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Finite-time blow-up of solutions for a Hartree type wave equation involving distributed delay, fractional conditions, and infinite memory effects

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Abstract. This paper studies a Hartree-type wave equation featuring a distributed delay term and memory effects governed by a past history. The problem is formulated with coupling through fractional boundary conditions. Under suitable assumptions and for negative initial energy, we prove that solutions blow up in finite time.

Keywords. Blow up, past history, fractional damping, distributed delay, Hartree-type nonlinearity.

1 Introduction

In this paper, we consider the following problem of Hartree-type wave equation:

$$\begin{cases} u_{tt} - \Delta u + (h * u) + a_1 u_t + \displaystyle\int_{t_1}^{t_2} a_2(\kappa)\partial_t^{\alpha,\eta} u\,(x, t-\kappa)\, d\kappa = \mathbb{F}(u), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,-t) = f_0(x,t), & in \ \ \Omega \times \mathbb{R}_+ \\ u_t(x,-t) = f_1(x,t), & x \in \Omega, t \in (0, t_2). \end{cases} \tag{1.1}$$

The function $\mathbb{F}$ is given by

$$\begin{aligned} \mathbb{F}(u) &:= \left(\frac{1}{|x|^{n-2}} * |u|^p\right)|u|^{p-2}u, \\ (h * u) &:= \int_0^\infty h(s)\Delta u(t-s)ds, \end{aligned}$$

where

$$\frac{1}{|x|^{n-2}} * |u|^p = \int_\Omega \frac{|u(y)|^p}{|x-y|^{n-2}} dy,$$

the expression $\mathbb{F}(u)$ is a generalization of the Hartree term $\left(\frac{1}{|x|} * |u|^2\right)u$. Here Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 5$ with a sufficiently smooth boundary $\partial\Omega$ of class C^2. $a_1 > 0, \ 1 < p < \frac{n+4}{n-4}$, and $a_2 \in L^\infty$, $0 < t_1 \leq t_2$ is referred to the function of distributed delay.

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On the other hand, the symbol $\partial_t^{\alpha,\eta}$ symbolizes the generalized fractional derivative of Caputo, where $0<\alpha<1$ [4, 15], given by the following expression:

$$\partial_t^{\alpha,\eta}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}e^{-\eta(t-s)}u_s(s)ds,\quad \eta\geq 0,$$

where

$$\partial_t^{\alpha,\eta}u(t)=I^{1-\alpha,\eta}u_t(t),\tag{1.2}$$

and

$$I^{\alpha,\eta}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}e^{-\eta(t-s)}u(s)ds,\quad \eta\geq 0.$$

Here $\Gamma, I^{\alpha,\eta}$ are the Euler gamma function and the operator of the exponential fractional integro-differential, respectively.

Of course, our equation (1.1) is closed related to the following stationary equation:

$$-\Delta u+V(x)u=\left(\frac{1}{|x|^v}*|u|^p\right)|u|^{p-2}u,\tag{1.3}$$

where $\frac{2n-v}{n}\leq p\leq\frac{2n-v}{n-2}$.

The expression (1.3) in the special case $((n,p)=(1,2))$ refers to the helium atom. For more information, see [26]. Pekar [28] in 1976 also considered it a description of the quantum theory of stationary polarons. Reference is made to the work of Choquard and Lieb in [22]. For other degrees, we refer the reader to [24] and the works of Petrovsky. This type of problem and equation arises in various branches of science and physics, such as acoustics, optics,... etc.

In recent years, many research papers have addressed this type of problem, including [33]. In this work, the authors studied the following equation.

$$u_{tt}-\Delta u=\left(\frac{1}{|x|^{n-2}}*|u|^p\right)|u|^{p-2}u,\tag{1.4}$$

they considered a class of wave equations of Hartree type on a bounded smooth convex domain with Dirichlet boundary condition. By applying the standard semigroup theory, they obtained the local existence result. Using potential well theory, they derived the condition of global existence of weak solutions. With the help of potential well theory and convexity method, they proved the blow-up results for solutions when the initial energy is nonnegative or negative.

