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Characterizations of $\mathcal{PR}$-slant warped submanifolds in Para-Kähler spaces via Weingarten maps

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Abstract. This paper investigates $\mathcal{PR}$-pseudo-slant warped product submanifolds in para-Kähler manifolds $\bar{F}^{2m}$. First, fundamental properties of $\mathcal{PR}$-pseudo-slant submanifolds in $\bar{F}^{2m}$ are obtained. Then non-existence and characterization results for $\mathcal{PR}$-pseudo-slant warped product submanifolds are derived in terms of the Weingarten and endomorphism mappings, including the corresponding slant factor in $\bar{F}^{2m}$. In addition, a numerical example of the warped product $F_\lambda \times_f F_\perp$ in $\bar{F}^{2m}$ is provided to illustrate the results.

Keywords. $\mathcal{PR}$-pseudo-slant submanifolds, para-Kähler manifolds, warped product manifolds

1 Introduction

In the field of differential geometry, the study of warped product submanifolds has been an active and fruitful area of research, owing to its deep connections with physics, geometry, and topology. This area of research gained substantial momentum after the successful application of warped product techniques in addressing theoretical physics problems, particularly in the domains of general relativity, space-time models, and black hole physics [1, 15, 18, 19]. The notion of warped product manifolds, also called warped bundles, was introduced by Bishop and O'Neill in 1969 [7]. They defined the warped product $B \times_f C$ of two pseudo-Riemannian manifolds B and C as the product manifold $B \times C$ equipped with a pseudo-Riemannian metric constructed from the metrics of B and C. For any tangent vector $E \in \mathfrak{X}(B \times_f C)$, the metric is defined by

$$\|E\|^2 = \|\pi^*(E)\|^2 + (f \circ \pi)^2 \|\sigma^*(E)\|^2,$$

where π and σ denote the natural projections from $B \times C$ onto B and C, respectively. In this context, B is called the base manifold, C is the fiber manifold, and f is a positive C^∞ warping function defined on B. Warped product manifolds hold great significance in differential geometry and mathematical physics due to their flexibility in constructing and analyzing a wide variety of geometric structures. These manifolds arise by modifying the metric of one manifold using

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a smooth function defined on another, providing a powerful framework for exploring curvature, geodesics, and other geometric properties in complex settings. They are especially valuable in general relativity, where they are used to model spacetimes with variable curvature, such as those encountered in cosmological models and black hole spacetimes. Additionally, warped products are essential in the study of Einstein manifolds, scalar curvature, and Riemannian submersions, underscoring their pivotal role in both theoretical and applied mathematics. Given the relevance of neutral metrics in para-Kähler manifolds and warped products, a natural question arises: what are the existence and characterization criteria for these structures?

Chen-Munteanu [10] initiated the study of pseudo-Riemannian warped products, specifically $\mathcal{PR}$-warped product submanifolds, providing key results regarding the existence and characterization of $\mathcal{PR}$-semi-invariant warped submanifolds in para-Kähler manifolds. Following this, Srivastava-Sharma [26] continued the exploration of this topic in para-contact settings. Since then, several differential geometers have investigated and analyzed the existence, characterization, and applications of such warped products from various perspectives in different manifolds [3, 4, 5, 6, 8, 12, 13, 16, 21, 24, 27].

The structure of this article is as follows: Section 2 provides an overview of fundamental concepts related to para-Kähler manifolds and their submanifolds in $\bar{F}$. Section 3 presents definitions and characterization results for slant and $\mathcal{PR}$-pseudo-slant submanifolds in brief PRPS, along with important findings and remarks for future reference. Finally, Section 4 proves the non-existence of a warped product of the form $F_\perp \times_f F_\lambda$, and establishes the necessary and sufficient conditions for a submanifold F to be locally a $\mathcal{PR}$-pseudo-slant warped product briefly PRWP of the form $F_\lambda \times_f F_\perp$ in $\bar{F}$. Additionally, a numerical example is provided to demonstrate the existence of $F_\lambda \times_f F_\perp$ and the characterization result in terms of the endomorphism in $\bar{F}$.

2 Preliminaries

Let $\bar{F}$ be a real smooth manifold. Denote by $\mathfrak{R}(\bar{F})$ the set of real-valued smooth functions on $\bar{F}$, and by $\mathfrak{X}(\bar{F})$ the set of all vector fields on $\bar{F}$. It is straightforward to verify that $\mathfrak{R}(\bar{F})$ is a module over the ring $\mathfrak{R}(\bar{F})$, and is also referred to as the set of derivations of $\mathfrak{R}(\bar{F})$. An almost product structure on a manifold $\bar{F}$ is defined as a tensor field $\mathcal{P}$ of type $(1,1)$, which satisfies the following condition:

$$\mathcal{P}^2 = \text{id}, \tag{2.1}$$

where id is the identity map on the tangent bundle of $\bar{F}$ and $\mathcal{P}^2$ denotes the composition of $\mathcal{P}$ with itself. This relation implies that $\mathcal{P}$ acts as a kind of "splitting" of the tangent bundle, dividing the manifold into two complementary distributions. The pair $(\bar{F}, \mathcal{P})$ is known as an almost product manifold. An almost product manifold is said to be almost para-Hermitian [17] if it admits a pseudo-Riemannian metric $\bar{g}$ satisfying:

$$\bar{g}(\mathcal{P}E_1, \mathcal{P}E_2) + \bar{g}(E_1, E_2) = 0. \tag{2.2}$$

It is evident that the signature of the metric tensor $\bar{g}$ is necessarily (m, m), regardless of the choice of vector fields E_1 and E_2 tangent to the manifold $\bar{F}$. Also, Eq. 2.2 implies that:

$$\bar{g}(\mathcal{P}E_1, E_2) + \bar{g}(E_1, \mathcal{P}E_2) = 0, \tag{2.3}$$

for any $E_1, E_2 \in \mathfrak{X}(\bar{F})$. The fundamental 2-form ω of $\bar{F}$ is defined by:

$$\omega(E_1, E_2) = \bar{g}(E_1, \mathcal{P}E_2), \tag{2.4}$$

for all $E_1, E_2 \in \mathfrak{X}(\bar{F})$.

