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Fredholm properties of a linear relation through invariant subspaces

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Abstract. In this paper, we present a sufficient conditions under which a linear relation (multivalued linear operator) admits an invariant subspace on a Banach space X. Also, we give some Fredholm properties of the linear relation T using properties of its restriction to an invariant subspace M and the linear relation induced by T over the quotient subspace X/M.

Keywords. Linear relation, invariant subspace, Fredholm properties

1 Introduction

Let X ba a complex Banach space, M a subspace of X and let T be a bounded and closed multivalued linear operator (linear relation) on X. We say that M is T-invariant (resp. weakly invariant under T) if $T(M) \subset M$ (resp. for all $x \in M$, $Tx \cap M \neq \varnothing$). The restriction T_M of T on M is defined by $G(T_M) = G(T) \cap (M \times M)$ where $G(T)$ stands for the graph of T. The invariant subspace problem is concerned with the existence of nontrivial closed subspaces which are invariant for a bounded operator T. One of the oldest results in the theory of invariant subspaces is the theorem of Aronszajn and Smith [1] published in 1954, that is every compact single valued operator on a Banach space X has a nontrivial invariant subspace. A much stronger result and probably one of the landmark theorems in the area of invariant subspaces was given in 1973 by Lomonosov [8], a special case of which states that if a bounded operator T on a Banach space commutes with a non-zero compact operator, then T has a nontrivial invariant subspace. In 2000, P. Saveliev [10] generalized Lomonosov's invariant subspace theorem to the case of linear relations whose multivalued parts are finite dimensional by using fixed point techniques. In particular, he proved that if S and T are linear relations defined on a Banach space X and having finite dimentional multivalued parts and if S right commutes with T and if S is compact then T has a nontrivial weakly invariant subspace. After that, in 2002 D. Wilcox [12] gives an alternative proofs and extend these results. She make the distinction between left and right commutation for linear relaions [12, Theorem 7.7.7]. In 2018, G. Wanjala [11] develop some operator representation techniques for linear relations and use them to solve the left commutativity case under the assumption that $ST(0) = S(0)$ and $TS(0) = T(0)$. We are

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concerned in the begining of this paper with giving some other sufficient conditions ensuring the existence of non-trivial invariant subspaces for some bounded linear relations.

In the following let us consider a bounded closed linear relation T that admits a complemented $T-$invariant subspace M. Pick P a bounded projection with $R(P) = M$. Then, we can show that T can be represented by an upper triangular matrix linear relation acting on the space $R(P) \times N(P)$. In order to verify this representation, we take $x, y \in X$. Then, $x = x_1 + x_2$ with $x_1 = Px$ and $x_2 = (I - P)x$ and $y = y_1 + y_2$ with $y_1 = Py$ and $y_2 = (I - P)y$. We claim that

$$y \in Tx \text{ if and only if } \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \begin{pmatrix} PTP & PT(I-P) \\ 0 & (I-P)T(I-P) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.1}$$

This matrix representation of the relation T justifies the importance given by several authors to study the upper triangular matrix linear relations of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ where the entries A and B are two linear relations [4, 7]. Under suitable conditions on the entries of the matrix, several properties of M_C are given from those of the diagonal A and B. However, for a relation T, the existence of complemented $T-$invariant subspace M is not always ensured. Therefore, the representation given in (1.1) is not always possible. This led us to associate to such relation T a pair of operators T_1 and T_2 which play the role that the diagonal plays for a matrix representation.

The paper is organized as follows. Section 2 contains basic notions and some auxiliary results which will be used in the following sections. In Section 3, we give sufficient conditions under which a linear relation admits an invariant subspace. In Section 4, we associate for a relation T and a weakly invariant subspace M a pair of linear relations T_M and $\widetilde{T_M}$ called the diagonal of T and we study the relationships between the spectral and Fredholm properties of T and those of the pair T_M and $\widetilde{T_M}$.

2 Preliminary

We begin by recalling some basics from the theory of linear relations in normed linear spaces. We adhere to the notations and terminology of the monographs [5, 9].

Let X, Y and Z be complex normed linear spaces. A linear relation T from X to Y is a mapping from a subspace $D(T) = \{x \in X : Tx \neq \emptyset\}$, called the domain of T, into the collection of nonempty subsets of Y verifying $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2$ for all non zero scalars α, β and vectors $x_1, x_2 \in D(T)$. We denote by $LR(X, Y)$ the class of all linear relations everywhere defined from X into Y and we write $LR(X) := LR(X, X)$. If T maps the points of its domain to singletons, then T is said to be an operator, that is equivalent to $T(0) = \{0\}$. The class of linear bounded operators is denoted by $L(X, Y)$ and we write $L(X) := L(X, X)$. A linear relation $T \in LR(X, Y)$ is uniquely determined by its graph defined by $G(T) := \{(x, y) \in X \times Y : x \in D(T), y \in Tx\}$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) := \{(y, x) \in Y \times X : (x, y) \in G(T)\}$. The linear subspaces $N(T) := T^{-1}(0) = \{x \in D(T) : 0 \in Tx\}$ and $R(T) := T(D(T)) := \{y : (x, y) \in G(T) \text{ for some } x \in D(T)\}$ are called the nullspace (or the kernel) and the range space of T respectively. We say that T is injective if $N(T) = \{0\}$ and surjective (or onto) if $R(T) = Y$. If T is injective and surjective then T is said to be bijective. The quantities $\alpha(T) = dim N(T), \beta(T) = codim R(T) = dim Y/R(T)$ are called the nullity and the conullity of T respectively, and the index of T is defined by $i(T) = \alpha(T) - \beta(T)$ provided that $\alpha(T)$ and $\beta(T)$ are not both infinite. If $\alpha(T)$ and $\beta(T)$ are both infinite, then T is said to have no index.

