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On the initial coefficient differences of Ma-Minda starlike functions and bounded turning functions

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Abstract. Let Φ be a univalent function in the unit disk $\mathbb{D}$ and let $\Phi(\mathbb{D})$ be symmetric with respect to the real axis and starlike with respect to $\Phi(0) = 1$, also $\Phi'(0) > 0$. For the given function Φ, let $\mathcal{S}^*(\Phi)$ and $\mathcal{R}(\Phi)$ be classes of Ma-Minda starlike functions and bounded turning functions, respectively. In this article, we establish the sharp lower and upper bounds of initial coefficient differences for the functions in $\mathcal{S}^*(\Phi)$ and $\mathcal{R}(\Phi)$. We also obtain the bounds for the inverse coefficients, logarithmic, and inverse logarithmic coefficients for these classes. All the results we study are sharp.

Keywords. Analytic functions, coefficient difference, Ma-Minda class, functions with positive real part

1 Introduction

Let $\mathcal{A}$ denote the class of functions analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be univalent in $\mathbb{D}$ if it is one-to-one. The class of all such functions is denoted by $\mathcal{S}$. The study of coefficient problems for functions in $\mathcal{S}$ has been a central theme in geometric function theory for more than a century, due to its deep connections with conformal mappings, extremal problems, and geometric properties of analytic functions. One of the most celebrated results in this theory is the Bieberbach conjecture, proved by de Branges, which asserts that $|a_n| \leq n,\ n \geq 2$, for all functions $f \in \mathcal{S}$, with equality attained by the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n$$

and its rotations. Following this outstanding result, it is natural to investigate coefficient functionals that measure the relative behavior of successive coefficients rather than their absolute

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size. In particular, a natural question that arises; whether the difference of successive coefficients satisfies a uniform bound, namely, $||a_{n+1}| - |a_n|| < 1, \quad n \geq 2$. Although this bound appears plausible in view of the linear growth of $|a_n|$, it was later shown that such an inequality does not hold in general for the entire class $\mathcal{S}$. In fact, Fekete and Szegö [6] was the first who gave negative answer to this assumption in 1933 by demonstrating the following sharp bound

$$-1 \leq |a_3| - |a_2| \leq \frac{3}{4} e^{-\beta} \left(2e^{-\beta} - 1\right) = 1.029,$$

where β is a unique root of the equation $e^{\beta} = 4\beta$ in $(0,1)$. Then, Hayman [8] showed that for some absolute constant $C > 1$, the following inequality $||a_{n+1}| - |a_n|| \leq C$ holds true for $f \in \mathcal{S}$. Despite the fact that Grinspan [7] calculated the best estimate of C to date, which is 3.61, the precise value of C remains unknown. This observation initiated a systematic study of coefficient differences and motivated the search for sharp bounds under additional structural assumptions or within specific subclasses of univalent functions.

Before proving the main results, we summarize the required notions and known results related to subclasses of univalent functions and various coefficient differences.

A univalent function f is said to be starlike if $f(\mathbb{D})$ is a starlike domain with respect to the origin. The class of all univalent starlike functions is denoted by $\mathcal{S}^*$. This subclass is defined analytically as follows: a function f is in $\mathcal{S}^*$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

Similarly, a univalent function f is said to have bounded turning if its derivative has positive real part. The class of all functions with bounded turning is denoted by $\mathcal{R}$ and analytically defined by the condition: a function f is in $\mathcal{R}$ if and only if

$$Re(f'(z)) > 0.$$

It is worth noting that each function $f \in \mathcal{R}$ is also close-to-convex.

In [12] it was demonstrated that $||a_{n+1}| - |a_n|| \leq 1$ is true for $n \geq 2$ when $f \in \mathcal{S}^*$. The same result for the class $\mathcal{K}$ of close-to-convex was proved by Koepf in [10]. Some results for the initial coefficient differences for some well-known subclasses of univalent functions are recently obtained by various authors, see [2, 5, 17].

The Caratheodory class, denoted by $\mathcal{P}$, is the collection of holomorphic functions p in the unit disk $\mathbb{D}$ satisfying the condition $Re\,(p(z)) > 0$ for $z \in \mathbb{D}$ and having series expansion of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.2}$$

Let $\mathcal{B}$ represent the class of all analytic (holomorphic) functions ω in $\mathbb{D}$ with the property that $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. These functions are called Schwarz functions. A number of problems in geometric function theory can be answered in an easy and precised way by using the concept of subordination. An analytic function f is said to be subordinate to an analytic function g if there exists $\omega \in \mathcal{B}$ such that $f(z) = g\left(\omega(z)\right)$ for $z \in \mathbb{D}$. If g is univalent and $f(0) = g(0)$, then $f(\mathbb{D}) \subset g(\mathbb{D})$.

By using subordination, Ma and Minda [13], defined a subclass of $\mathcal{S}^*$ denoted by $\mathcal{S}^*(\Phi)$ by using a function Φ which is univalent in $\mathbb{D}$, $\Phi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\Phi(0)=1$, and $\Phi'(0)>0$. This class is defined analytically as

$$\mathcal{S}^*(\Phi)=\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\Phi(z)\right\}.$$

Similarly, the subclass of bounded turning function $\mathcal{R}(\Phi)$ can be defined as

$$\mathcal{R}(\Phi)=\left\{f\in\mathcal{A}:f'(z)\prec\Phi(z)\right\}.$$

The function $\Phi(z)$ has a series expansion of the form

$$\Phi(z)=1+B_1z+B_2z^2+B_3z^3+\dots.$$

These classes generalize various classes of starlike and bounded turning functions respectively by taking particular function Φ (see, [9, 15, 18, 20, 21] and the references therein).