Definition 1. An almost para-Hermitian manifold $\bar{F}$ is referred to as a para-Kähler manifold if the almost product structure $\mathcal{P}$ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$, meaning that the following condition holds:

$$(\bar{\nabla}_{E_1}\mathcal{P})E_2 = 0, \tag{2.5}$$

for each $E_1, E_2 \in \mathfrak{X}(\bar{F})$.

This condition implies that the tensor field $\mathcal{P}$ is covariantly constant, ensuring that the almost product structure is preserved under parallel transport. In other words, the para-Kähler structure remains invariant along the manifold's geodesics, making the manifold a rich object of study in both differential geometry and theoretical physics. Suppose F is an isometrically immersed submanifold in $\bar{F}$. Then, at each point $p \in F$, the tangent space $\mathfrak{X}_p(\bar{F})$ of $\bar{F}$ can be decomposed into a direct sum:

$$\mathfrak{X}_p(\bar{F}) = \mathfrak{X}_p(F) \oplus \mathfrak{X}_p(F)^{\perp}, \tag{2.6}$$

where $\mathfrak{X}_p(F)$ is the tangent subspace of $\mathfrak{X}_p(\bar{F})$ and $\mathfrak{X}_p(F)^{\perp}$ is the normal subspace of $\mathfrak{X}_p(\bar{F})$. We use the same notation g for the induced metric on F as for the original metric on $\bar{F}$. With this convention, the important formulas are expressed as follows:

Gauss formula: For any vector fields $E_1, E_2 \in \mathfrak{X}(F)$, the second fundamental form h and the induced connection ∇ are related by:

$$\bar{\nabla}_{E_1}E_2 = \nabla_{E_1}E_2 + h(E_1, E_2), \tag{2.7}$$

where h is the second fundamental form that captures how the submanifold curves within the ambient manifold.

Weingarten formula: For any vector field ζ normal to F, the derivative of a normal vector is decomposed into tangential and normal components:

$$\bar{\nabla}_{E_1}\zeta = -\mathcal{A}_{\zeta}E_1 + \nabla^{\perp}_{E_1}\zeta. \tag{2.8}$$

We use the symbol ∇ (resp., $\nabla^{\perp}$) for the induced tangent (resp., normal) connection on $\mathfrak{X}(F)$ (resp., $\mathfrak{X}(F^{\perp})$) and $\mathcal{A}_{\zeta}$ is the shape operator (or Weingarten map) associated with the normal vector ζ. The shape operator describes how the normal vector field changes along the manifold. The Weingarten map and h, the second fundamental form, are related by the following relation [9]:

$$g(\mathcal{A}_{\zeta}E_1, E_2) = g(h(E_1, E_2), \zeta), \tag{2.9}$$

These formulas are fundamental in the study of submanifold geometry, relating the intrinsic geometry of F to its embedding in a larger ambient space.

Consider the orthonormal frame $\{e_1, \cdots, e_m, e_{m+1}, \cdots, e_{2m}\}$ of the tangent space $\mathfrak{X}_p(\bar{F})$, such that the vectors $\{e_1, \cdots, e_m\}$ span the subspace $\mathfrak{X}_p(F)$, while the vectors $\{e_{m+1}, \cdots, e_{2m}\}$ span the orthogonal complement $\mathfrak{X}_p(F)^{\perp}$, for an arbitrary point p on the manifold F. This structure facilitates a clear distinction between the components of the tangent space and its orthogonal complement, crucial for subsequent analysis.

Then the mean curvature vector $\mathcal{H}$ of F at a point p is given by $\mathcal{H}(p) = \frac{1}{m}\operatorname{trace} h$, where h denotes the second fundamental form. The squared length of the second fundamental form, denoted by $||h||^2$, is defined as the sum of the squared norms of the shape operator components specifically:

$$||h||^2 = \sum_{x,y=1}^{m} g(h(e_x, e_y), h(e_x, e_y)), \tag{2.10}$$

where e_x and e_y are elements of the orthonormal frame of the tangent space at p. By setting $h^r_{xy} = g(h(e_x, e_y), e_r)$, $e_x, e_y \in \{e_1, \cdots, e_m\}$, $e_r \in \{e_{m+1}, \cdots, e_{2m}\}$, Eq. (2.10) can be represented as:

$$||h||^2 = \sum_{r=m+1}^{2m} \sum_{x,y=1}^{m} g(h(e_x, e_y), e_r)^2. \tag{2.11}$$

An immersed submanifold is umbilical if its Weingarten map is directly proportional to the identity map, and F is totally umbilical (resp., geodesic) if every point is umbilical and h fulfils

$$h(E_1, E_2) = g(E_1, E_2)\mathcal{H} \quad (\text{resp.,} \quad h(E_1, E_2) = 0, \forall E_1, E_2 \in \mathfrak{X}(F)).$$