For $T, S \in LR(X,Y)$ and $\lambda \in \mathbb{C}$, the linear relations $T+S, \lambda T, T-\lambda I$ are respectively defined by: $G(T+S) := \{(x, y+z) \in X \times Y : (x,y) \in G(T) \text{ and } (x,z) \in G(S)\}$; $G(\lambda T) := \{(x, \lambda y) \in X \times Y : (x,y) \in G(T)\}$ and $G(T-\lambda I) := \{(x, y-\lambda x) \in X \times X : (x,y) \in G(T)\}$. For $T \in LR(X,Y)$ and $S \in LR(Y,Z)$, the product linear relation $ST \in LR(X,Z)$ is defined by

$$G(ST) := \{(x,z) \in X \times Z : (x,y) \in G(T) \text{ and } (y,z) \in G(S) \text{ for some } y \in Y\}.$$

A linear operator A is called a selection of T if $T = A + T - T$ and $D(A) = D(T)$. For a given closed subspace E of X, let Q_E^X or simply Q_E denote the quotient map with domain X onto X/E. We shall denote $Q_{\overline{T(0)}}$ by Q_T for a given linear relation $T \in LR(X,Y)$. It is easy to see that $Q_T T$ is an operator and thus we can define $\|Tx\| := \|Q_T T x\|$ $x \in D(T)$ and $\|T\| := \|Q_T T\|$. We say that T is continuous if $\|T\| < \infty$, bounded if it is continuous and everywhere defined, open if its inverse is continuous, and bounded below if it is injective and open. The class of all bounded linear relations will be denoted by $BR(X,Y)$. A linear relation T is said to be *closed* if its graph is a closed subspace of $X \times Y$. The class of all closed linear relations will be denoted by $CR(X,Y)$. It is well known that T^{-1} is closed if and only if T is closed if and only if $Q_T T$ is a closed operator and $T(0)$ is a closed subspace. If T is closed, then $N(T)$ is closed. It is also known that if X and Y are Banach spaces and T is closed with $D(T) = X$, then T is bounded. This suggests the following notation

$$BCR(X,Y) := \{T \in LR(X,Y) : T \text{ is closed }\}.$$

An everywhere defined closed linear relation in a Banach space X is said to be invertible if T^{-1} is a bounded operator, or equivalenty, if T is injective and onto. Let T be a closed linear relation. The resolvent set of T is defined by

$$\rho(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective and onto}\}$$

and $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the spectrum of T. For $T \in LR(X)$, the kernels and the ranges of the iterates T^n, $n \in \mathbb{N}$, form two increasing and decreasing chains respectively, i.e., the chain of kernels

$$N(T^0) = \{0\} \subset N(T) \subset N(T^2) \subset .....$$

and the chain of ranges

$$R(T^0) = X \supset R(T) \supset R(T^2) \supset .....$$

The ascent and the descent of a linear relation T are respectively defined by:

$$asc(T) := \min\{k \in \mathbb{N} : \ N(T^k) = N(T^{k+1})\},$$

$$des(T) := \min\{k \in \mathbb{N} : \ R(T^k) = R(T^{k+1})\}.$$

If no such k exists, then $asc(T) = \infty$, respectively $des(T) = \infty$.
The singular chain manifold $R_c(T)$ of T is defined as:

$$R_c(T) := (\bigcup_{n=1}^{\infty} N(T^n)) \cap (\bigcup_{n=1}^{\infty} T^n(0)).$$

We say that a closed linear relation T is upper semi-Fredholm if $R(T)$ is closed and $\alpha(T)$ is finite, T is lower semi Fredholm linear relation if its range is closed and has a finite codimensional and T is Fredholm if it is both upper and lower semi Fredholm linear relation.

The sets of upper, lower semi Fredholm and Fredholm linear relations are denoted respectively by :

- $\phi_+(X,Y) = \{T \in CR(X,Y)$ such that $R(T)$ is closed and $\alpha(T)$ is finite$\}$,
- $\phi_-(X,Y) = \{T \in CR(X,Y)$ such that $R(T)$ is closed and $\beta(T)$ is finite$\}$,
- $\phi(X,Y) = \{T \in CR(X,Y)$ such that $R(T)$ is closed and $\alpha(T)$ and $\beta(T)$ are finite$\}$.

The essential (Fredholm) spectrum $\sigma_e(T)$ is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \ is \ not \ Fredholm\}.$$

In [6], F. Fakhfakh and M. Mnif established a relationship between the conullity, $\beta(T)$, and the closure of the range $R(T)$.

Lemma 2.1. [6, Corollary 3.1] Let X and Y be two Banach spaces and $T \in CR(X,Y)$. If $\beta(T)$ is finite then $R(T)$ is closed.

In the following lemma, we recall somme results related the ascent and the descent of a linear relation.

Lemma 2.2. [9] Let T be a linear relation in a linear space X.
(1) If $asc(T) < \infty$ and $R_c(T) = \{0\}$, then $\alpha(T) \leqslant \beta(T)$.
(2) If $D(T) = X$, $des(T) < \infty$, then $\beta(T) \leqslant \alpha(T)$.
(3) If $R_c(T) = \{0\}$, $D(T) = X$, $asc(T) < \infty$ and $des(T) < \infty$, then $\alpha(T) = \beta(T)$.
(4) If $\rho(T) \neq \varnothing$ then $R_c(T) = \{0\}$.

Definition 2.1. Let X, Y be two Banach spaces. A closed linear relation $T \in CR(X,Y)$ is called Weyl if it is Fredholm of index zero and is called Browder if it is Fredholm of index zero and has finite ascent and descent. The Browder spectrum $\sigma_b(T)$, and the Weyl spectrum $\sigma_w(T)$ of T are the sets

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : \lambda - T \ is \ not \ Browder\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : \lambda - T \ is \ not \ Weyl\}.$$

3 The invariant subspace problem

This section explores the question: Under which sufficient conditions a linear relation admits an invariant subspace ? But to carry out this question, we first give an overview of the concept of invariance in the context of linear relations.

Definition 3.1. [10, Section 1][3, Definition 2.1] Let $T \in LR(X)$. A subspace M of X is said to be $T-$invariant if $T(M) \subset M$. M is said to be weakly invariant under T if for all $x \in M$ we have $T(x) \cap M \neq \emptyset$, and is called nontrivial if $\{0\} \neq M \neq X$ and $M \neq T(0)$.

A closed subspace M of X is said to be strongly invariant under T with nonempty resolvent set $\rho(T)$ if for all $\lambda \in \rho(T)$ we have $(\lambda - T)^{-1}(M) \subset M$.

Now, we give the meaning of commutativity for linear relations.

Definition 3.2. Let T, $S \in LR(X)$. We say that S left commutes with T if $G(ST) \subset G(TS)$, and S right commutes with T if $G(TS) \subset G(ST)$. We define the sets of left commutants and right commutants of T denoted respectively by $comm_L(T)$ and $comm_R(T)$ as follows: $comm_L(T) = \{S \in LR(X) : G(ST) \subset G(TS)$ and $D(S) = D(T)\}$ and $comm_R(T) = \{S \in LR(X) : G(TS) \subset G(ST)$ and $D(S) = D(T)\}$.

In this section we propose some theorems which aim to respond favorably but partially to the question of the extension of Lomonosov's theorem to the case of linear relations.

Notation: In the following we will use the above notation:
$LR_0(X,Y) := \{T \in LR(X,Y); T \text{ is bounded and } \dim T(0) < \infty\}$ and $LR_0(X) = LR_0(X,X)$.

Lemma 3.1. If $S \in LR_0(X,Z)$ and $T \in LR_0(Z,Y)$, then $TS \in LR_0(X,Y)$.