In [17], the authors consedered the following Hartree-type Petrovsky equations

$$z_{tt}+\Delta^2z-\Delta z=\left(\frac{1}{|x|^{n-2}}*|z|^p\right)|z|^{p-2}z,\tag{1.5}$$

they proved the global existence of weak solutions by applying the potential well theory. Furthermore, they investigated the blow-up phenomena of solutions under nonnegative or negative initial energy conditions.

Delay plays a significant role in many natural phenomena and arises in a wide range of problems in the form of constant, distributed, or time-varying delays. Concerning the distributed delay considered in the present study, we refer to the work of Nicaise et al. [27], where the following problem was investigated:

$$u_{tt}-\Delta u+\mu_1u_t+\int_{\tau_1}^{\tau_2}\mu_2(s)u(x,t-s)ds=0.$$

In that work, the authors established well-posedness and general decay results for solutions under appropriate assumptions on the delay function. Subsequently, numerous researchers extended this line of research by incorporating delay effects into various models and examining issues such as well-posedness, global existence, general decay, blow-up, and exponential growth of solutions (see, for instance, [11]).

Several analytical approaches have been used to study such problems, including the energy method combined with semigroup theory, as well as the Faedo–Galerkin method, which is particularly effective for nonlinear problems that cannot be formulated in a linear framework. We mention here some representative contributions.

In [12], the authors investigated a nonlinear viscoelastic Kirchhoff-type equation with distributed delay and variable exponents. Under suitable assumptions, they proved the blow-up of solutions. Moreover, in the absence of a source term, general decay results were derived using an integral inequality due to Komornik. Additional related works can be found in [7].

Over the past few years, there has been an increasing interest in fractional derivatives of partial differential equations. Some physical phenomena are transformed into successful models through initial boundary value and fractional boundary conditions. They are applied in various fields of science, the following papers can be viewed Magin [24]. This book deals with fractional calculus and some of its applications. Tarasov [32]. Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields, and Media, this book presents the applications of fractional calculus (integrals and derivatives of non-integer order) to describe physical systems exhibiting long-time memory, non-local spatial interactions, and fractal properties.

As for the term logarithmic source, we can address some works related to the wave equation, including. In [10], the authors considered a nonlinear viscoelastic plate equation with logarithmic nonlinearity and variable-exponents. They showed the global existence and by using Komornik the general decay result are obtained. Finally, the blow-up of solutions are proved with negative initial energy.

Within the framework of works similar to ours, there is the work [21] where the authors considered the following:

$$y_{tt} - \Delta y + \partial_t^{\alpha} y = |y|^{p-2} y.$$

They studied the exponential growth of solutions. Regarding the good positioning and explosion of solutions, there is the work [20] where the authors studied the well-posedness and then proved the blow-up of solutions in a certain time.

$$u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u(x, t-\tau) = |y|^{p-2} y \ln |y|^{k}.$$

Then came the work [4] in which the authors considered the following problem:

$$y_{tt} - \Delta y + a_1 \partial_t^{\alpha,\beta} y(t-s) + a_2 y_t = |y|^{p-2} y.$$

By the semi-group theory, they established the existence of solutions of the problem and they proved a decay rate estimate for the energy. Also, they proved that the solution blows up in finite time if the initial energy is negative. As for the introduction of the fractional condition into the Timoshenko system, we refer the reader to the work [5] where the authors proved the existence and uniqueness of solutions using the Faedo-Galarkin method and then proved the general decay of solutions under appropriate assumptions on the coefficients.

The relationship between the stress and strain history in the beam inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term of memory (finite or infinite) is the function h. There are many works that talk about this topic with a lot of new and innovative results, especially the hypotheses on the kernel and the initial conditions, we have two type of the memory (finite or infinite).

In the case of finite memory reccalling the following works: in [9] the authors studied the blow-up result of the solution of a coupled nonlocal singular viscoelastic equation with general source and localized frictional damping terms under some suitable conditions.

In the case of infinite memory we have a some works from them: in [14] this work it is the first work to convert the term infinite memory into linear writing based on changing an appropriate variable that is easy to deal with with this term in the researchs that came after it. We mention them for example: in [18] the authors considered in this paper the problem of asymptotic behavior of solutions for two viscoelastic wave equations with infinite memory, they showed that the stability of the system, and [2] the authors are concerned with a coupled system of viscoelastic wave equations in the presence of infinite-memory terms. they showed that the stability of the system holds for a much larger class of kernels, in [19] the authors considered a class of second order abstract linear hyperbolic equations with infinite memory and distributed time delay. Under appropriate assumptions on the infinite memory and distributed time delay convolution kernels, they proved the results of the well-posedness and the stability of the system. For more information see [13].