If $\mathcal{H}$ vanishes then F is minimal. Moreover, if we admit $\forall E_1 \in \mathfrak{X}(F)$ and $\zeta \in \mathfrak{X}(F^\perp)$ with

$$\mathcal{P}E_1 = tE_1 + nE_1, \tag{2.12}$$
$$\mathcal{P}\zeta = t'\zeta + n'\zeta, \tag{2.13}$$

where tE_1 (respectively, nE_1) is the tangential (respectively, normal) part of $\mathcal{P}E_1$ and $t'\zeta$ (respectively, $n'\zeta$) is the tangential (respectively, normal) part of $\mathcal{P}\zeta$, then for any $E_1, E_2 \in \mathfrak{X}(F)$ we obtain a direct consequence of the decomposition (2.13) and (2.12) of the vector fields into their tangential and normal components under the action of the endomorphism $\mathcal{P}$ that

$$\bar{g}(E_1, tE_2) = -\bar{g}(tE_1, E_2).$$

Now from straightforward consequence of Eqs. (2.12) and (2.13), we arrive at

$$(\nabla_{E_1} t)E_2 = \mathcal{A}_{nE_2}E_1 + t'h(E_1, E_2), \tag{2.14}$$
$$(\nabla_{E_1} n)E_2 = -h(E_1, tE_2) + n'h(E_1, E_2), \tag{2.15}$$

where $\mathcal{A}_{nE_2}$ denotes the shape operator associated with the normal component nE_2, and t' and n' represent the tangential and normal parts of the endomorphism $\mathcal{P}$, respectively.

3 $\mathcal{PR}$-pseudo-slant submanifolds

The current study of submanifolds extend the concept of slant submanifolds by incorporating additional geometric constraints and properties, making them a rich subject of study in differential geometry. In this section, we give some definitions, characterization results for slant and such submanifolds F along with important results and remarks for later references.

Definition 2. [2, 21] Let $\bar{F}^{2m}$ be an almost para-Kähler manifold and F be its isometrically immersed submanifolds. Consider $\mathfrak{D}_\lambda$ be the distribution on F. Then

- A distribution $\mathfrak{D}_\lambda$ is referred to as a slant distribution on F, and consequently, F is called a slant submanifold, if there exists a real constant λ such that

  $$t^2 = \lambda\, \mathrm{id}, \quad g(tE_1, E_2) = -g(E_1, tE_2),$$

  for any non-degenerate tangent vectors $E_1, E_2 \in \mathfrak{D}_\lambda$ on F. In this context, we refer to λ as the slant coefficient, which is globally constant, meaning that λ does not depend on the choice of any point on F in $\bar{F}^{2m}$.

- A submanifold F is called a $\mathcal{PR}$-pseudo-slant (in short PRPS) submanifold if its tangent space can be decomposed into two non-degenerate, orthogonal distributions $(\mathfrak{D}_\perp, \mathfrak{D}_\lambda)$, i.e., $\mathfrak{X}(F) = \mathfrak{D}_\perp \oplus \mathfrak{D}_\lambda$, such that the distribution $\mathfrak{D}_\lambda$ is a slant distribution with slant coefficient λ and the distribution $\mathfrak{D}_\perp$ is an anti-invariant distribution under the action of $\mathcal{P}$, i.e., $\mathcal{P}(\mathfrak{D}_\perp) \subseteq \mathfrak{X}(F^\perp)$.

If we denote d_1 and d_2 as the dimensions of $\mathfrak{D}^\perp$ and $\mathfrak{D}_\lambda$, respectively, then from the aforementioned definition, the following conclusions have been identified:

Remark 1. A PRPS submanifold F of $\bar{F}^{2m}$ is *invariant submanifold* if $d_1 = 0$, $d_2 \neq 0$ with slant coefficient $\lambda = 1$, *anti-invariant submanifold* if $d_1 \neq 0$ and $d_2 = 0$, and *$\mathcal{PR}$-submanifold* if $d_1, d_2 \neq 0$ with slant coefficient $\lambda = 1$ [10].

Remark 2. A submanifold F is a *proper* PRPS of $\bar{F}^{2m}$ if $d_1, d_2 \neq 0$ and $\mathfrak{D}_\lambda$ is a proper slant distribution.

Further, if we use the symbols $\mathfrak{D}_\perp$ and $\mathfrak{D}_\lambda$ by $P_\perp$ and P_λ, respectively, for the projections on F, then we have the following representation

$$E_1 = P_\perp E_1 + P_\lambda E_1$$

for every $E_1 \in \mathfrak{X}(F)$.

Here, we present some important results for F:

Theorem 3.1. *For a para-Kähler manifold $\bar{F}^{2m}$ and a proper PRPS submanifold F of $\bar{F}^{2m}$, the endomorphism n is parallel if and only if*

$$\mathcal{A}_{n'\zeta} tE_4 + \lambda \mathcal{A}_\zeta E_4 = 0, \quad \textit{for all } E_3, E_4 \in \mathfrak{X}(F_\lambda).$$

Proof. If n is parallel, then by Eq. (2.15), we have

$$n'h(E_3, E_4) = h(E_3, tE_4), \quad \text{for all } E_3, E_4 \in \mathfrak{X}(F_\lambda).$$

Now, replace E_4 by tE_4 in the previous expression, we have

$$n'h(E_3, tE_4) = \lambda h(E_3, E_4).$$

By taking the inner product with $\zeta \in \mathfrak{X}(F_\perp)$ in the above expression and employing (2.7) and (2.8), we get

$$\mathcal{A}_{n'\zeta} tE_4 + \lambda \mathcal{A}_\zeta E_4 = 0.$$

□

Theorem 3.2. *Let F be a proper PRPS submanifold of a para-Kähler manifold $\bar{F}^{2m}$ with $\lambda \in \mathbb{R}$. If the endomorphism n is parallel, then one of the following conditions must hold:*

(i) the slant distribution is totally geodesic.