Proof: Since T and S are bounded relations, then by [5, Corollary II.3.13], TS is also a bounded relation. We claim that $\dim TS(0) < \infty$. Indeed, let $(e_1, ..., e_n)$ be a basis of $S(0)$ and let $y \in TS(0)$. Then there exist $\alpha_1, ..., \alpha_n \in \mathbb{K}$ such that $y \in \alpha_1 T(e_1) + ... + \alpha_n T(e_n)$. Let $u_1, ..., u_n$ be vectors such that $u_i \in T(e_i)$. Then we have $y \in \alpha_1 u_1 + ... + \alpha_n u_n + T(0)$. If $(v_1, ..., v_m)$ is a basis of $T(0)$, then $TS(0)$ is generated by the family $(u_1, ..., u_n, v_1, ..., v_m)$ which is finite dimension.□

Proposition 3.1. [5, Proposition I.5.2] Let X and Y be two Banach spaces and let $T \in LR(X,Y)$. If $T(0)$ is topologically complemented in Y with topological complement $T(0)^{\oplus}$, then, T has a selection A such that $R(A) \subset T(0)^{\oplus}$.

In the following, for every $T \in LR_0(X)$, we denote by A_T the selection of T such that $R(A_T) \subset T(0)^{\oplus}$, where $T(0)^{\oplus}$ is a topological complement of $T(0)$.

Lemma 3.2. [11, Lemma 3.3] Let $T \in LR_0(X)$. Then, T is compact if and only if the operator A_T is compact.

Lemma 3.3. Let $T \in LR_0(X)$. If A_T has a nontrivial invariant subspace then T has a nontrivial weakly invariant subspace.

Proof: Let M be an A_T-invariant nontrivial subspace. As we have $R(A_T) \subset T(0)^{\oplus}$ then, $M \neq T(0)$. Moreover, let $x \in M$. We have $Tx = A_T x + T(0)$. So, $Tx \cap M \neq \emptyset$. Hence, M is a nontrivial weakly invariant subspace under T. □

Lemma 3.4. [11, Lemma 4.1] Let X be a Banach space and let $S, T \in LR_0(X)$. We consider P_{ST} the projection of X onto $ST(0)^{\oplus}$ with kernel $ST(0)$. Then for all $x \in X$ we have,
(i) $A_{ST}(x) = P_{ST} A_S A_T(x)$;
(ii) If $S(0) = ST(0)$ then $A_S A_T(x) = A_{ST}(x)$.

Theorem 3.1. Let X be a Banach space. We consider T a compact operator and $S \in LR_0(X)$. Suppose that $S \in comm_R(T)$ and that $T(S(0)^{\oplus}) \subset S(0)^{\oplus}$. Then we have,
(i) $TA_S = A_S T$.
(ii) There exists a nontrivial weakly invariant subspace M under S.

Proof: (i) Let $x \in D(S) = X$ and $y = TA_S x$. Then, $y \in STx$. Now, as A_S is a selection of S then,

$$TA_S x \in A_S Tx + S(0).$$

Therefore, there is $\beta \in S(0)$ such that $TA_S x = A_S Tx + \beta$. But, $TA_S x - A_S Tx \in S(0)^{\oplus}$. Then, $TA_S x = A_S Tx$.

(ii) Since T is a compact operator, A_S is a bounded operator and we have $TA_S = A_S T$, then by Lomonosov's theorem, A_S has a nontrivial invariant subspace.Therefore, using Lemma 3.3, the relation S has a nontrivial weakly invariant subspace. □

Theorem 3.2. Let X be a Banach space and let $S, T \in LR_0(X)$. Suppose that S and T commute, that is, $ST = TS$, and suppose that $T(0) \subset N(S)$ and $S(0) \subset N(T)$.

If T is compact then, there exists a nontrivial weakly invariant subspace under S.

Proof: Let $S, T \in LR_0(X)$ be satisfying the assumption. Then, it follows from Lemma 3.4 that for all $x \in X$ we have:

$$ST(x) = ST(0) + A_S A_T(x) \text{ and } TS(x) = TS(0) + A_T A_S(x).$$

Since $TS = ST$, then $ST(0)+A_S A_T(x) = TS(0)+A_T A_S(x)$. Hence, $A_S A_T(x)-A_T A_S(x) \in S(0)$. But as $S(0) = T(0)$, then we can take $S(0)^{\oplus} = T(0)^{\oplus}$ and therefore $A_S A_T(x)-A_T A_S(x) \in S(0)^{\oplus}$. So, $A_S A_T(x) - A_T A_S(x) = 0$. Thus, $A_S A_T = A_T A_S$. Then by Lomonosov's theorem, A_S has a nontrivial invariant subspace. Therefore, the relation S has a nontrivial weakly invariant subspace. □

Theorem 3.3. Let X be a Banach space and let $T \in LR(X)$ with nonempty resolvent set $\rho(T)$ and $K \in LR(X)$ be such that K is compact. Suppose further that $TK = KT$ and $K(0) \subset N(T)$. Then, T has a strongly invariant subspace.

Proof: Let $\lambda_0 \in \rho(T)$. Since $TK = KT$, it follows that $K(\lambda_0-T)^{-1} = (\lambda_0-T)^{-1}K$. Indeed, $TK = KT$ implies that $(\lambda_0 - T)^{-1}(\lambda_0 - T)K = (\lambda_0 - T)^{-1}K(\lambda_0 - T)$. Then, $K(\lambda_0 - T)^{-1} = (\lambda_0 - T)^{-1}K(\lambda_0 - T)(\lambda_0 - T)^{-1}$. So, $K(\lambda_0 - T)^{-1} = (\lambda_0 - T)^{-1}K(I + T(0)) = (\lambda_0 - T)^{-1}K + (\lambda_0 - T)^{-1}KT(0)$. Since $TK = KT$ and $K(0) \subset N(T)$ we get that $K(\lambda_0 - T)^{-1} = (\lambda_0 - T)^{-1}K$. Lomonosov's Theorem implies that $(\lambda_0 - T)^{-1}$ has a nontrivial hyperinvariant subspace M, i.e. for all bounded operator S verifying $S(\lambda_0 - T)^{-1} = (\lambda_0 - T)^{-1}S$ we have $S(M) \subset M$. But, for all $\lambda \in \rho(T)$, we have

$$(\lambda - T)^{-1}(\lambda_0 - T)^{-1} = (\lambda_0 - T)^{-1}(\lambda - T)^{-1}.$$

Then, for all $\lambda \in \rho(T)$ we have $(\lambda-T)^{-1}M \subset M$. Therefore, we get that M is a strongly invariant subspace under T. □