Recently, in [6] the authors considered a viscoelastic wave equation with a time delay term in internal fractional feedback.

$$w_{tt} - \Delta w + \int_0^t f(t-\sigma)\Delta w(\sigma)d\sigma + \mu_1 w_t + \mu_2 \partial_t^{\alpha,\beta} w(t-\tau) = 0.$$

By employing the energy method along with the Faedo-Galerkin procedure, they established the global existence of solutions, subject to certain conditions. Additionally, they demonstrated how appropriate Lyapunov functionals can lead to general decay results of the energy.

In [10], this work deals with a wave equation with acoustic and fractional boundary conditions coupled by source and delay terms as follows:

$$\begin{aligned} &u_{tt} - \Delta u + \mathfrak{A}_1 u_t + \mathfrak{A}_2 u(t-\tau) = |u|^{p-2}u, \\ &\frac{\partial u}{\partial \nu} = -\mathfrak{B}\partial_t^{\alpha,\eta} u + \chi_t, \\ &u_t + P(x)\chi_t + Q(x)\chi = 0. \end{aligned}$$

Under some hypotheses, they studied the global existence of the solution and by suitable Lyapunov functions the general decay results are proved.

Inspired by these studies, we introduce a novel formulation by integrating the fractional condition with distributed delay and in presence of the past history, leading to a problem distinct from prior investigations. This paper is dedicated to its analysis.

The work is structured as follows: Section 2 presents the necessary preliminaries, including essential concepts, lemmas, assumptions, and the definition of an energy functional. Section 3 establishes the main blow-up result. Finally, a general conclusion and directions for future work are provided.

Throughout this paper, the symbols c and C denote generic positive constants.

2 Preliminaries

This section presents the key notations, assumptions, and lemmas necessary for establishing our main results.

Lemma 2.1. *[1]*
Let p be a number with $1 \leq p \leq +\infty$ $(N = 1, 2)$ or $1 \leq p \leq \frac{N+2}{N-2}$, $(N \geq 3)$. Then, $\exists C_* = B_{p,\Omega} > 0$ such that

$$\|u\|_{p+1} \leq C_* \|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

Lemma 2.2. *(Hardy-Littlewood-Sobolev inequality) [23],[31]*
Suppose that $\sigma; p > 1$ and $0 < \mu < n$ with $\frac{1}{\sigma} + \frac{\mu}{n} + \frac{1}{p} = 2$; $f \in L^\sigma(\mathbb{R}^n)$ and $h \in L^s(\mathbb{R}^n)$: There is a sharp constant $C(\sigma; n; \mu; s)$; independent of f, h so that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\mu} dxdy \leq C(\sigma, n, \mu, s) \|f\|_\sigma \|h\|_p. \tag{2.1}$$

For all $u \in H^1(\mathbb{R}^n)$. $\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy$ is well defined by $\frac{2n-\mu}{n} \leq p \leq \frac{2n-\mu}{n-2}$.
From Lemma 2.2, for $u \in H_0^1(\Omega)$, we define $u(x) = 0$ by $x \in \mathbb{R}^n / \Omega$:
Therefore $u \in H^2(\mathbb{R}^n)$; i.e. for a general field, for $u \in L^\sigma(\mathbb{R}^n)$ and $\omega \in L^s(\mathbb{R}^n)$, we find the following the Hardy-Littlewood-Sobolev inequality:

$$\int_\Omega \int_\Omega \frac{u(x)\omega(y)}{|x-y|^\mu} dxdy \leq C(\sigma, n, \mu, s, \Omega) \|u\|_\sigma \|\omega\|_p, \tag{2.2}$$