(ii) $h(E_3, E_4)$ is an eigenvector of n'^2 with eigenvalue $\sqrt{\lambda}$ when $\lambda \in (0, \infty)$ and with imaginary eigenvalue when $\lambda \in (-\infty, 0)$.

Proof. Suppose that the endomorphism n is parallel, then by Eq. (2.15), we have

$$n'h(E_3, E_4) = h(E_3, tE_4), \tag{3.1}$$

for all $E_3, E_4 \in \mathfrak{X}(F_\lambda)$. Operating n' on Eq. (3.1), we have

$$n'^2 h(E_3, E_4) = n'h(E_3, tE_4).$$

Now utilizing Theorem 3.1 into the last expression, we achieve

$$n'^2 h(E_3, E_4) = \lambda h(E_3, E_4).$$

This implies the following

$$(n'^2 - \lambda)h(E_3, E_4) = 0.$$

This completes the proof. □

4 On $\mathcal{PR}$-pseudo-slant singly warped product submanifold

Initially, we review several foundational principles from [7], which will be relevant for the subsequent discussion. Let ∇ be the Levi-Civita connection on $B \times_f C$, then for any $E_1, E_2 \in \mathfrak{X}(B)$ and $E_3, E_4 \in \mathfrak{X}(F)$, we see that

$$\begin{aligned}
\nabla_{E_1} E_2 &\in \mathfrak{X}(TB), \\
\nabla_{E_1} E_3 &= \nabla_{E_3} E_1 = \left(\frac{E_1 f}{f}\right) E_3, \\
\nabla_{E_3} E_4 &= \frac{-g(E_3, E_4)}{f} \nabla f,
\end{aligned}$$

where $g(\nabla f, E_1) = E_1 f$ and ∇f is the gradient of f [11, 22, 23].

Remark 3. Since lift plays a crucial role in computations on product manifolds, we will, for the sake of simplicity, focus our analysis on $F = B \times_f C$ and unify the notations for a vector $E_1 \in \mathfrak{X}(B)$ with the *lift* $\tilde{E}_1$ and for a vector $E_3 \in \mathfrak{X}(C)$ with the *lift* $\tilde{E}_3$. Moreover, it is indeed worth remembering that for the warped product $B \times_f F$ such that B (resp., F) is totally geodesic (resp., umbilical) in F.

Like the formulations presented in [10, 20], we introduce the concept of a PRPS singly warped product submanifold within the context of a para-Kähler manifold $\bar{F}^{2m}$:

Definition 3. Let F be a PRPS submanifold within a para-Kähler manifold $\bar{F}^{2m}$. F is classified as a *$\mathcal{PR}$-pseudo-slant singly warped product* in short *PRWP*, if it can be represented as a singly warped product of the form $F_\lambda \times_f F_\perp$ or $F_\perp \times_f F_\lambda$ or both. Here, F_λ denotes a proper slant submanifold, and $F_\perp$ signifies the anti-invariant submanifold of $\bar{F}^{2m}$. The function f is a non-constant positive smooth function defined on the first factor. If the warping function f is constant, the submanifold F is referred to as a PRPS product or *trivial product*.

Now, we shall prove some existence and non-existence results for such a type of warped product submanifolds.

Theorem 4.1. *There are no PRWP submanifold of the form $F = F_\perp \times_f F_\lambda$ in a para-Kähler manifold $\bar{F}^{2m}$.*

Proof. Consider $F = F_\perp \times_f F_\lambda$ as a PRWP submanifold embedded in a para-Kähler manifold $\bar{F}^{2m}$. Then by the utilization of Eqs. (2.2), (2.12), (2.7), and (2.8), we concede that

$$g(\mathcal{A}_{nE_1}E_3, tE_3) = E_1(\ln f)g(tE_3, tE_3) + g(\mathcal{A}_{nE_3}E_1, tE_3), \tag{4.1}$$

for any $E_1 \in \mathfrak{X}(F_\perp)$ and $E_3 \in \mathfrak{X}(F_\lambda)$, where E_2 and E_2 are non-degenerate vector fields. Using definition (2) in (4.1), we deduce that

$$g(\mathcal{A}_{nE_1}E_3, tE_3) = -\lambda E_1(\ln f)g(E_3, E_3) + g(\mathcal{A}_{nE_3}E_1, tE_3). \tag{4.2}$$

By interchanging E_3 by tE_3 into the above relation, we concede that

$$g(\mathcal{A}_{nE_1}tE_3, t^2E_3) = -\lambda E_1(\ln f)g(tE_3, tE_3) + g(\mathcal{A}_{ntE_3}E_1, t^2E_3). \tag{4.3}$$

Using some algebraic manipulation in the above expression, we get

$$g(\mathcal{A}_{nE_1}tE_3, E_3) = \lambda E_1(\ln f)g(E_3, E_3) + g(\mathcal{A}_{ntE_3}E_1, E_3). \tag{4.4}$$

In the other direction, we get the following relation by the direct consequence of Eqs. (2.3) and (2.5)-(2.12)

$$g(\mathcal{A}_{nE_3}E_1, tE_3) = g(\nabla_{E_1}E_3, t^2E_3) + g(\mathcal{A}_{ntE_3}E_1, E_3) + g(\nabla_{E_1}tE_3, tE_3). \tag{4.5}$$