4 Properties of T related to those of the diagonal of T

This section explores the question: How to define a pair of linear relations T_1 and T_2 that play the role of the relations PTP and $(I - P)T(I - P)$ given in (1.1) ?
Let T be a linear relation and let M be a closed subspace of X. We say that T and M satisfy the condition $(\mathcal{H})$ if we have

$$(\mathcal{H}) : \begin{cases} \bullet\ T \in BCR(X), \\ \bullet\ M \text{ is a closed weakly invariant subspace under } T \text{ satisfying the condition (c):} \\ \qquad\qquad x \in M \text{ if and only if } Tx \cap M \neq \varnothing, \\ \bullet\ M + T(0) \text{ is closed.} \end{cases}$$

In the sequel of this section we suppose that T and M satisfy the condition $(\mathcal{H})$. We denote by T_M the relation defined by

$$G(T_M) = G(T) \cap (M \times M). \tag{4.1}$$

It is clear that $T_M \in BCR(M)$. Now, we denote by $\widetilde{T_M}$ the linear relation induced by T on X/M and defined by

$$\begin{aligned} \widetilde{T_M} : X/M &\longrightarrow X/M \\ \overline{x} &\longmapsto \widetilde{T_M}\overline{x} = \{\overline{y} : y \in Tx\}. \end{aligned} \tag{4.2}$$

The linear relation $\widetilde{T_M}$ is well defined. Indeed, for $\overline{x_1} = \overline{x_2}$, we prove that $\widetilde{T_M}\overline{x_1} = \widetilde{T_M}\overline{x_2}$. Let $\overline{y} \in \widetilde{T_M}\overline{x_1}$. Then $y \in Tx_1$. On the other hand, let $z \in Tx_2$. Then, $y - z \in T(x_1 - x_2)$. As

$x_1 - x_2 \in M$, we get $T(x_1 - x_2) \cap M \neq \varnothing$. Set $\alpha_m \in T(x_1 - x_2) \cap M$. Thus, $y - z - \alpha_m \in T(0)$. Therefore, $y - \alpha_m \in Tx_2$. Hence $\overline{y - \alpha_m} = \overline{y} \in \widetilde{T_M}(\overline{x_2})$. This shows that $\widetilde{T_M}\overline{x_1} = \widetilde{T_M}\overline{x_2}$.

We claim that $\widetilde{T_M}$ is a closed relation. Indeed, let $(\overline{x_n}, \overline{y_n}) \in G(\widetilde{T_M})$ such that $(\overline{x_n}, \overline{y_n}) \xrightarrow[n\to+\infty]{} (\overline{x}, \overline{y})$. Therefore, for all $n \in \mathbb{N}$, there exists α_n, β_n and γ_n in M such that $x_n - \alpha_n \xrightarrow[n\to+\infty]{} x$, $y_n - \gamma_n \xrightarrow[n\to+\infty]{} y$ and $y_n - \beta_n \in T(x_n - \alpha_n)$. Let $z \in Tx$. We have, $y_n - \beta_n - z \in T(x_n - \alpha_n - x)$. So, $\|T(x_n - \alpha_n - x)\| = d(y_n - \beta_n - z, T(0)) \leqslant \|T\| \, \|x_n - \alpha_n - x\|$. Thus, there exists $\alpha'_n \in T(0)$ such that $\lim_{n\to+\infty} y_n - \beta_n - z - \alpha'_n = 0$. Hence, $\lim_{n\to+\infty} y_n - \gamma_n - y_n + \beta_n + z + \alpha'_n = y$. Therefore, $y - z = \lim_{n\to+\infty} \beta_n - \gamma_n + \alpha'_n$. Since, $M + T(0)$ is closed, then $y - z \in M + T(0)$ and so, $y \in M + Tx$. Thus, there exists $m \in M$ such that $y - m \in Tx$. This implies that $\overline{y} \in \widetilde{T_M}\overline{x}$. So $(\overline{x}, \overline{y}) \in G(\widetilde{T_M})$ and $\widetilde{T_M}$ is closed as claimed. Now, as $\widetilde{T_M}$ is everywhere defined then it is bounded.

The pair $(T_M, \widetilde{T_M})$ is called the generalized diagonal of the relation T. This appellation is justified by the fact that when, in addition we have, M is $T-$invariant and complemented we can see that T_M is nothing other than the relation PTP and $\widetilde{T_M}$ is a bounded operator similar to the operator $(I - P)T(I - P)$ given in (1.1), in the following sense: There exists an invertible linear operator $V \in B(X/M, N(P))$ such that $(I - P)T(I - P) = V \, \widetilde{T_M} \, V^{-1}$. To see this we just have to consider the operator

$$\begin{array}{rcl} V : X/M & \longrightarrow & N(P) \\ \overline{x} & \longmapsto & (I - P)x. \end{array}$$

Remark 4.1. The condition (c) is equivalent to $T^{-1}(M) = M$.

Indeed, giving T and M satisfying the condition (c). Let $x \in T^{-1}(M)$. Then, $Tx \cap M \neq \varnothing$. Thus, $x \in M$. Therefore $T^{-1}(M) \subset M$. For the reverse inclusion, let $x \in M$. Then, $Tx \cap M \neq \varnothing$. So $x \in T^{-1}(M)$. This shows that $T^{-1}(M) = M$. For the opposite way, let $x \in M$. Then, $x \in T^{-1}(M)$. Thus, $Tx \cap M \neq \varnothing$. On the other hand, let x be such that $Tx \cap M \neq \varnothing$. Then, $x \in T^{-1}(M)$. So $x \in M$. □

Lemma 4.1. Let T and M be satisfying the condition $(\mathcal{H})$ and let T_M, $\widetilde{T_M}$ be defined as in (4.1) and (4.2). Then, we have
(1) $N(T_M) = N(T) \cap M = N(T)$ and $R(T_M) = R(T) \cap M$.
(2) $N(\widetilde{T_M}) = \dfrac{N(T) + M}{M} = \{0\}$ and $R(\widetilde{T_M}) = \dfrac{R(T) + M}{M}$.

Proposition 4.1. Let T and M be satisfying the condition $(\mathcal{H})$ and let T_M, $\widetilde{T_M}$ be defined as in (4.1) and (4.2).
If $T \in \phi(X)$, then $T_M \in \phi_+(M)$ and $\widetilde{T_M} \in \phi_-(X/M)$.