and the integral if $\frac{2n-\mu}{n} \leq p \leq \frac{2n-\mu}{n-2}$ then $\int_\Omega \int_\Omega \frac{|u(x)|^p |\omega(y)|^p}{|x-y|^\mu} dxdy$ is well defined for $u \in H_0^1(\Omega)$. Hence, for $u \in H_0^1(\Omega)$, applying the Sobolev embedding theorem and the Hardy- Littlewood-Sobolev inequality (2.2), gives

$$\int_\Omega \int_\Omega \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dxdy \leq C_1(n, p, \Omega) \|u\|^{2p}_{\frac{2np}{n+2}} \leq C_2 \|\nabla u\|^{2p}, \tag{2.3}$$

here $C_2 = C_1(n, p, \Omega) c_*^{2p}$, where C_1 and c_* are the Hardy-Littlewood-Sobolev and the Sobolev embedding constants respectively. Next, by applying the Sobolev embedding theorem,

$$\frac{n+2}{n} < p < \frac{n+2}{n-2}.$$

Theorem 2.1. *[25]*

Let μ be the function

$$\mu(\xi) = |\xi|^{\frac{(2\alpha-1)}{2}}, \quad 0 < \alpha < 1, \ \xi \in \mathbb{R}. \tag{2.4}$$

Then, we can get

$$O = I^{1-\alpha,\eta} U. \tag{2.5}$$

where represents the relationship between U the "input" and the "output" O of the following system

$$\partial_t \phi(x, \kappa, \xi, t) + (\xi^2 + \eta)\phi(x, \kappa, \xi, t) - U(x, \kappa, t)\mu(\xi) = 0, \quad t > 0, \eta \geq 0, \ \xi \in \mathbb{R}, \tag{2.6}$$

$$\phi(x, \kappa, \xi, 0) = 0, \tag{2.7}$$

$$O(x, \kappa, t) = \frac{\sin(\alpha\pi)}{\pi} \int_{-\infty}^{+\infty} \phi(x, \kappa, \xi, t)\mu(\xi) d\xi, \ \xi \in \mathbb{R}, \kappa \in [\tau_1, \tau_2], \ t > 0. \tag{2.8}$$

Lemma 2.3. *[8] For all* $\lambda \in D_\eta := \{\lambda \in \mathbb{C} : \Re e\lambda + \eta > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$,

$$A_\lambda = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda + \eta + \xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)} (\lambda + \eta)^{\alpha - 1}. \tag{2.9}$$

To help us achieve our objective, we put this important supposition on a_2:
(H1) $a_2 : [t_1, t_2] \to \mathbb{R}$ is a bounded function satisfying

$$a_1 > 2bA_0 \int_{t_1}^{t_2} |a_2(\kappa)| d\kappa. \tag{2.10}$$

(**H2**) $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing C^1 function satisfing

$$h_0 = \int_0^\infty h(s) ds > 0, \quad 1 - h_0 = \Im > 0. \tag{2.11}$$

(**H3**)There exists a positive constant Ξ such that

$$h'(t) \leq -\Xi h(t), \ \forall t \geq 0. \tag{2.12}$$

As in [27], taking the following new variables

$$z(x, \varpi, \kappa, t) = u_t(x, t - \kappa\varpi),$$

where

$$(x, \varpi, \kappa, t) \in \mathfrak{D} := \Omega \times (0, 1) \times (t_1, t_2) \times \mathbb{R}_+,$$

which satisfy

$$\begin{cases} \kappa z_t(x, \varpi, \kappa, t) + z_\varpi(x, \varpi, \kappa, t) = 0 \\ z(x, 0, \kappa, t) = u_t(x, t). \end{cases} \tag{2.13}$$

Set an auxiliary variable as in [14]