By utilizing the definition of a slant submanifold and the covariant derivative identities in (4.5), we easily deduce that

$$g(\mathcal{A}_{nE_3}E_1, tE_3) = \lambda E_1(\ln f)g(E_3, E_3) + g(\mathcal{A}_{ntE_3}E_1, E_3) + E_1(\ln f)g(tE_3, tE_3). \tag{4.6}$$

The above equation, by employing certain calculations, yields

$$g(\mathcal{A}_{nE_3}E_1, tE_3) = g(\mathcal{A}_{ntE_3}E_1, E_3). \tag{4.7}$$

From Eqs. (4.2), (4.4) and (4.7), we conclude that

$$g(\mathcal{A}_{nE_1}E_3, tE_3) + \lambda E_1(\ln f)g(E_3, E_3) = g(\mathcal{A}_{nE_1}E_3, tE_3) - \lambda E_1(\ln f)g(E_3, E_3). \tag{4.8}$$

By virtue of Eq. (2.9) and the symmetry of h, we get

$$2\lambda E_1(\ln f)g(E_3, E_3) = 0. \tag{4.9}$$

The only possibility is that f must be constant since E_3 is a non-degenerate vector field and F_λ is a proper slant submanifold; this occurs against our hypothesis. Thus, the theorem's proof is finished. □

Next, we derive some important results as lemmas for later use by considering submanifolds of the form $F_\lambda \times_f F_\perp$:

Lemma 4.1. *If $F = F_\lambda \times_f F_\perp$ is a PRWP submanifold embedded in a para-Kähler manifold $\bar{F}^{2m}$, then for any $U, V \in \mathfrak{X}(F)$ and $E_1 \in \mathfrak{X}(F_\perp)$, we have*

$$g(h(U, E_1), nV) - g(h(U, V), \mathcal{P}E_1) = tV(\ln f)g(P_\perp U, E_1).$$

Proof. From Eq. (3), we have

$$g(\nabla_U tV, E_1) = g(\nabla_{P_\perp U} tV, E_1) + g(\nabla_{P_\lambda U} tV, E_1).$$

By the application of the covariant derivative identity, the second term vanishes, i.e.,

$$g(\nabla_{P_\lambda U} tV, E_1) = 0,$$

and the remaining term reduces to the following form:

$$g(\nabla_U tV, E_1) = tV(\ln f) g(P_\perp U, E_1). \tag{4.10}$$

Further, by applying Eq. (2.3) and the Gauss formula, we obtain

$$g(h(U, V), \mathcal{P}E_1) = -g(\mathcal{P}\nabla_U V, E_1).$$

Using Eqs. (2.7), (2.8), and (2.12) in the above, we derive

$$g(\nabla_U tV, E_1) = g(h(U, E_1), nV) - g(h(U, V), \mathcal{P}E_1). \tag{4.11}$$

Thus, the desired result follows directly from Eqs. (4.10) and (4.11). □

Lemma 4.2. *If $F = F_\lambda \times_f F_\perp$ is a PRWP submanifold embedded in a para-Kähler manifold $\bar{F}^{2m}$, then for any $U \in \mathfrak{X}(F)$, $E_3 \in \mathfrak{X}(F_\lambda)$, and $E_1 \in \mathfrak{X}(F_\perp)$, we have*

$$(\nabla_U t)E_1 = t\nabla(\ln f)\, g(P_\perp U, E_1),$$
$$(\nabla_U t)E_3 = tE_3(\ln f) P_\perp U.$$

Proof. By the relation (3), we have $(\nabla_U t)E_1 = (\nabla_{P_\perp U} t)E_1 + (\nabla_{P_\lambda U} t)E_1$. Since $P_\perp U$ is tangential to $F_\perp$, then by the utilization of the covariant derivative identities, we have

$$(\nabla_{P_\perp U} t)E_1 = t\nabla(\ln f) g(P_\perp U, E_1). \tag{4.12}$$

In the light of the covariant derivative identities, we easily see that

$$(\nabla_{P_\lambda U} t)E_1 = 0. \tag{4.13}$$

From Eqs. (4.12) and (4.13), we obtain the first part. For second part we reuse the Eq. (3), and consequently we have $(\nabla_U t)E_3 = (\nabla_{P_\perp U} t)E_3 + (\nabla_{P_\lambda U} t)E_3$. By the consequence of the covariant derivative identities, we have

$$(\nabla_{P_\perp U} t)E_3 = tE_3(\ln f) P_\perp U. \tag{4.14}$$

Since F_λ is totally geodesic then by the use Eq. (2.14), we have

$$(\nabla_{P_\lambda U} t)E_3 = 0. \tag{4.15}$$

In light of Eqs. (4.14) and (4.15), we achieved the second part. □

Further, we present our main results:

Theorem 4.2. *Let $F \to \bar{F}^{2m}$ be an isometric immersion of a proper PRPS submanifold F into a para-Kähler manifold $\bar{F}^{2m}$. Then F is locally a PRWP submanifold of the form $F_\lambda \times_f F_\perp$ if and only if:*

(*i*) *The Weingarten operator $\mathcal{A}$ of F satisfies*

$$\mathcal{A}_{\mathcal{P}E_2}tE_3 - \mathcal{A}_{ntE_3}E_2 = -\lambda(E_3 \ln f)E_2, \quad \forall\, E_2 \in \mathfrak{X}(\mathfrak{D}_\perp),\ E_3 \in \mathfrak{X}(\mathfrak{D}_\lambda), \tag{4.16}$$

(*ii*) *The endomorphism t of F satisfies*

$$(\nabla_U t)V = t\nabla(\mu)\, g(P_\perp U, P_\perp V) + tP_\lambda V(\mu)P_\perp U, \quad \forall\, U, V \in \mathfrak{X}(F), \tag{4.17}$$

for some function μ on F such that $E_1(\mu) = 0$ for all $E_1 \in \mathfrak{X}(\mathfrak{D}_\perp)$.