In recent years, attention has also been directed toward coefficient differences of inverse functions. If $f\in\mathcal{S}$ and its inverse f^{-1} exists in a neighborhood of the origin, then it admits the expansion

$$f^{-1}(z)=z+\sum_{n=2}^{\infty}A_nz^n \tag{1.3}$$

at least in the disk $|z|\leq\frac{1}{4}$. The behavior of inverse coefficients and their differences provides further insight into the geometric nature of univalent mappings.

Since $f(f^{-1}(z))=z$, so from (1.1) and (1.3), we have

$$A_2=-a_2 \quad\text{and}\quad A_3=2a_2^2-a_3. \tag{1.4}$$

Many authors have recently explored coefficient bounds for inverse functions (see [19, 23]). Specifically, Sim and Thomas showed that $-1\leq|A_3|-|A_2|\leq 3$ for $f\in\mathcal{S}$, [19]. They also studied the sharp bound for the same coefficient difference for the class $\mathcal{S}^*$ and a few other subclasses of univalent functions.

The logarithmic coefficients γ_n for a function $f\in\mathcal{S}$are defined as

$$\frac{F_f(z)}{2}=\frac{1}{2}\log\frac{f(z)}{z}=\sum_{n=1}^{\infty}\gamma_nz^n,\qquad z\in\mathbb{D}. \tag{1.5}$$

Now, by comparing (1.1) and (1.5), we get

$$\gamma_1=\frac{a_2}{2}\quad\text{and}\quad\gamma_2=\frac{1}{4}\left(2a_3-a_2^2\right). \tag{1.6}$$

In function theory, the logarithmic coefficients γ_n are essential. The importance of these functions was brought to light by Milin [14] in his article about the well-known Bieberbach conjecture. The study of logarithmic coefficients for univalent functions has recently attracted the attention of several authors (see [1, 22]). In 2023, Lecko and Partyka [11] used the Loewner technique to show that for $f\in\mathcal{S}$

$$-\frac{\sqrt{2}}{2}\leq|\gamma_2|-|\gamma_1|\leq\frac{1}{2}.$$

The upper bound of the result can be obtained by the function $f(z)=z/\left(1+\sigma^2z^2\right)$, where $|\sigma|=1$ and the lower bound is attained by the function $f(z)=z/\left(1-\sqrt{2}\sigma z+\sigma^2z^2\right)$.

For $f \in \mathcal{S}$, the inverse logarithmic coefficients Γ_n are give as

$$\frac{F_{f^{-1}}(z)}{2} = \frac{1}{2}\log\frac{f^{-1}(z)}{z} = \sum_{n=1}^{\infty}\Gamma_n z^n, \qquad z \in \mathbb{D}. \tag{1.7}$$

Differentiating (1.7) and by using (1.4), we have

$$\Gamma_1 = \frac{-a_2}{2}, \quad \text{and} \quad \Gamma_2 = \frac{1}{4}(-2a_3 + 3a_2^2). \tag{1.8}$$

In 2018 Ponnusamy et al. [16] established a sharp bound on the logarithmic inverse coefficients for the class of univalent functions. Allu and Shaji [3] recently demonstrated that for $f \in \mathcal{S}$,

$$\Delta \leq |\Gamma_2| - |\Gamma_1| \leq \frac{1}{2} \tag{1.9}$$

where

$$\Delta = \begin{cases} -\frac{1}{2} & \text{if} \;\; a_2 \leq 1, \\ -0.6353\ldots & \text{if} \;\; a_2 > 1. \end{cases}$$

The result is sharp for the second inequality in (1.9) for the function $f(z) = z/(1+z+z^2)$ whereas the sharpness of the first inequality is an open problem.

In this article, we present the bounds on $|a_3|-|a_2|$, $|A_3|-|A_2|$, $|\gamma_2|-|\gamma_1|$ and $|\Gamma_2|-|\Gamma_1|$ for the class $\mathcal{S}^*(\Phi)$. The similar kind of results are obtained for the class $\mathcal{R}(\Phi)$. All the results proved are best possible. Moreover, various results can be obtained by choosing particular function Φ. We will use the following lemmas to prove our general result.

Lemma 1.1. *[4] Let $p \in \mathcal{P}$. Then*

$$p_1 = 2t_1,$$

and

$$p_2 = 2t_1^2 + 2(1-|t_1|^2)t_2,$$

where $t_i \in \overline{\mathbb{D}}$ for $i \in \{1,2\}$.
If $|t_1| = 1$, then there exists a unique function $p \in \mathcal{P}$ given as

$$p(z) = \frac{1+t_1 z}{1-t_1 z}.$$

If $t_1 \in \mathbb{D}$ and $|t_2| = 1$, then there exists a unique function $p \in \mathcal{P}$ defined as

$$p(z) = \frac{1+(\overline{t_1}t_2 + t_1)z + t_2 z^2}{1+(\overline{t_1}t_2 - t_1)z - t_2 z^2}. \tag{1.10}$$

Lemma 1.2. *[19] Let $p \in \mathcal{P}$. Then for any numbers a, b and c such that $a \geq 0$, $b \in \mathbb{C}$, and $c \in \mathbb{R}$ with*

$$\Psi = |bp_1^2 + cp_2| - |ap_1|,$$

we have

$$\Psi \leq \begin{cases} m - 2a & \text{if} \;\; |2b+c| \geq a + |c|, \\ 2|c| & \text{otherwise,} \end{cases}$$

and

$$\Psi \geq \begin{cases} m-2a & \text{if } \ a \geq m+2|c|, \\ -2a\sqrt{\frac{2|c|}{m+2|c|}} & \text{if } \ a^2 \leq 2|c|(m+2|c|), \\ -2|c|-\frac{a^2}{m+2|c|} & \text{otherwise}, \end{cases}$$

where $m=|4b+2c|$.