Proof: As T and M satisfy the condition $(\mathcal{H})$ then $N(T_M) = N(T)$. Since, $T \in \phi(X)$ we get that $\alpha(T)$ is finite. Therefore, $dim \, N(T_M)$ is finite. On the other hand, since $R(T)$ is closed in X and M is a closed subspace of X we get that $R(T_M)$ is closed in M. Thus, $T_M \in \phi_+(M)$. We claim now that $\widetilde{T_M} \in \phi_-(X/M)$. Indeed, since $T \in \phi(X)$ we get that $\beta(T) = dim(X/R(T))$ is finite. Therefore, there exists a finite dimentional subspce X_0 such that $R(T) \oplus X_0 = X$. Then $R(\widetilde{T_M}) + \dfrac{X_0 + M}{M} = \dfrac{R(T) + X_0 + M}{M} = \dfrac{X}{M}$. This verifies that $\beta(\widetilde{T_M})$ is finite. Using Lemma 2.1 we get that $R(\widetilde{T_M})$ is closed. Therefore, $\widetilde{T_M} \in \phi_-(X/M)$. □

Proposition 4.2. Let T and M be satisfying the condition $(\mathcal{H})$ and let T_M, $\widetilde{T_M}$ be defined as in (4.1) and (4.2). Then,
(1) $T(M)\cap M = R(T)\cap M$.
(2) $\alpha(\widetilde{T_M}) = 0$ and $\alpha(T_M) = \alpha(T)$.
(3) Assume that $\beta(T_M)$ and $\beta(\widetilde{T_M})$ are finite. Then,

$$\beta(T) = \beta(T_M) + \beta(\widetilde{T_M}).$$

Proof: (1) Clearly that $T(M)\cap M \subset R(T)\cap M$. Let $y \in R(T)\cap M$.Then, there exists $x \in X$ such that $y \in Tx \cap M$. Hence, $x \in M$. Thus $y \in T(M)\cap M$.

(2) By Lemma 4.1, we have $N(\widetilde{T_M}) = \{0\}$. Hence, $\alpha(\widetilde{T_M}) = 0$. On the other hand, by Remark 4.1, we have $T^{-1}(M) = M$. Thus $N(T) \subset M$. Therefore $N(T_M) = N(T)\cap M = N(T)$. Thus, $\alpha(T_M) = \alpha(T)$.

(3) Since $R(T_M)$ has a finite codimension in M, then there exists a finite dimensional subspace M_0 such that $M = R(T_M)\oplus M_0$. Let $\{m_1, m_2, ..., m_n\}$ be a base of M_0. Similary, since $R(\widetilde{T_M})$ has finite codimension in X/M, we can choose a linearly independant set $\{x_1+M, x_2+M, ..., x_{n'}+M\}$ in X/M such that $X/M = R(\widetilde{T_M}) \oplus vect\{x_1 + M, x_2 + M, ..., x_{n'} + M\}$. We claim now that $X = R(T) + vect\{x_1, x_2, ..., x_{n'}, m_1, m_2, ..., m_n\}$. Let $x \in X$. Then, since $R(\widetilde{T_M}) = \dfrac{R(T)+M}{M}$ and $X/M = R(\widetilde{T_M})\oplus vect\{x_1+M, x_2+M, ..., x_{n'}+M\}$ there exist $y_1 \in R(T)$ and scalars λ_k such that $x + M = y_1 + \sum_{i=1}^{n'} \lambda_i x_i + M$. Therefore, $y_1 + \sum_{i=1}^{n'} \lambda_i x_i - x \in M$. So, there existe $y_2 \in R(T_M)$ and scalars μ_k such that $y_1 + \sum_{i=1}^{n'} \lambda_i x_i - x = y_2 + \sum_{i=1}^{n} \mu_i m_i$. Thus, $x = y_1 - y_2 + \sum_{i=1}^{n'} \lambda_i x_i - \sum_{i=1}^{n} \mu_i m_i$. It follows that $\beta(T) \leqslant \beta(T_M) + \beta(\widetilde{T_M})$.
Now we claim that $R(T) \cap vect\{x_1, x_2, ..., x_{n'}, m_1, m_2, ..., m_n\} = \{0\}$. Let y be an element of the intersection. Then there exist $x \in X$, $(\lambda_k)_{1\leqslant k\leqslant n'}$ and $(\mu_k)_{1\leqslant k\leqslant n}$ such that $y = \sum_{k=1}^{n'} \lambda_k x_k + \sum_{k=1}^{n} \mu_k m_k \in Tx$. Therefore, $\overline{y} = \sum_{k=1}^{n'} \lambda_k \overline{x_k} \in X/M$. On the other hand, as $y \in Tx$ then $\overline{y} \in R(\widetilde{T_M})$. But $R(\widetilde{T_M}) \cap vect\{\overline{x_1}, \overline{x_2}, ..., \overline{x_{n'}}\} = \{\overline{0}\}$, then $\overline{y} = \overline{0}$ and hence $\lambda_1 = \lambda_2 = ... = \lambda_{n'} = 0$. So, $y \in Tx \cap M \subset R(T)\cap M = T(M)\cap M$. Then, there exists $m \in M$ such that $y \in T(m)$. Hence, $y \in R(T_M)\cap M_0 = \{0\}$ as claimed. This entails that $\beta(T) = \beta(T_M) + \beta(\widetilde{T_M})$. □

In the following theorem we prove that T is Fredholm if and only if T_M and $\widetilde{T_M}$ are Fredholm.

Theorem 4.1. Let T and M be satisfying the condition $(\mathcal{H})$ and let T_M, $\widetilde{T_M}$ be defined as in (4.1) and (4.2). Then, the following assertions are equivalent.
(1) $T \in \phi(X)$;
(2) $T_M \in \phi(M)$ and $\widetilde{T_M} \in \phi(X/M)$.

Proof: (1)⇒(2) By Proposition 4.1 it remains to prove that $\beta(T_M) < \infty$ and that $\alpha(\widetilde{T_M}) < \infty$. But, $N(\widetilde{T_M}) = \{0\}$ then $\alpha(\widetilde{T_M}) = 0 < \infty$. For the proof of $\beta(T_M) < \infty$, let consider $m_1, m_2, ..., m_n \in M$ be such that $\{\overline{m_1}, ..., \overline{m_n}\}$ is linearly independant in $\dfrac{M}{R(T_M)} = \dfrac{M}{R(T)\cap M}$.

We claim that $\{\overline{\overline{m_1}}, ..., \overline{\overline{m_n}}\}$ is linearly independant in $X/R(T)$. Indeed, let $\alpha_1, ..., \alpha_n$ be such that $\alpha_1\overline{\overline{m_1}}+...+\alpha_n\overline{\overline{m_n}} = \overline{0}$. Then, $\alpha_1 m_1+...+\alpha_m m_n \in R(T)$. Hence, $\alpha_1 m_1+...+\alpha_m m_n \in R(T)\cap M$. Therefore, $\alpha_1\overline{m_1}+...+\alpha_m\overline{m_n} = \overline{0}$. So, $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. Thus, $n \leqslant \beta(T)$. This entails that $\beta(T_M) \leqslant \beta(T)$.