Proof. The proof of part (*i*) is analogous to Theorem 4.2 in [25] and is thus omitted.

For part (*ii*), suppose F is a PRWP submanifold of $\bar{F}^{2m}$. Then, by applying Lemma 4.2, we directly obtain the condition (4.17) by taking $\mu = \ln f$.

Conversely, suppose F is a proper PRPS submanifold of $\bar{F}^{2m}$ satisfying (4.17). Replacing U by E_3 and V by E_1 in (4.17), we get

$$g(\nabla_{E_3}E_1, tE_4) = 0.$$

Using the Gauss formula (2.7), this implies

$$g(h^\lambda(E_3, tE_4), E_1) = 0.$$

Hence, we deduce that the integral manifold F_λ of the distribution $\mathfrak{D}_\lambda$ is totally geodesic in F.

Now, replacing U by E_1 and V by E_2 in (4.17), we obtain

$$(\nabla_{E_1}t)E_2 = t\nabla(\mu)\, g(E_1, E_2), \quad \forall\, E_1, E_2 \in \mathfrak{X}(\mathfrak{D}_\perp). \tag{4.18}$$

Taking the inner product of both sides with $E_3 \in \mathfrak{X}(\mathfrak{D}_\lambda)$ gives

$$g((\nabla_{E_1}t)E_2, E_3) = g(t\nabla(\mu), E_3)\, g(E_1, E_2). \tag{4.19}$$

Using the identity (2) and (4.19), we derive

$$g(\nabla_{E_1}E_2, tE_3) = g(t\nabla(\mu), E_3)\, g(E_1, E_2),$$

which reduces to

$$\nabla_{E_1}E_2 = -\nabla(\mu)\, g(E_1, E_2). \tag{4.20}$$

Therefore, from (4.20) and the Gauss formula, we get

$$h_\perp(E_1, E_2) = -g(E_1, E_2)\, \nabla\mu.$$

This relation indicates that the distribution $\mathfrak{D}_\perp$ is totally umbilical in F with non-zero parallel mean curvature. Then, by Hiepko's theorem [14], we conclude that F is a PRWP submanifold of $\bar{F}^{2m}$.

This completes the proof of part (*ii*). □

Next, we present a numerical example illustrating PRWP submanifold of the form $F = F_\lambda \times_f F_\perp$ and verifies Theorem 4.2 in $\bar{F}^{2m}$.

Example 1. Let $\bar{F} = \mathbb{R}^6$ be a 6-dimensional pseudo-Euclidean space of index three with the standard Cartesian coordinates $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6)$ and pseudo-metric

$$\bar{g} = \sum_{i=1}^{3}(d\bar{x}_i)^2 - \sum_{j=4}^{6}(d\bar{x}_j)^2.$$

Define a structure $(\mathcal{P}, \bar{g})$ on $\bar{F}$ by

$$\mathcal{P}e_1 = e_4,\ \mathcal{P}e_2 = e_5,\ \mathcal{P}e_3 = e_6,\ \mathcal{P}e_4 = e_1,\ \mathcal{P}e_5 = e_2, \mathcal{P}e_6 = e_3, \tag{4.21}$$

where $e_1 = \frac{\partial}{\partial \bar{x}_1}$, $e_2 = \frac{\partial}{\partial \bar{x}_2}$, $e_3 = \frac{\partial}{\partial \bar{x}_3}$, $e_4 = \frac{\partial}{\partial \bar{x}_4}$, $e_5 = \frac{\partial}{\partial \bar{x}_5}$ and $e_6 = \frac{\partial}{\partial \bar{x}_6}$. Simple calculations can demonstrate that the structure is an almost para-Hermitian manifold. We can easily determine that the manifold $(\bar{F}, \mathcal{P}, \bar{g})$ is a para-Kähler manifold for the Levi-Civita connection $\bar{\nabla}$ with regard to $\bar{g}$. Now, let F is an isometrically immersed smooth submanifold in $\mathcal{R}^6$ defined by

$$\Omega(x_1, x_2, x_3) = (x_1, x_1\cos(x_2), x_1\sin(x_2), x_3, k_1, k_2)$$

where k_1, k_2 are constants and $x_1 \in \mathbb{R}_+,\ x_2 \in (0, \pi/2)$ and x_3 is non-zero. Then the TF spanned by the vectors

$$\begin{aligned} E_{x_1} &= \frac{\partial}{\partial \bar{x}_1} + \cos(x_2)\frac{\partial}{\partial \bar{x}_2} + \sin(x_2)\frac{\partial}{\partial \bar{x}_3}, \\ E_{x_2} &= -x_1\sin(x_2)\frac{\partial}{\partial \bar{x}_2} + x_1\cos(x_2)\frac{\partial}{\partial \bar{x}_3}, \\ E_{x_3} &= \frac{\partial}{\partial \bar{x}_4}, \end{aligned} \tag{4.22}$$

where $E_{x_1}, E_{x_2}, E_{x_3} \in \mathfrak{X}(F)$. Using Eq. (4.21), we obtain that

$$\begin{aligned} \mathcal{P}(E_{x_1}) &= \frac{\partial}{\partial \bar{x}_4} + \cos(x_2)\frac{\partial}{\partial \bar{x}_5} + \sin(x_2)\frac{\partial}{\partial \bar{x}_6}, \\ \mathcal{P}(E_{x_2}) &= -x_1\sin(x_2)\frac{\partial}{\partial \bar{x}_5} + x_1\cos(x_2)\frac{\partial}{\partial \bar{x}_6}, \\ \mathcal{P}(E_{x_3}) &= \frac{\partial}{\partial \bar{x}_1}. \end{aligned} \tag{4.23}$$