2 Coefficient differences for the class $\mathcal{S}^*(\Phi)$

Let $f \in \mathcal{S}^*(\Phi)$. Then by the definition of subordination, there exists a Schwarz function $w(z)$ such that

$$\frac{zf'(z)}{f(z)} = \Phi\left(w(z)\right). \tag{2.1}$$

Let $p \in \mathcal{P}$. Then

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1+p_1z+p_2z^2+p_3z^3+\cdots. \tag{2.2}$$

Now, by equating the coefficients of (2.1) and (2.2), we get

$$a_2 = \frac{B_1}{2}p_1 \quad \text{and} \quad a_3 = \left(\frac{-B_1+B_2+B_1^2}{8}\right)p_1^2 + \frac{B_1}{4}p_2. \tag{2.3}$$

In this theorem, we establish the sharp upper and lower bounds on $|a_3|-|a_2|$ for the function $f \in \mathcal{S}^*(\Phi)$.

Theorem 2.1. *Let $f \in \mathcal{S}^*(\Phi)$ be of the form (1.1) with $B_1>0$. Then*

$$-B_1\sqrt{\frac{B_1}{B_1+|B_2+B_1^2|}} \leq |a_3|-|a_2| \leq \begin{cases} \frac{\left|B_2+B_1^2\right|}{2}-B_1 & \text{if } \ |B_2+B_1^2| \geq 3B_1, \\ \frac{B_1}{2} & \text{if } \ |B_2+B_1^2| < 3B_1. \end{cases}$$

These inequalities are sharp.

Proof. By applying (2.3), we have

$$|a_3|-|a_2| = |bp_1^2+cp_2|-|ap_1|, \tag{2.4}$$

with

$$a=\frac{B_1}{2} \quad , \quad b=\frac{-B_1+B_2+B_1^2}{8} \quad , \quad c=\frac{B_1}{4}.$$

Let $\left|B_2+B_1^2\right| \geq 3B_1$. Then

$$|2b+c|-a-|c| = \frac{\left|B_2+B_1^2\right|}{4}-\frac{3}{4}B_1 \geq 0.$$

It implies $|2b+c| \geq a+|c|$ and thus by Lemma 1.2 and (2.4), we have

$$|a_3|-|a_2| \leq \frac{\left|B_2+B_1^2\right|}{2}-B_1.$$

The bound is sharp for the function f_1 given by (2.1) with $p(z)=(1+z)/(1-z)$. Then $f_1 \in \mathcal{S}^*(\Phi)$ with

$$f_1(z)=z+B_1z^2+\frac{B_2+B_1^2}{2}z^3+\cdots . \tag{2.5}$$

Now, for $\left|B_2+B_1^2\right|<3B_1$, we have

$$|2b+c|-a-|c|=\frac{\left|B_2+B_1^2\right|}{4}-\frac{3}{4}B_1<0,$$

and it gives $|2b+c|<a+|c|$. Therefore, Lemma 1.2 and (2.4) imply

$$|a_3|-|a_2| \leq \frac{B_1}{2}.$$

The equality holds for the function f_2 defined by (2.1) with $p(z)=(1+z^2)/(1-z^2)$. Then $f_2 \in \mathcal{S}^*(\Phi)$ with

$$f_2(z)=z+\frac{B_1}{2}z^3+\cdots . \tag{2.6}$$

Now, we will find the lower bound on $|a_3|-|a_2|$. Since

$$a^2-2|c|\left(m+2|c|\right)=-\frac{|B_1B_2+B_1^3|}{4} \leq 0,$$

it implies $a^2 \leq 2|c|\left(m+2|c|\right)$. So, by Lemma 1.2 and (2.4), we have

$$|a_3|-|a_2| \geq -2a\sqrt{\frac{2|c|}{m+2|c|}}=-B_1\sqrt{\frac{B_1}{B_1+|B_2+B_1^2|}}.$$

In order to show that the inequality is sharp, consider the function f_3 defined by (2.1) where $p(z)$ is given by (1.10) with

$$t_1=\sqrt{\frac{B_1}{B_1+|B_2+B_1^2|}} \quad \text{and} \quad t_2=\begin{cases}-\dfrac{B_2+B_1^2}{|B_2+B_1^2|} & \text{if } B_2 \neq -B_1^2,\\ 1 & \text{if } B_2=-B_1^2.\end{cases}$$

Since $p_1=2t_1$ and $p_2=2t_1^2+2(1-t_1^2)t_2$, so it gives $bp_1^2+cp_2=0$. Therefore, by Lemma 1.2, we have

$$|a_3|-|a_2|=|bp_1^2+cp_2|-|ap_1|=-2at_1=-B_1\sqrt{\frac{B_1}{B_1+|B_2+B_1^2|}}.$$

It completes the proof. □

In the following theorem, we find the bounds on $|A_3|-|A_2|$ for $f \in \mathcal{S}^*(\Phi)$.

Theorem 2.2. *Let $f \in \mathcal{S}^*(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-B_1\sqrt{\frac{B_1}{B_1+|3B_1^2-B_2|}} \leq |A_3|-|A_2| \leq \begin{cases} \frac{|3B_1^2-B_2|}{2} - B_1 & \text{if } |3B_1^2-B_2| \geq 3B_1, \\ \frac{B_1}{2} & \text{if } |3B_1^2-B_2| < 3B_1. \end{cases}$$

These inequalities are sharp.