(2)⇒(1) Assume that (2) holds. Then, we have $\alpha(T_M)$, $\alpha(\widetilde{T_M})$, $\beta(T_M)$ and $\beta(\widetilde{T_M})$ are all finite. By Proposition 4.2, it follows that $\alpha(T)$ and $\beta(T)$ are finite. Now the desired result follows from Lemma 2.1. □

Lemma 4.2. Let $T \in LR(X,Y)$ and $x \in D(T)$. If T is continuous then for all $y \in Tx$ and for all $(x_n)_{n\in\mathbb{N}} \subset D(T)$ such that $\lim\limits_{n\to+\infty} x_n = x$ there exists $\rho(n)$ such that $y_{\rho(n)} \in Tx_{\rho(n)}$ and $y_{\rho(n)} \xrightarrow[n\to+\infty]{} y$.

Proof: Let $T \in LR(X,Y)$ be continuous and let $(x_n)_{n\in\mathbb{N}} \subset D(T)$ be such that $\lim\limits_{n\to+\infty} x_n = x$. Let $y \in Tx$. For all neighbourhood V of $y \in Tx$, there exists a neighbourhood U of x such that for all $x \in U \cap D(T)$ we have $Tx \cap V \neq \varnothing$. In particular, $B(y,1)$ is a neighbourhood of y, therefore there exists a neighbourhood U_1 of x such that for all $x \in U_1 \cap D(T)$, $Tx \cap B(y,1) \neq \varnothing$. Then, there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$, $x_n \in U_1 \cap D(T)$. Thus, $T(x_{n_1}) \cap B(y,1) \neq \varnothing$. Pick $y_{n_1} \in T(x_{n_1}) \cap B(y,1)$. For $n = 2$ we have $B(y, \frac{1}{2})$ is a neighbourhood of y, then there exists U_2 a neighbourhood of x such that for all $x \in U_2 \cap D(T)$ we have $Tx \cap B(y, \frac{1}{2}) \neq \varnothing$. There exists $n_2 > n_1$ such that for all $n \geqslant n_2$, $x_n \in U_2 \cap D(T)$. Therefore, $T(x_{n_2}) \cap B(y, \frac{1}{2}) \neq \varnothing$. Pick $y_{n_2} \in T(x_{n_2}) \cap B(y, \frac{1}{2})$ then $\|y_{n_2} - y\| \leqslant \frac{1}{2}$. By iteration for $n \geqslant 2$, $B(y, \frac{1}{n})$ is a neighbourhood of y so there exists $x_{n_n} \in U_n \cap D(T)$ such that $T(x_{n_n}) \cap B(y, \frac{1}{2}) \neq \varnothing$. Pick $y_{n_n} \in T(x_{n_n}) \cap B(y, \frac{1}{n})$. We set $\rho(n) = n_n$. We have $y_{\rho(n)} \in Tx_{\rho(n)}$ and $y_{\rho(n)} \in B(y, \frac{1}{n})$. Then, $\|y_{\rho(n)} - y\| \leqslant \frac{1}{n}$. Thus, $y_{\rho(n)} \xrightarrow[n\to+\infty]{} y$. □

Lemma 4.3. Let $T \in BCR(X)$ and define $\overline{R^\infty}(T) := \bigcap\limits_{n\in\mathbb{N}} \overline{R(T^n)}$. Then, $T(\overline{R^\infty}(T)) \subset \overline{R^\infty}(T)$ and for all $\lambda \in \mathbb{C}$, $(\lambda - T)(\overline{R^\infty}(T)) \subset \overline{R^\infty}(T)$.

Proof: Without loss of generality we can suppose that $\lambda = 0$. Let $n \in \mathbb{N}^*$ and $y \in T(\overline{R(T^n)})$. We claim that $y \in \overline{R(T^n)}$. Indeed, pick $x \in \overline{R(T^n)}$ such that $y \in Tx$ and let $(x_p)_{p\in\mathbb{N}} \subset R(T^n)$ such that $x_p \xrightarrow[p\to+\infty]{} x$. Since $T \in BCR(X)$, T is closed and everywhere defined on the Banach space X hence by the closed graph theorem for linear relations (Cross, Theorem III.4.1 [5]), T is continuous. Then by Lemma 4.2, there exists $\rho(p)$ and $y_{\rho(p)} \in Tx_{\rho(p)}$ such that $y_{\rho(p)} \xrightarrow[p\to+\infty]{} y$. Thus, $y \in \overline{R(T^n)}$. Now, we have $T(\overline{R^\infty}(T)) \subset \bigcap\limits_{n\in\mathbb{N}} T(\overline{R(T^n)})$. Then, $T(\overline{R^\infty}(T)) \subset \bigcap\limits_{n\in\mathbb{N}} \overline{R(T^n)} = \overline{R^\infty}(T)$. □

Lemma 4.4. Let $\lambda \in \mathbb{C}^*$. Then, we have:

$$(\lambda - T)x \cap \overline{R^\infty}(T) \neq \varnothing \;\Rightarrow\; x \in \overline{R^\infty}(T).$$

Proof: Let $y_1 \in (\lambda - T)x \cap \overline{R^\infty}(T)$. Then, $\lambda x - y_1 \in T(x)$. We claim that for all $n \geqslant 1$, there exists $y_n \in \overline{R^\infty}(T)$ such that $\lambda^n x - y_n \in T^n x$. Indeed, we prove this by induction.

For $n = 1$, the result holds. Now, suppose that for some $n \geqslant 1$, $\lambda^n x - y_n \in T^n x$ for some $y_n \in \overline{R^\infty}(T)$. Then, $\lambda^n Tx - Ty_n \subset T^{n+1}x$. Therefore, by the result for $n = 1$, we can see that $\lambda^n(\lambda x - y_1 + T(0)) - Ty_n \subset T^{n+1}x$. Thus, $\lambda^{n+1}x - \lambda^n y_1 - Ty_n \subset T^{n+1}x$. Let $z_n \in Ty_n$. Then, $\lambda^{n+1}x - y_{n+1} \subset T^{n+1}x$ with $y_{n+1} = \lambda^n y_1 - z_n$. By Lemma 4.3, we get $y_{n+1} \in \overline{R^\infty}(T)$ as desired. Now, as $\lambda \neq 0$, then $x \in \overline{R(T^n)}$ for all $n \geqslant 1$. Therefore $x \in \overline{R^\infty}(T)$. □

We now prove the following inequalities linking the ascent and the descent of T, T_M and $\widetilde{T_M}$ for all linear relation T and subspace M satisfying the condition $(\mathcal{H})$:

Proposition 4.3. Let T and M be satisfying the condition $(\mathcal{H})$. Then the following properties hold.

$$asc(T_M) \leqslant asc(T) \leqslant asc(T_M) + asc(\widetilde{T_M}),$$
$$des(T_M) \leqslant des(T) \leqslant des(T_M) + des(\widetilde{T_M}).$$

Proof:
Let $n \geqslant asc(T)$. Using that $N((T_M)^n) = N(T^n) \cap M$, for all $n \geq 1$, we deduce that $N((T_M)^{n+1}) = N((T_M)^n)$. This proves that $asc(T_M) \leqslant asc(T)$.

