha}$ p-harmonic map u_0 from S^i into N minimizing p-energy in its homotopy class for any $p > i$.*

Further applications and homotopically vanishing theorems are given in [13, 16]. As a p-harmonic morphism by definition, carries germs of p-harmonic functions to germs of p-harmonic functions, and can be characterized as a horizontally weak conformal p-harmonic map (cf. Theorem 4.), a p-harmonic morphism should be also linked with topology in a certain way.

On the other hand, B.-Y. Chen provides the following link between Riemannian submersions and minimal immersions with cohomology class (cf. [5]):

Theorem B. *Let $\pi : M \to B$ be a Riemannian submersion with minimal fibers and orientable base manifold B. If M is a closed manifold with cohomology class $H^b(M, \mathbf{R}) =$*

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0, $b = \dim B$, then the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is never integrable. Thus the submersion π is always non-trivial.

Let M and B be differentiable manifolds and let $\phi : M \to B$ be a differentiable map between them. The map ϕ is called a *submersion* if, at each point $x \in M$, the differential $d\phi_x$ is a surjective linear map. A closed manifold is a compact manifold without boundary.

Let $\pi : M \to B$ be a submersion between two Riemannian manifolds. For each $x \in B$, $\pi^{-1}(x)$ is an $(n-b)$-dimensional submanifold of M, which is called a fiber. A vector field on M is called *vertical* if it is always tangent to fibers; and it is called *horizontal* if it is always orthogonal to fibers in B.

The simplest type of Riemannian submersions is the projection of a Riemannian product manifold on one of its factors. For such Riemannian submersions, both horizontal and vertical distributions are *totally geodesic distributions*, i.e., both distributions are completely integrable and their leaves (i.e., integrable submanifolds) are totally geodesic submanifolds.

A submersion $\pi : M \to B$ between two Riemannian manifolds is said to be *nontrivial* if its horizontal and vertical distributions are not both totally geodesic distribution. And a submersion $\pi : M \to B$ between two Riemannian manifolds is called a *Riemannian submersion* if the differential $d\pi$ preserves the length of horizontal vector fields.

The purpose of this article is to connect and extend the two seemingly unrelated areas of p-harmonic morphisms and cohomology classes. More precisely, we prove the following.

Theorem 1. *Let $u : M \to N$ be an n-harmonic morphism which is a submersion. If M is a closed manifold with cohomology class $H^n(M, \mathbf{R}) = 0$ with $n = \dim N$, then the horizontal distribution $\mathcal{H}$ of u is never integrable. Thus the submersion u is always non-trivial.*

Theorem 1. recaptures Theorem B, when $u : M \to B$ is a Riemannian submersion with minimal fibers and orientable base manifold B. To see this, we note that in general, n-harmonic morphisms into n-manifolds which are submersions but not necessarily Riemannian submersions with minimal fibers(cf. Section 5). On the other hand, a Riemannian submersion into an n-manifold with minimal fibers is an n-harmonic morphism which is a submersion by a result of [2] (cf. Theorem 4.).

Theorem 2. *Let $u : M \to N$ be an n-harmonic morphism which is a submersion from a closed manifold M with $n = \dim N$. Then the pull back of the volume element of N is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable.*

This Theorem recaptures a result of [5] (cf. Theorem 3.).

2. Minimal submanifolds, submersions and p-harmonic morphisms

We recall some related basic facts, notations, definitions, and formulas (see [4, 5, 6] for details).

2.1. Basic formulas and equations

Let $\tilde{M}$ be a Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$. The tangent bundle of $\tilde{M}$ is denoted by $T\tilde{M}$, and the (infinite dimensional) vector space of smooth sections of a smooth vector bundle E is denoted by $\Gamma(E)$.

Let M be a submanifold of dimension $n \geq 2$ in $\tilde{M}$. Denote by ∇ and D, the Levi-Civita connection and the normal connection of M, respectively. For each normal vector $\xi \in T_p^{\perp}M, p \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM at p. Then the shape operator and the second fundamental form h are related by

$$\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle \tag{1.1}$$

for X, Y tangent to M and ξ normal to M.

The formulas of Gauss and Weingarten are given respectively by (cf. [4])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{1.3}$$

for tangent vector fields X, Y and normal vector field ξ on M.

2.2. Definitions

The *mean curvature vector field* of a submanifold M in a Riemannian manifold is defined by

$$H = \left(\frac{1}{n}\right) \operatorname{trace} h. \tag{1.4}$$

A submanifold M in $\tilde{M}$ is called *totally geodesic* (respectively, *minimal*) if the second fundamental form h (respectively, the mean curvature vector field H) of M vanishes identically.

2.3. Riemannian submersions

For Riemannian submersions we have the following result from [5].

Theorem 3. *Let $\pi : M \to B$ be a Riemannian submersion from a closed manifold M onto an orientable base manifold B. Then the pull back of the volume element of B is harmonic if and only if the horizontal distribution $\mathcal{H}$ is integrable and fibers are minimal.*

3. p-harmonic morphisms

A C^2 map $u : M \to N$ between two Riemannian manifolds is said to be a *p-harmonic morphism* if, for any p-harmonic function f defined on an open set V of N, the composition $f \circ u$ is p-harmonic on $u^{-1}(V)$.

A C^2 map $u : M \to N$ between two Riemannian manifolds is called *horizontally weak conformal* if, for any x such that $du(x) \neq 0$, the restriction of $du(x)$ to the orthogonal complement H of $Ker(du(x))$ is conformal and surjective.