Proof. By using (2.3) and (1.4), we have

$$|A_3|-|A_2| = |bp_1^2+cp_2|-|ap_1|, \tag{2.7}$$

with

$$a=\frac{B_1}{2} \quad , \quad b=\frac{B_1-B_2+3B_1^2}{8} \quad , \quad c=-\frac{B_1}{4}.$$

Let $\left|3B_1^2-B_2\right| \geq 3B_1$. Then

$$|2b+c|-a-|c| = \frac{|3B_1^2-B_2|-3B_1}{4} \geq 0.$$

It implies $|2b+c| \geq a+|c|$. By Lemma 1.2 and (2.7), we have

$$|A_3|-|A_2| \leq \frac{|3B_1^2-B_2|}{2} - B_1.$$

The bound is sharp for the function f_1 given by (2.5). In this case,

$$A_2=-a_2=-B_1 \quad \text{and} \quad A_3 = 2a_2^2-a_3 = \frac{3B_1^2-B_2}{2}.$$

Now, for $\left|3B_1^2-B_2\right| < 3B_1$, we have

$$|2b+c|-a-|c| = \frac{|3B_1^2-B_2|-3B_1}{4} < 0,$$

and it gives $|2b+c| < a+|c|$. Therefore, Lemma 1.2 and (2.7) imply

$$|A_3|-|A_2| \leq \frac{B_1}{2}.$$

The equality exists for the function f_2 defined by (2.6). Consequently,

$$A_2=-a_2=0 \quad \text{and} \quad A_3 = 2a_2^2-a_3 = -\frac{B_1}{2}.$$

Now, we will establish the lower bound on $|A_3|-|A_2|$.
Since

$$a^2-2|c|\,(m+2|c|) = -\frac{|3B_1^3-B_1B_2|}{4} \leq 0,$$

and it implies $a^2 \leq 2|c|\,(m+2|c|)$. Lemma 1.2 and (2.7) lead to

$$|A_3|-|A_2| \geq -2a\sqrt{\frac{2|c|}{m+2|c|}} = -B_1\sqrt{\frac{B_1}{B_1+|3B_1^2-B_2|}}.$$

In order to show that the inequality is sharp, consider the function f_4 defined by (2.1) where $p(z)$ is given by (1.10) with

$$t_1 = \sqrt{\frac{B_1}{B_1 + |3B_1^2 - B_2|}} \quad \text{and} \quad t_2 = \begin{cases} \dfrac{3B_1^2 - B_2}{|3B_1^2 - B_2|} & \text{if } B_2 \neq 3B_1^2, \\ 1 & \text{if } B_2 = 3B_1^2. \end{cases}$$

Since $p_1 = 2t_1$ and $p_2 = 2t_1^2 + 2(1 - t_1^2)t_2$, so $bp_1^2 + cp_2 = 0$. Therefore, by Lemma 1.2, we have

$$\begin{aligned} |A_3| - |A_2| &= |bp_1^2 + cp_2| - |ap_1| = -2at_1 \\ &= -B_1\sqrt{\frac{B_1}{B_1 + |3B_1^2 - B_2|}}. \end{aligned}$$

It completes the proof. □

The next theorem deals with the bounds on difference of successive initial logarithmic coefficients, $|\gamma_2| - |\gamma_1|$ for $f \in \mathcal{S}^*(\Phi)$.

Theorem 2.3. *Let $f \in \mathcal{S}^*(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |B_2|}} \leq |\gamma_2| - |\gamma_1| \leq \begin{cases} \frac{|B_2| - 2|B_1|}{4} & \text{if } |B_2| \geq 3B_1, \\ \frac{B_1}{4} & \text{if } |B_2| < 3B_1. \end{cases}$$

These inequalities are sharp.

Proof. By applying (2.3) and (1.6), we get

$$|\gamma_2| - |\gamma_1| = |bp_1^2 + cp_2| - |ap_1|, \tag{2.8}$$

with

$$a = \frac{B_1}{4} \quad , \quad b = \frac{B_2 - B_1}{16} \quad , \quad c = \frac{B_1}{8}.$$

Let $|B_2| \geq 3B_1$. Then

$$|2b + c| - a - |c| = \frac{|B_2| - 3B_1}{8} \geq 0.$$

It implies $|2b + c| \geq a + |c|$ and, by Lemma 1.2 and (2.8), we have

$$|\gamma_2| - |\gamma_1| \leq \frac{|B_2| - 2B_1}{4}.$$

The bound is sharp for the function f_1 given by (2.5). Here,

$$\gamma_1 = \frac{a_2}{2} = \frac{B_1}{2} \quad \text{and} \quad \gamma_2 = \frac{2a_3 - a_2^2}{4} = \frac{B_2}{4}.$$

Now, for $|B_2| < 3B_1$, we have

$$|2b + c| - a - |c| = \frac{|B_2| - 3B_1}{4} < 0,$$

and it gives $|2b + c| < a + |c|$. Therefore, Lemma 1.2 and (2.8) imply

$$|\gamma_2| - |\gamma_1| \le \frac{B_1}{4}.$$

The equality holds for the function f_2 defined by (2.6). In this case,

$$\gamma_1 = \frac{a_2}{2} = 0 \quad \text{and} \quad \gamma_2 = \frac{2a_3 - a_2^2}{4} = \frac{B_1}{4}.$$