From Eqs. (4.22) and (4.23), it can be observed that $\mathfrak{D}_\lambda$ is a proper slant distribution spanned by $\{E_{x_1}, E_{x_3}\}$, with a slant coefficient $\lambda = \frac{1}{\sqrt{2}}$. Additionally, $\mathfrak{D}^\perp$ is an anti-invariant distribution spanned by $\{E_{x_2}\}$, with a non-zero dimension. Consequently, F becomes a proper PRPS submanifold. The induced pseudo-Riemannian non-degenerate metric tensor g on F is described by

$$g_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & x_1^2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

that is, $g = 2dx_1^2 - dx_3^2 + x_1^2 dx_2^2$. Thus, F is a 3-dimensional PRWP submanifold of $\mathcal{R}^6$ with warping function $f = x_1$. Next, assume the normal bundle of F is expressed by the following vectors:

$$W_1 = \frac{\partial}{\partial \bar{x}_6},\ W_2 = \frac{\partial}{\partial \bar{x}_5},\ W_3 = (-\cos x \cot x - \sin x)\frac{\partial}{\partial \bar{x}_1} + \cot x_2 \frac{\partial}{\partial \bar{x}_2} + \frac{\partial}{\partial \bar{x}_3},$$

By doing some computation, we have

$$tE_{x_1} = E_{x_3},\ tE_{x_3} = \frac{1}{2}E_{x_1},\ nE_{x_2} = y\cos xW_1 - y\sin xW_2, \tag{4.24}$$

$$tE_{x_2} = 0,\ ntE_{x_3} = \frac{\sin x}{2}W_1 + \frac{\sin x}{2}W_2,\ ntE_{x_1} = -\frac{\sin x}{2}W_3. \tag{4.25}$$

By using the Weingarten formula, we have

$$\mathcal{A}_{ntE_{x_1}}E_{x_2} = -\frac{1}{2x_1}E_{x_2},\ \mathcal{A}_{ntE_{x_1}}E_{x_2} = 0, \tag{4.26}$$

$$\mathcal{A}_{nE_{x_3}}tE_{x_1} = 0,\ \mathcal{A}_{nE_{x_2}}tE_{x_3} = 0. \tag{4.27}$$

Combining Eqs. 4.24-4.27, we verifies the $part-(i)$ of Theorem 4.2. Furthermore, let us consider the vector fields U and V as follows

$$U = AE_{x_1} + BE_{x_2} + CE_{x_3},\ V = A_1E_{x_1} + B_1E_{x_2} + C_1E_{x_3},$$

for $A, B, C, A_1, B_1, C_1 \in C^\infty(F)$. By the application of Eq. 3, we obtain

$$P_\perp U = BE_{x_2},\ P_\lambda U = A_1E_{x_1} + C_1E_{x_3}, \tag{4.28}$$

$$P_\perp V = B_1E_{x_2},\ P_\lambda U = AE_{x_1} + CE_{x_3}. \tag{4.29}$$

By doing simple computation, we compute the following expression

$$t\nabla\mu = \frac{1}{2x_1},\ g(P_\perp U, P_\perp V) = BB_1x_1^2, \tag{4.30}$$

$$tP_\lambda V(\mu) = \frac{C_1}{2},\ tP_\lambda V(\mu)P_\perp U = \frac{BC_1}{2}E_{x_2}, \tag{4.31}$$

where $\mu = \ln x_1$. Now, we compute the expression $(\nabla_U t)V$ for F; by using the definition of covariant derivative, we have

$$(\nabla_U t)V = \nabla_U tV - t\nabla_U V. \tag{4.32}$$

By using Eq. 4.24, we obtain

$$\nabla_U tV = \nabla_{(AE_{x_1}+BE_{x_2}+CE_{x_3})}\left(A_1E_{x_3} + \frac{C_1}{2}E_{x_3}\right).$$

Now, using the property of Levi-Civita connection, we compute

$$\begin{aligned}\nabla_U tV =& AE_{x_1}(A_1)E_{x_3} + BE_{x_2}(A_1)E_{x_3} + AE_{x_1}\left(\frac{C_1}{2}\right)E_{x_1} + CE_{x_3}(A_1)E_{x_3}\\ &+ BE_{x_2}\left(\frac{C_1}{2}\right)E_{x_1} + CE_{x_3}\left(\frac{C_1}{2}\right)E_{x_1} + \frac{BC_1}{2x_1}E_{x_2}.\end{aligned} \tag{4.33}$$

Similarly, we compute

$$\begin{aligned}t\nabla_U V =& AE_{x_1}(A_1)E_{x_3} + BE_{x_2}(A_1)E_{x_3} + AE_{x_1}\left(\frac{C_1}{2}\right)E_{x_1} + CE_{x_3}(A_1)E_{x_3}\\ &+ BE_{x_2}\left(\frac{C_1}{2}\right)E_{x_1} + CE_{x_3}\left(\frac{C_1}{2}\right)E_{x_1} - \frac{BB_1}{x_1}E_{x_3}.\end{aligned} \tag{4.34}$$

Combining Eqs. (4.32)-(4.34), we obtain

$$(\nabla_U t)V = \frac{BB_1}{x_1}E_{x_3} + \frac{BC_1}{2x_1}E_{x_2}. \tag{4.35}$$

By the consequence of Eqs. (4.30)-(4.35), we verifies the Theorem 4.2.