Recently, E. Loubeau and Burel-Loubeau obtain a characterization of a p-harmonic morphism:

Theorem 4. [1, 9] *A C^2 map $u : M \to N$ is a p-harmonic morphism with $p \in (1, \infty)$ if and only if it is a p-harmonic and horizontally weak conformal map.*

In [2], P. Baird and S. Gudmundsson link n-harmonic morphism with minimal fibers:

Theorem 5. [2] *Let $u : M \to N$ be a horizontally conformal submersion. Then u is n-harmonic with $n = \dim N$ if and only if the fibers of u are minimal in M.*

4. Proof of Theorems 1. and 2.

By virtue of Theorems 4. and 5., an n-harmonic morphism u which is a submersion with $n = \dim N$ has minimal fibers. Thus, we may proceed as in the proof of Theorem 3. (see [5] for details): Since the fibers are minimal submanifold of N (hence (6.6) in [5] holds), the pull back of the volume element ω of N is a co-closed n-form if and only if the horizontal distribution $\mathcal{H}$ of the submersion u is integrable (hence (6.7) in [5] also holds). Furthermore, the pull back of the volume element ω of N is also a closed form, since the exterior differentiation d and the pull back u^* commutes. Therefore, it follows that ω is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable. In case, ω is harmonic, it represents a nontrivial cohomology class $H^n(M, \mathbf{R})$ by Hodge Theorem. This proves Theorem 2..

Since each nonzero harmonic form represents a non-trivial cohomology class by Hodge Theorem, and since on a closed manifold a differential form is harmonic if and only if it is closed and co-closed, Theorem 1. follows from Theorem 2..

5. Examples of p-harmonic morphisms which are submersion, but not Riemannian submersions

In this section, we provide some simple examples of p-harmonic morpisms which are submersion, but not Riemannian submersions.

Example 5.1. Let $\pi : \mathbb{E}^k \times \mathbb{E}^n \to \mathbb{E}^n$ be an orthogonal projection and let $\sigma_n : \mathbb{E}^n \to \mathbb{E}^n$ be a non-isometric conformal diffeomorphism, where $\mathbb{E}^n$ is the Euclidean n-space. Then $\sigma_n \circ \pi$ is an n-harmonic morphism which is a submersion with minimal fibers but is not a Riemannian submersion (cf. Example 4.9 in [11]).

Analogously, we have on compact manifolds

Example 5.2. Let $\pi_2 : S^k \times S^n \to S^n$ be an orthogonal projection and let $\gamma_n : S^n \to S^n$ be a non-isometric conformal diffeomorphism. Then $\gamma_n \circ \pi_2$ is an n-harmonic morphism which is a submersion with minimal fibers but is not a Riemannian submersion.

These examples are based on the fact that the composition of p-harmonic morphisms is a p-harmonic morphism, and a conformal map between equal dimensional n-manifolds, such as stereographic projections $u : \mathbb{E}^n \to S^n$ is an n-harmonic morphism (cf. [15, 11]).

References

[1] J. M. Burel and E. Loubeau, *p-harmonic morphisms: the* $1 < p < 2$ *case and a non-trivial example*, Contemp. Math. **308** (2002), 21–37.

[2] P. Baird and S Gudmundsson, *p-harmonic maps and minimal submanifolds*, Math. Ann. **294** (1992), 611–624.

[3] E. Cartan, Lecons sur la Geométrie des Espaces de Riemann, 2nd edition, Gauthier-Villars, Paris, 1951.

[4] B.-Y. Chen, *Geometry of Submanifolds*, M. Dekker, New York, 1973.

[5] B.-Y. Chen, *Riemannian submersions, minimal immersions and cohomology class* Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), 162–167.

[6] B.-Y. Chen and S. W. Wei, *Submanifolds of warped product manifolds* $I \times_f S^{m-1}(k)$ *from a p-harmonic viewpoint,* Bull. Transilv. Univ. Braşov Ser. III, **1**(50) (2008), 59–78.

[7] H. Federer and W. Fleming, Normal and integral currents, Ann. of Math., 72 (1960) 458–520.

[8] W.V.D. Hodge, *The theory and application of harmonic intergrals*, Cambridge University Press, New York (1941) (2nd ed., 1952)

[9] E. Loubeau, *On p-harmonic morphisms*, Diff. Geom. Appl. **12** (2000), 219-229.

[10] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1982.

[11] Y. L. Ou and S. W. Wei, *A classification and some constructions of p-harmonic morphisms*, Beiträge Algebra Geom. **45** (2004) 637–647.

[12] A. V. Pogorelov, *On the stability of minimal surfaces*, Soviet Math. Dokl. **24** (1981), 274-276.

[13] S. W. Wei, *Representing Homotopy Groups and Spaces of Maps by p-harmonic maps*, Indiana Univ. Math. J. **47** (1998), 625–670.

[14] S. W. Wei, *p-Harmonic geometry and related topics*, Bull. Transilv. Univ. Brasov Ser. III **1**(50) (2008), 415-453.

[15] S.W. Wei, J.F. Li, and L. Wu, *Generalizations of the Uniformization Theorem and Bochner's Method in p-Harmonic Geometry*, Proceedings of the 2006 Midwest Geometry Conference, Commun. Math. Anal. 2008, Conference 1, 46-68

[16] S. W. Wei, and L. Wu, *Homotopy groups and p-Harmonic Maps*, Commun. Math. Anal. 2008, Conference 1, 40-45.

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