Now, we discuss the lower bound on $|\gamma_2| - |\gamma_1|$. Since

$$a^2 - 2|c|\,(m + 2|c|) = -\frac{|B_1 B_2|}{16} \le 0,$$

then $a^2 \le 2|c|\,(m + 2|c|)$. By Lemma 1.2 and (2.8)

$$|\gamma_2| - |\gamma_1| \ge -2a\sqrt{\frac{2|c|}{m + 2|c|}} = -\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |B_2|}}.$$

In order to show that the inequality is sharp, consider the function f_5 defined by (2.1) where $p(z)$ is given by (1.10) with

$$t_1 = \sqrt{\frac{B_1}{B_1 + |B_2|}} \quad \text{and} \quad t_2 = \begin{cases} -\dfrac{B_2}{|B_2|} & \text{if } \; B_2 \neq 0, \\ 1 & \text{if } \; B_2 = 0. \end{cases}$$

Since $p_1 = 2t_1,\ p_2 = 2t_1^2 + 2(1 - t_1^2)t_2$, after some simplifications, $bp_1^2 + cp_2 = 0$. Therefore, by Lemma 1.2, we have

$$\begin{aligned} |\gamma_2| - |\gamma_1| &= |bp_1^2 + cp_2| - |ap_1| = -2at_1 \\ &= -\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |B_2|}}. \end{aligned}$$

It completes the proof. □

We end this section by discussing the bounds on $|\Gamma_2| - |\Gamma_1|$ for $f \in \mathcal{S}^*(\Phi)$.

Theorem 2.4. *Let $f \in \mathcal{S}^*(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |2B_1^2 - B_2|}} \le |\Gamma_2| - |\Gamma_1| \le \begin{cases} \frac{|2B_1^2 - B_2| - 2B_1}{4} & \text{if } \; |2B_1^2 - B_2| \ge 3B_1, \\ \frac{B_1}{4} & \text{if } \; |2B_1^2 - B_2| < 3B_1. \end{cases}$$

These inequalities are sharp.

Proof. By using (2.3) in (1.8), we deduce

$$|\Gamma_2| - |\Gamma_1| = |bp_1^2 + cp_2| - |ap_1|, \tag{2.9}$$

with
$$a=\frac{B_1}{4}\quad,\quad b=\frac{2B_1^2+B_1-B_2}{16}\quad,\quad c=-\frac{B_1}{8}.$$
Let $\left|2B_1^2-B_2\right|\geq 3B_1$. Then
$$|2b+c|-a-|c|=\frac{|2B_1^2-B_2|-3B_1}{8}\geq 0.$$
It results in $|2b+c|\geq a+|c|$ and, by Lemma 1.2 and (2.9),
$$|\Gamma_2|-|\Gamma_1|\leq\frac{|2B_1^2-B_2|-2B_1}{4}.$$
The bound is sharp for the function f_1 given by (2.5). Here,
$$\Gamma_1=-\frac{a_2}{2}=-\frac{B_1}{2}\quad\text{and}\quad\gamma_2=\frac{-2a_3+3a_2^2}{4}=\frac{2B_1^2-B_2}{4}.$$

Now, for $\left|2B_1^2-B_2\right|<3B_1$, we have
$$|2b+c|-a-|c|=\frac{|2B_1^2-B_2|-3B_1}{4}<0,$$
which gives $|2b+c|<a+|c|$. Therefore, Lemma 1.2 and (2.9) yield
$$|\Gamma_2|-|\Gamma_1|\leq\frac{B_1}{4}.$$
The equality holds for the function f_2 given by (2.6). We have
$$\Gamma_1=-\frac{a_2}{2}=0\quad\text{and}\quad\Gamma_2=\frac{-2a_3+3a_2^2}{4}=-\frac{B_1}{4}.$$

Finally, we find the lower bound on $|\Gamma_2|-|\Gamma_1|$.We have
$$a^2-2|c|\left(m+2|c|\right)=-\frac{|2B_1^3-B_1B_2|}{16}\leq 0,$$
so $a^2\leq 2|c|\left(m+2|c|\right)$. Thus, by Lemma 1.2 and (2.9)
$$|\Gamma_2|-|\Gamma_1|\geq -2a\sqrt{\frac{2|c|}{m+2|c|}}=-\frac{B_1}{2}\sqrt{\frac{B_1}{B_1+|2B_1^2-B_2|}}.$$
In order to show that the inequality is sharp, consider the function f_6 defined by (2.1), where $p(z)$ is given by (1.10) with
$$t_1=\sqrt{\frac{B_1}{B_1+|2B_1^2-B_2|}}\quad\text{and}\quad t_2=\begin{cases}\dfrac{2B_1^2-B_2}{|2B_1^2-B_2|} & \text{if } 2B_1^2\neq B_2,\\ 1 & \text{if } 2B_1^2=B_2.\end{cases}$$
Since $p_1=2t_1,\ p_2=2t_1^2+2(1-t_1^2)t_2$, we obtain $bp_1^2+cp_2=0$. Therefore, by Lemma 1.2, we have
$$\begin{aligned}|\Gamma_2|-|\Gamma_1|&=|bp_1^2+cp_2|-|ap_1|=-2at_1\\&=-\frac{B_1}{2}\sqrt{\frac{B_1}{B_1+|2B_1^2-B_2|}}.\end{aligned}$$
It completes the proof. □

3 Coefficient differences for the class $\mathcal{R}(\Phi)$

Let $f \in \mathcal{R}(\Phi)$. Then, by the definition of subordination, there exists a Schwarz function $w(z)$ such that

$$f'(z) = \Phi(w(z)). \tag{3.1}$$

Let $p \in \mathcal{P}$ is given by (2.2). Then, by comparing the coefficients of (3.1) and (2.2), we have

$$a_2 = \frac{B_1}{4} p_1 \quad \text{and} \quad a_3 = \left(\frac{B_2 - B_1}{12}\right) p_1^2 + \frac{B_1}{6} p_2. \tag{3.2}$$

In our first theorem of this section, we will establish bounds on $|a_3| - |a_2|$ for $f \in \mathcal{R}(\Phi)$.