Let $n \geqslant asc(T_M) + asc(\widetilde{T_M})$. We have that $n \geqslant asc(T_M)$ and $n \geqslant asc(\widetilde{T_M})$. So, $N((T_M)^n) = N((T_M)^{n+1})$ and $N((\widetilde{T_M})^n) = N((\widetilde{T_M})^{n+1})$. Hence, $N(T^n) \cap M = N(T^{n+1}) \cap M$ and $\dfrac{N(T^n) + M}{M} = \dfrac{N(T^{n+1}) + M}{M}$. We claim that $N(T^{n+1}) = N(T^n)$. Suppose, on the contrary, that $N(T^n) \subsetneq N(T^{n+1})$. Then, there exists $x \in N(T^{n+1})$ such that $x \notin N(T^n)$. On another hand we have $\overline{x} \in \dfrac{N(T^{n+1}) + M}{M} = \dfrac{N(T^n) + M}{M}$. So, $x \in N(T^n) + M$. Therefore, there exists $\alpha \in N(T^n) \subset N(T^{n+1})$ such that $x - \alpha \in M$. Hence $x - \alpha \in N(T^{n+1}) \cap M$. Therefore $x - \alpha \in N(T^n) \cap M$ and so, $x \in N(T^n)$ which is absurd. Thus, $N(T^{n+1}) = N(T^n)$ and so $asc(T) \leqslant asc(T_M) + asc(\widetilde{T_M})$.

The result for the descent can be proved by similar arguments. □

Lemma 4.5. Let T and M be satisfying the condition $(\mathcal{H})$. Then, for all $\lambda \in \mathbb{C}$ we have:

$$(\lambda - T)_M = \lambda - T_M \text{ and } \widetilde{(\lambda - T)}_M = \lambda - \widetilde{T_M}.$$

Proof: We have
$$\begin{aligned} G((\lambda - T)_M) &= G(\lambda - T) \cap (M \times M) \\ &= \{(x, \lambda x - y); (x, y) \in G(T)\} \cap (M \times M) \\ &= \{(x, \lambda x - y); (x, y) \in G(T) \cap (M \times M)\} \\ &= \{(x, \lambda x - y); (x, y) \in G(T_M)\} \\ &= G(\lambda - T_M). \end{aligned}$$
Hence, $(\lambda - T)_M = \lambda - T_M$.

We claim that $\widetilde{(\lambda - T)}_M = \lambda - \widetilde{T_M}$. Indeed, let $(\overline{x}, \overline{y}) \in G(\widetilde{(\lambda - T)}_M)$. Thus, $x = x_1 + m_x$ and $y = y_1 + m_y$ with $m_x, m_y \in M$ and $y_1 \in (\lambda - T)x_1$. Therefore $-y_1 + \lambda x_1 \in Tx_1$. Thus, $(x_1, -y_1 + \lambda x_1) \in G(T)$. Hence, $(\overline{x_1}, \overline{-y_1 + \lambda x_1}) \in G(\widetilde{T_M})$. But $(\overline{x_1}, \overline{-y_1 + \lambda x_1}) = (\overline{x}, \overline{-y + \lambda x})$. Therefore, $(\overline{x}, \overline{y}) \in G(\lambda - \widetilde{T_M})$.

For the reverse inclusion, let $(\overline{x}, \overline{y}) \in G(\lambda - \widetilde{T_M})$. Then, $\overline{y} \in (\lambda - \widetilde{T_M})\overline{x}$. So, $-\overline{y} + \lambda\overline{x} \in \widetilde{T_M}\overline{x}$. Hence, there exists $z \in X$ such that $-\overline{y} + \lambda\overline{x} = \overline{z}$ and $z \in Tx$. Thus, $z = -y + \lambda x + m$ with $m \in M$. Therefore, $-y + \lambda x + m \in Tx$. Then, $y + m \in (\lambda - T)x$. Hence, $(\overline{x}, \overline{y + m}) \in G(\widetilde{(\lambda - T)}_M)$. But, $(\overline{x}, \overline{y + m}) = (\overline{x}, \overline{y})$. Therefore, $(\overline{x}, \overline{y}) \in G(\widetilde{(\lambda - T)}_M)$. □

Now, we are able to state and prove our main result.

Theorem 4.2. Let $T \in BCR(X)$ be such that $M = \overline{R^\infty}(T) \neq X$. Let $\lambda \in \mathbb{C}^*$. Then,
(1) $\lambda - T \in \phi(X)$ if and only if $\lambda - T_M \in \phi(M)$ and $\lambda - \widetilde{T_M} \in \phi(X/M)$.
In this case we have $\alpha(\lambda - T) = \alpha(\lambda - T_M)$ and $\beta(\lambda - T) = \beta(\lambda - T_M) + \beta(\lambda - \widetilde{T_M})$.
(2) $\lambda - T$ is invertible if and only if $\lambda - T_M$ is invertible and $\lambda - \widetilde{T_M}$ is invertible.
(3) (i) If $\lambda - T_M$ and $\lambda - \widetilde{T_M}$ are Browder then $\lambda - T$ is Browder.
(ii) If moreover $\rho(T) \neq \varnothing$ then

$$\lambda - T_M \text{ and } \lambda - \widetilde{T_M} \text{ are Browder if and only if } \lambda - T \text{ is Browder.}$$

Proof: By Lemmas 4.3 and 4.4 we have, for $\lambda \neq 0$, $\lambda - T$ and $M = \overline{R^\infty}(T)$ satisfy the condition $(\mathcal{H})$. Therefore:
(1) The results follow immediately from Theorem 4.1 and Lemma 4.2.

(2) Assume that $\lambda - T$ is invertible. Then, $(\lambda - T)^{-1}$ is a bounded operator and M is a $(\lambda - T)^{-1}$-invariant. So, by [2, Proposition 4] we get that $(\lambda - T)_M^{-1}$ and $\widetilde{(\lambda - T)}_M^{-1}$ are bounded operators and we have $(\lambda - T)_M^{-1}$ is the inverse of $(\lambda - T)_M$ and $\widetilde{(\lambda - T)}_M^{-1}$ is the inverse of $\widetilde{(\lambda - T_M)}$. Therefore, by Lemma 4.2 we get that $\lambda - T_M$ and $\lambda - \widetilde{T}_M$ are invertible.