Finally, we establish a criterion for a warped product submanifold of the form $F_\lambda \times_f F_\perp$ to be a PRPS product.

Proposition 4.1. *A PRWP submanifold of the form $F = F_\lambda \times_f F_\perp$ of a para-Kähler manifold $\bar{F}^{2m}$ is locally a PRPS product if $g(h(E_1, E_2), ntE_3)$ vanishes, for any $E_1, E_2 \in \mathfrak{X}(\mathfrak{D}_\perp)$ and $E_3 \in \mathfrak{X}(\mathfrak{D}_\lambda)$.*

Proof. By the consequence of (2.7), we have $g(\nabla_{E_1}E_3, E_2) = \bar{g}(\bar{\nabla}_{E_1}E_3, E_2)$. By the application of (2.1),(2.3), (2.12), (2.13) and (2.9) in the previous relation, we concede that

$$\begin{aligned} g(\nabla_{E_1}E_3, E_2) &= -\,\bar{g}(\mathcal{P}\bar{\nabla}_{E_1}E_3, \mathcal{P}E_2) = -\bar{g}(\bar{\nabla}_{E_1}(tE_3), \mathcal{P}E_2) - \bar{g}(\bar{\nabla}_{E_1}(nE_3), \mathcal{P}E_2) \\ &= \bar{g}(\bar{\nabla}_{E_1}(t^2E_3), E_2) + \bar{g}(\bar{\nabla}_{E_1}(ntE_3), E_2) - \bar{g}(\bar{\nabla}_{E_1}nE_3, \mathcal{P}E_2). \end{aligned}$$

Using Eq. (2.8), definition of slant submanifold and orthogonality in the above relation then we receive that

$$g(\nabla_{E_1}E_3, E_2) = \lambda\bar{g}(\bar{\nabla}_{E_1}E_3, E_2) - g(h(E_1, E_2), ntE_3) + g(\nabla^\perp_{E_1}nE_3, \mathcal{P}E_2). \tag{4.36}$$

Thus, from Eqs. (4.36) and the covariant derivative identities, we observe that

$$(1 - \lambda)(E_3 \ln f)g(E_1, E_2) = -g(h(E_1, E_2), ntE_3) + g(\nabla^\perp_{E_1}nE_3, \mathcal{P}E_2). \tag{4.37}$$

We obtain the following relation by subtracting from (4.37) after exchanging E_1 and E_2 in Eq. (4.37)

$$g(\nabla^\perp_{E_1}nE_3, \mathcal{P}E_2) = g(\nabla^\perp_{E_2}nE_3, \mathcal{P}E_1). \tag{4.38}$$

By the virtue of (2.1), (2.3), (2.12) and (2.7)-(2.8), we derive that

$$g(\nabla^\perp_{E_1}nE_3, \mathcal{P}E_2) = -(E_3 \ln f)g(E_1, E_2) - \bar{g}(\bar{\nabla}_{E_1}tE_3, \mathcal{P}E_2). \tag{4.39}$$

By exchanging E_1 with E_2 in Eq. (4.39) concludes that Eq. (4.38) holds only if

$$\bar{g}(\bar{\nabla}_{E_1}tE_3, \mathcal{P}E_2) = -\bar{g}(\bar{\nabla}_{E_1}\mathcal{P}E_2, tE_3) = 0 \tag{4.40}$$

Using Eqs. (2.7), (2.8), (2.1),(2.3) and (2.12) in Eq. (4.40), we achieve

$$\lambda(E_3 \ln f)g(E_1, E_2) + g(h(E_1, E_2), ntE_3) = 0. \tag{4.41}$$

Since F_λ is proper slant submanifold and E_i $\forall i \in \{1, 2, 3\}$ are non-null vector fields. Thus from Eq. (4.41), we can say that f is constant when $g(h(E_1, E_2), ntE_3) = 0$. This completes the proof. ☐

5 Conclusion

This paper presents several significant findings related to parallel canonical structures on PRPS submanifolds in para-Kähler manifolds $\bar{F}^{2m}$. We provide conditions that determine both the existence and non-existence of PRWP submanifolds possessing a genuine slant factor within the complex manifold $\bar{F}^{2m}$. By carefully analyzing these criteria, we offer a comprehensive understanding of when such submanifolds can or cannot exist in this particular geometric setting. Additionally, we present characterization results concerning the Weingarten operator and endomorphism in the setting of PRWP submanifolds of the specified form $F_\lambda \times_f F_\perp$. These results provide deeper insights into the geometric structure and behavior of such submanifolds, offering a detailed description of how the Weingarten operator interacts with the endomorphism in this specific context. A numerical example is provided to demonstrate and clarify the concept of the warped product $F_\lambda \times_f F_\perp$ and validate the characterization results in $\bar{F}^{2m}$. Finally, we formulate a condition that determines when a warped product submanifold of the form $F_\lambda \times_f F_\perp$ can be classified as a $\mathcal{PR}$-pseudo-slant product within the manifold $\bar{F}^{2m}$.

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