Theorem 3.1. *Let $f \in \mathcal{R}(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |B_2|}} \leq |a_3| - |a_2| \leq \begin{cases} \frac{2|B_2| - 3B_1}{6} & \text{if } 2|B_2| - 5B_1 \geq 0, \\ \frac{B_1}{3} & \text{if } 2|B_2| - 5B_1 < 0. \end{cases}$$

These inequalities are sharp.

Proof. By using (3.2), we get

$$|a_3| - |a_2| = |bp_1^2 + cp_2| - |ap_1|, \tag{3.3}$$

with

$$a = \frac{B_1}{4} \quad , \quad b = \frac{B_2 - B_1}{12} \quad , \quad c = \frac{B_1}{6}.$$

Let $2|B_2| - 5B_1 \geq 0$. Then

$$|2b + c| - a - |c| = \frac{|B_2|}{6} - \frac{5|B_1|}{12} \geq 0.$$

It implies $|2b + c| \geq a + |c|$. By Lemma 1.2 and (3.3)we have

$$|a_3| - |a_2| \leq \frac{2|B_2| - 3B_1}{6}.$$

The bound is sharp for the function f_7 given by (3.1) with $p(z) = (1+z)/(1-z)$. Then $f_7 \in \mathcal{R}(\Phi)$ with

$$f_7(z) = z + \frac{B_1}{2} z^2 + \frac{B_2}{3} z^3 + \cdots. \tag{3.4}$$

Now, for $2|B_2| - 5B_1 < 0$, we have

$$|2b + c| - a - |c| = \frac{|B_2|}{6} - \frac{5|B_1|}{12} < 0,$$

which gives $|2b + c| < a + |c|$. Therefore, Lemma 1.2 and (3.3) lead to

$$|a_3| - |a_2| \leq \frac{B_1}{3}.$$

The equality holds for the function f_8 defined by (3.1) with $p(z) = (1 + z^2)/(1 - z^2)$. Then $f_8 \in \mathcal{R}(\Phi)$ with

$$f_8(z) = z + \frac{B_1}{3} z^3 + \cdots. \tag{3.5}$$

Now, we will find the lower bound on $|a_3| - |a_2|$. Since

$$a^2 - 2|c|\,(m + 2|c|) = -\frac{7B_1^2 + 16B_1|B_2|}{144} \leq 0,$$

it implies that$a^2 \leq 2|c|\,(m + 2|c|)$. So, by Lemma 1.2 and (3.3), we have

$$|a_3| - |a_2| \geq -2a\sqrt{\frac{2|c|}{m + 2|c|}} = -\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |B_2|}}.$$

In order to show that the inequality is sharp, consider the function f_9 defined by (3.1), where $p(z)$ is given by (1.10) with

$$t_1 = \sqrt{\frac{B_1}{B_1 + |B_2|}} \quad \text{and} \quad t_2 = \begin{cases} \dfrac{-B_2}{|B_2|} & \text{if } B_2 \neq 0, \\ 1 & \text{if } B_2 = 0. \end{cases}$$

Since $p_1 = 2t_1$ and $p_2 = 2t_1^2 + 2(1 - t_1^2)t_2$, so it gives $bp_1^2 + cp_2 = 0$. Therefore, by Lemma 1.2, we have

$$|a_3| - |a_2| = |bp_1^2 + cp_2| - |ap_1| = -2at_1 = -\frac{B_1}{2}\sqrt{\frac{B_1}{B_1 + |B_2|}}.$$

It completes the proof. □

In the next theorem we will discuss the bounds on $|A_3| - |A_2|$ for $f \in \mathcal{R}(\Phi)$.

Theorem 3.2. *Let $f \in \mathcal{R}(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-\frac{B_1}{2}\sqrt{\frac{2B_1}{2B_1 + |3B_1^2 - 2B_2|}} \leq |A_3| - |A_2| \leq \begin{cases} \frac{|3B_1^2 - 2B_2| - 3B_1}{6} & \text{if } |3B_1^2 - 2B_2| \geq 5B_1, \\ \frac{B_1}{3} & \text{if } |3B_1^2 - 2B_2| < 5B_1. \end{cases}$$

These inequalities are sharp.