Assume now that $\lambda - T_M$ and $\lambda - \widetilde{T_M}$ are invertible. First we claim that $\lambda - T$ is surjective. Indeed, let $y \in X$. Then there exists $\overline{x} \in X/M$ such that $\overline{y} \in (\lambda - \widetilde{T_M})\overline{x}$. Thus there exists $z \in (\lambda - T)x$ such that $y - z \in M$. As $\lambda - T_M$ is surjective, then there exists $m \in M$ such that $y - z \in (\lambda - T)m$. Then, $y \in (\lambda - T)(m + x)$. So, $\lambda - T$ is surjective. Now, using Lemma 4.1 we get that $N(\lambda - T_M) = N(\lambda - T) = \{0\}$.

(3)(i) Assume that $\lambda - T_M$ and $\lambda - \widetilde{T_M}$ are Browder. Then, $\lambda - T_M \in \phi(M)$, $\lambda - \widetilde{T_M} \in \phi(X/M)$, $i(\lambda - T_M) = i(\lambda - \widetilde{T_M}) = 0$ and $asc(\lambda - T_M)$, $des(\lambda - T_M)$, $asc(\lambda - \widetilde{T_M})$, $des(\lambda - \widetilde{T_M})$ are all finite. Hence, from (1) we get that $\lambda - T \in \phi(X)$. On the other hand, by Proposition 4.3, it follows that $asc(\lambda - T)$ and $des(\lambda - T)$ are finite. Moreover, by (1) it follows that $i(\lambda - T) = i(\lambda - T_M) + i(\lambda - \widetilde{T_M})$. Therefore, $i(\lambda - T) = 0$. This shows that $\lambda - T$ is a Browder linear relation.

(ii) The first implication is given by (i). For the converse, since $\lambda - T \in \phi(X)$ then, we get by (1) that $\lambda - T_M \in \phi(M)$ and $\lambda - \widetilde{T_M} \in \phi(X/M)$. On the other hand, using the Proposition 4.3 we get that $asc(\lambda - T_M)$ and $des(\lambda - \widetilde{T_M})$ are all finite. We claim now that $i(\lambda - T_M) = i(\lambda - \widetilde{T_M}) = 0$. Indeed, since $des(\lambda - \widetilde{T_M}) < \infty$ and $\lambda - \widetilde{T_M}$ is a bounded linear relation then it follows, by Lemma 2.2, that $\beta(\lambda - \widetilde{T_M}) \leqslant \alpha(\lambda - \widetilde{T_M})$. Hence, $i(\lambda - \widetilde{T_M}) \geqslant 0$. On the other hand, we have $N(\lambda - \widetilde{T_M}) = \{0\}$ then $asc(\lambda - T_M) = 0 < \infty$. By Theorem 4.2 (2) and hypthesis we get that $\rho(\widetilde{T_M}) \neq \varnothing$. Then, it follows by (1) and (4) of Lemma 2.2 that $i(\lambda - \widetilde{T_M}) \leqslant 0$. So, $i(\lambda - \widetilde{T_M}) = 0$. Moreover, by (1) it follows that $i(\lambda - T) = i(\lambda - T_M) + i(\lambda - \widetilde{T_M})$. Thus, $i(\lambda - T_M) = 0$. Therofore, $\lambda - T_M$ is a Weyl linear relation. On the other hand, since, $\rho(T_M) \neq \varnothing$ then, $R_c(T_M) = \{0\}$. Thus, it follows by [4, Proposition 11] that $\lambda - T_M$ is Browder. □

As a direct consequence, we get the following corollary.

Corollary 4.1. Let $T \in BCR(X)$ be such that $M = \overline{R^\infty}(T) \neq X$. Then, the following properties hold.
(i) $\sigma_e(T) \setminus \{0\} = (\sigma_e(T_M) \cup \sigma_e(\widetilde{T_M})) \setminus \{0\}$;
(ii) $\sigma(T) \setminus \{0\} = (\sigma(T_M) \cup \sigma(\widetilde{T_M})) \setminus \{0\}$;

(iii) $\sigma_b(T) \setminus \{0\} \subset (\sigma_b(T_M) \cup \sigma_b(\widetilde{T_M})) \setminus \{0\}$. If moreover we suppose that $\rho(T) \neq \varnothing$ then, $\sigma_b(T) \setminus \{0\} = (\sigma_b(T_M) \cup \sigma_b(\widetilde{T_M})) \setminus \{0\}$.

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References

[1] N. Aronszajn, K.T. Smith, : *Invariant subspaces of completely continuous operators.* Ann.of Math. 60(1954), pp. 345-350.

[2] B.A. Barnes: *Spectral and Fredholm theory involving the diagonal of bounded linear operators,* Acta Math.73 (2007), pp. 237-250.

[3] A. G. Baskakov, A. S. Zagorskii: *Spectral theory of linear relations on real Banach spaces,* Matematicheskie Zametki, Vol. 81 (2007), No. 1, pp. 17-31.

[4] Y. Chamkha, M. Mnif: *Browder Spectra of Upper Triangular Matrix Linear Relations,* Publ. Math. Debrecen 82 (2013), pp. 569-590.

[5] R. W. Cross: *Multivalued linear operators,* Pure and Applied Mathematics. Marcel Dekker, New York, (1998).

[6] F. Fakhfakh, M. Mnif: *Perturbation theory of lower semi-Browder multivalued linear operators,* Publ. Math. Debrecen. 78 (2011), pp. 595-606.

[7] A. Ghorbel, M. Mnif: *Drazin inverse of multivalued operators and its applications,* Mon. Math. 189 (2019), pp. 273-293.

[8] V.I. Lomonosov: *Invariant subspaces for the family of operators which commute with a completely continuous operator,* Funct. Anal. Appl., 7 (1973), pp. 213-214.

[9] A. Sandovici, H. S. V. De Snoo and H. Winkler: *Ascent, descent, nullity, defect and related notions for linear relations in linear spaces,* Line. Alge. Appli. 423 (2007), pp. 456-497.

[10] P. Saveliev: *Lomonosov's invariant subspace theorem for multivalued linear operators,* Proc. Amer. Math. Soc.,131 (2000), No. 3, pp. 825-834.

[11] G. Wanjala: *Weakly invariant subspaces for multivalued linear operators on Banach spaces*, J. Nonlinear Sci.App.,11 (2018), pp. 877-884.

[12] D.L. Wilcox: *Multivalued semi-Fredholm Operators in Normed Linear Spaces*, Thesis, University of Cape Town, (2002).

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