Proof. By using (3.2) and (1.4), we have

$$|A_3| - |A_2| = |bp_1^2 + cp_2| - |ap_1|, \tag{3.6}$$

with

$$a = \frac{B_1}{4} \quad , \quad b = \frac{3B_1^2 + 2B_1 - 2B_2}{24} \quad , \quad c = -\frac{B_1}{6}.$$

Let $\left|3B_1^2 - 2B_2\right| \geq 5B_1$. Then

$$|2b + c| - a - |c| = \frac{|3B_1^2 - 2B_2| - 5B_1}{12} \geq 0.$$

It implies $|2b + c| \geq a + |c|$ and thus, by Lemma 1.2 and (3.6), we have

$$|A_3| - |A_2| \leq \frac{|3B_1^2 - 2B_2| - 3B_1}{6}.$$

The bound is sharp for the function f_7 given by (3.4). Here, $A_2 = -a_2 = -\frac{B_1}{2}$, $A_3 = 2a_2^2 - a_3 = \frac{3B_1^2 - 2B_2}{6}$.
Now, for $\left|3B_1^2 - 2B_2\right| < 5B_1$, we have

$$|2b + c| - a - |c| = \frac{|3B_1^2 - 2B_2| - 5B_1}{12} < 0,$$

and it gives, $|2b + c| < a + |c|$. Therefore, Lemma 1.2 and (3.6) imply

$$|A_3| - |A_2| \le \frac{B_1}{3}.$$

The equality exists for the function f_8 defined by (3.5). Hence,

$$A_2 = -a_2 = 0 \quad \text{and} \quad A_3 = 2a_2^2 - a_3 = -\frac{B_1}{3}.$$

Now, we will establish the lower bound on $|A_3| - |A_2|$. Since

$$a^2 - 2|c|\,(m + 2|c|) = -\frac{7B_1^2 + 8|3B_1^3 - 2B_1B_2|}{144} \le 0,$$

so $a^2 \le 2|c|\,(m + 2|c|)$. Thus, by Lemma 1.2 and (3.6)

$$|A_3| - |A_2| \ge -2a\sqrt{\frac{2|c|}{m + 2|c|}} = -\frac{B_1}{2}\sqrt{\frac{2B_1}{2B_1 + |3B_1^2 - 2B_2|}}.$$

In order to show that the inequality is sharp, consider the function f_{10} defined by (3.1) where $p(z)$ is given by (1.10) with

$$t_1 = \sqrt{\frac{2B_1}{2B_1 + |3B_1^2 - 2B_2|}} \quad \text{and} \quad t_2 = \begin{cases} \dfrac{3B_1^2 - 2B_2}{|3B_1^2 - 2B_2|} & \text{if } 2B_2 \neq 3B_1^2, \\ 1 & \text{if } 2B_2 = 3B_1^2. \end{cases}$$

Since $p_1 = 2t_1$ and $p_2 = 2t_1^2 + 2(1 - t_1^2)t_2$, it gives $bp_1^2 + cp_2 = 0$. Therefore, by Lemma 1.2, we have

$$\begin{aligned} |A_3| - |A_2| &= |bp_1^2 + cp_2| - |ap_1| = -2at_1 \\ &= -\frac{B_1}{2}\sqrt{\frac{2B_1}{2B_1 + |3B_1^2 - 2B_2|}}. \end{aligned}$$

It completes the proof. □

In the following theorem we will find bounds on $|\gamma_2| - |\gamma_1|$ for $f \in \mathcal{R}(\Phi)$.

Theorem 3.3. *Let $f \in \mathcal{R}(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-\frac{B_1}{4}\sqrt{\frac{8B_1}{|3B_1^2 - 8B_2| + 8B_1}} \le |\gamma_2| - |\gamma_1| \le \begin{cases} \frac{|3B_1^2 - 8B_2| - 12B_1}{48} & \text{if } |3B_1^2 - 8B_2| \ge 20B_1, \\ \frac{B_1}{6} & \text{if } |3B_1^2 - 8B_2| < 20B_1. \end{cases}$$

These inequalities are sharp.

Proof. By applying (3.2) and (1.6), we get

$$|\gamma_2| - |\gamma_1| = |bp_1^2 + cp_2| - |ap_1|, \tag{3.7}$$

with

$$a = \frac{B_1}{8} \quad , \quad b = \frac{-3B_1^2 - 8B_1 + 8B_2}{192} \quad , \quad c = \frac{B_1}{12}.$$

Let $|3B_1^2 - 8B_2| \geq 20B_1$. Then

$$|2b + c| - a - |c| = \frac{|3B_1^2 - 8B_2| - 20B_1}{96} \geq 0$$

and $|2b + c| \geq a + |c|$. By Lemma 1.2 and (3.7), we have

$$|\gamma_2| - |\gamma_1| \leq \frac{|3B_1^2 - 8B_2| - 12B_1}{48}.$$

The bound is sharp for the function f_7 given by (3.4). Consequently,

$$\gamma_1 = \frac{a_2}{2} = \frac{B_1}{4} \quad \text{and} \quad \gamma_2 = \frac{2a_3 - a_2^2}{4} = \frac{-3B_1^2 + 8B_2}{48}.$$

For $|3B_1^2 - 8B_2| < 20B_1$, we have

$$|2b + c| - a - |c| = \frac{|3B_1^2 - 8B_2| - 20B_1}{96} < 0,$$

so $|2b + c| < a + |c|$. Therefore, Lemma 1.2 and (3.7) result in

$$|\gamma_2| - |\gamma_1| \leq \frac{B_1}{6}.$$

The equality holds for the function f_8 defined by (3.5). Therefore,

$$\gamma_1 = \frac{a_2}{2} = 0 \quad \text{and} \quad \gamma_2 = \frac{2a_3 - a_2^2}{4} = \frac{B_1}{6}.$$

Now, we discuss the lower bound on $|\gamma_2| - |\gamma_1|$.We have

$$a^2 - 2|c|\,(m + 2|c|) = -\frac{7B_1^2 + |6B_1^3 - 16B_1B_2|}{576} \leq 0,$$

which implies $a^2 \leq 2|c|\,(m + 2|c|)$. Thus, by Lemma 1.2 and (2.8)

$$|\gamma_2| - |\gamma_1| \geq -2a\sqrt{\frac{2|c|}{m + 2|c|}} = -\frac{B_1}{4}\sqrt{\frac{8B_1}{|3B_1^2 - 8B_2| + 8B_1}}.$$

In order to show that the inequality is sharp, consider the function f_{11} defined by (3.1), where $p(z)$ is given by (1.10) with

$$t_1 = \sqrt{\frac{8B_1}{|3B_1^2 - 8B_2| + 8B_1}} \quad \text{and} \quad t_2 = \begin{cases} \dfrac{3B_1^2 - 8B_2}{|3B_1^2 - 8B_2|} & \text{if } \ 3B_1^2 \neq 8B_2, \\ 1 & \text{if } \ 3B_1^2 = 8B_2. \end{cases}$$

Since $p_1 = 2t_1,\ p_2 = 2t_1^2 + 2(1-t_1^2)t_2$, after some computation, $bp_1^2 + cp_2 = 0$. Therefore, by Lemma 1.2, we have

$$\begin{aligned}|\gamma_2| - |\gamma_1| &= |bp_1^2 + cp_2| - |ap_1| = -2at_1 \\ &= -\frac{B_1}{4}\sqrt{\frac{8B_1}{|3B_1^2 - 8B_2| + 8B_1}}.\end{aligned}$$

It completes the proof. □

At the end, we will establish bounds on $|\Gamma_2| - |\Gamma_1|$ for $f \in \mathcal{R}(\Phi)$.

Theorem 3.4. *Let $f \in \mathcal{R}(\Phi)$ be of the form (1.1) with $B_1 > 0$. Then*

$$-\frac{B_1}{4}\sqrt{\frac{8B_1}{|9B_1^2 - 8B_2| + 8B_1}} \le |\Gamma_2| - |\Gamma_1| \le \begin{cases} \frac{|9B_1^2 - 8B_2| - 12B_1}{48} & \text{if } |9B_1^2 - 8B_2| \ge 20B_1, \\ \frac{B_1}{6} & \text{if } |9B_1^2 - 8B_2| < 20B_1. \end{cases}$$

These inequalities are sharp.

Proof. By using (3.2) in (1.8), we deduce

$$|\Gamma_2| - |\Gamma_1| = |bp_1^2 + cp_2| - |ap_1|, \tag{3.8}$$

with

$$a = \frac{B_1}{8} \quad , \quad b = \frac{9B_1^2 + 8B_1 - 8B_2}{192} \quad , \quad c = -\frac{B_1}{12}.$$

Let $\left|9B_1^2 - 8B_2\right| \ge 20B_1$. Then

$$|2b + c| - a - |c| = \frac{|9B_1^2 - 8B_2| - 20B_1}{96} \ge 0.$$

Hence $|2b + c| \ge a + |c|$ and, by Lemma 1.2 and (3.8), we have

$$|\Gamma_2| - |\Gamma_1| \le \frac{|9B_1^2 - 8B_2| - 12B_1}{48}.$$

The bound is sharp for the function f_7 given by (3.4). Here,

$$\Gamma_1 = -\frac{a_2}{2} = -\frac{B_1}{4} \quad \text{and} \quad \Gamma_2 = \frac{-2a_3 + 3a_2^2}{4} = \frac{9B_1^2 - 8B_2}{48}.$$

For $\left|9B_1^2 - 8B_2\right| < 20B_1$, we have

$$|2b + c| - a - |c| = \frac{|9B_1^2 - 8B_2| - 20B_1}{96} < 0,$$

and $|2b + c| < a + |c|$. Therefore,

$$|\Gamma_2| - |\Gamma_1| \le \frac{B_1}{6}.$$

The equality holds for the function f_8 given by (3.5). Now,

$$\Gamma_1 = -\frac{a_2}{2} = 0 \quad \text{and} \quad \Gamma_2 = \frac{-2a_3 + 3a_2^2}{4} = -\frac{B_1}{6}.$$

Finally, we find the lower bound on $|\Gamma_2| - |\Gamma_1|$. Since

$$a^2 - 2|c|\,(m + 2|c|) = -\frac{7B_1^2 + |18B_1^3 - 16B_1B_2|}{576} \leq 0,$$

we get $a^2 \leq 2|c|\,(m + 2|c|)$. Thus, Lemma 1.2 and (2.9) imply

$$|\Gamma_2| - |\Gamma_1| \geq -2a\sqrt{\frac{2|c|}{m + 2|c|}} = -\frac{B_1}{4}\sqrt{\frac{8B_1}{|9B_1^2 - 8B_2| + 8B_1}}.$$

In order to show that the inequality is sharp, consider the function f_{12} defined by (3.1), where $p(z)$ is given by (1.10) with

$$t_1 = \sqrt{\frac{8B_1}{|9B_1^2 - 8B_2| + 8B_1}} \quad \text{and} \quad t_2 = \begin{cases} \dfrac{9B_1^2 - 8B_2}{|9B_1^2 - 8B_2|} & \text{if } 9B_1^2 \neq 8B_2, \\ 1 & \text{if } 9B_1^2 = 8B_2. \end{cases}$$

Since $p_1 = 2t_1,\ p_2 = 2t_1^2 + 2(1 - t_1^2)t_2$, we get $bp_1^2 + cp_2 = 0$. Therefore, by Lemma 1.2, we have

$$\begin{aligned} |\Gamma_2| - |\Gamma_1| &= |bp_1^2 + cp_2| - |ap_1| = -2at_1 \\ &= -\frac{B_1}{4}\sqrt{\frac{8B_1}{|9B_1^2 - 8B_2| + 8B_1}}. \end{aligned}$$

It completes the proof. □

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