$$\varphi^t(x, s) = u(x, t) - u(x, t - s), s \geq 0.$$

Then,

$$\varphi_t^t(x, s) + \varphi_s^t(x, s) = u_t(x, t). \tag{2.14}$$

Hence, thank's the relation (1.2) and theorem 2.1 we find

$$\begin{cases} u_{tt} - \Im\Delta u - \int_0^\infty h(s)\Delta\varphi^t(s) ds + a_1 u_t + b \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} a_2(\kappa)\phi(x, \xi, \kappa, t)\mu(\xi) d\xi d\kappa = \mathbb{F}(u), \\ \partial_t \phi(x, \xi, \kappa, t) + (\xi^2 + \eta)\phi(x, \xi, \kappa, t) - z(x, 1, \kappa, t)\mu(\xi) = 0, \\ \kappa z_t(x, \varpi, \kappa, t) + z_\varpi(x, \varpi, \kappa, t) = 0, \\ \varphi_t^t(x, s) + \varphi_s^t(x, s) = u_t(x, t), \\ u(x, t) = 0, \qquad x \in \partial\Omega, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \\ z(x, \varpi, \kappa, 0) = f_1(x, \kappa\varpi), \quad \kappa \in (0, t_2), \\ u(x, t) = \varphi^t(x, s) = 0, \quad x \in \ \partial\Omega, \quad t, s \in (0, \infty), \\ \varphi^t(x, 0) = 0, \quad in \ \ \Omega \times (0, \infty), \\ \varphi^0(x, s) = \varphi_0(x, s) = f_0(x, 0) - f_0(x, s), \quad in \ \ \Omega \times (0, \infty). \end{cases} \tag{2.15}$$

where

$$(x, \varpi, \kappa, t) \in \mathfrak{D}, \quad \xi \in \mathbb{R} \quad , \quad s \in \mathbb{R}_+ \quad \text{and} \quad b = \frac{\sin(\alpha\pi)}{\pi}.$$

Now, we give the well posedness result for the problem (2.15), which can be established by energy method and combination between the results [4], [20] and [30] with the necessary changes.

Theorem 2.2. *Suppose that (2.10)-(2.12) are satisfied. Then, for any* $(u_0, u_1, \phi_0, f_1, \varphi^0) \in \mathcal{H}$*, there exists a weak solution* (u, ϕ, z, φ^t) *of problem (2.15) such that*

$$\begin{aligned}
&u, u_t \in C(]0,T[, H_0^1(\Omega)) \cap C^1(]0,T[, L^2(\Omega)),\\
&u_{tt} \in C(]0,T[, L^2(\Omega)),\\
&\phi \in C([0,T]; L^2(\Omega \times \mathbb{R} \times (t_1,t_2))),\\
&z \in C([0,T]; L^2(\Omega \times (0,1) \times (t_1,t_2))),\\
&\varphi^t \in C([0,T]; L_h^2(\mathbb{R}_+, H_0^1(\Omega))),
\end{aligned}$$

where

$$\mathcal{H} := H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times \mathbb{R} \times (t_1,t_2)) \times L^2(\Omega \times (0,1) \times (t_1,t_2)) \times L_h^2(\mathbb{R}_+, H_0^1(\Omega)),$$

with

$$L_h^2(\mathbb{R}_+, H_0^1(\Omega)) = \{\chi : \mathbb{R}_+ \to H_0^1(\Omega), \quad \int_\Omega \int_0^\infty h(s)|\nabla \chi(s)|^2 ds dx < \infty\}.$$

Also, we indroduce the following notation for simplicity:

$$(h \circ u)(t) := \int_\Omega \int_0^\infty h(s)|u(t) - u(t-s)|^2 ds dx = \int_\Omega \int_0^\infty h(s)|\varphi^t(s)|^2 ds dx.$$

Next step we introduce E the energy function of the system (2.15) with definition and proof.

Lemma 2.4. *Let* (u, ϕ, z, φ^t) *be the solution of (2.15). Then, we give the energy functional by*

$$\begin{aligned}
E(t) \quad &= \quad \frac{1}{2}\|u_t\|_2^2 + \frac{\Im}{2}\|\nabla u\|_2^2 - \frac{1}{2p}\int_\Omega \int_\Omega \frac{|u(x)|^p |u(y)|^p}{|x-y|^{n-2}} dx dy\\
&+\frac{b}{2}\int_\Omega \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} |a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx + \frac{1}{2}(h \circ \nabla u)(t)\\
&+bA_0 \int_\Omega \int_0^1 \int_{t_1}^{t_2} \kappa |a_2(\kappa)||z(x,\varpi,\kappa,t)|^2 d\kappa d\varpi dx, \qquad (2.16)
\end{aligned}$$

satisfies

$$\begin{aligned}
E'(t) \quad &\leq \quad -C_0\|u_t\|_2^2 + \frac{1}{2}(h' \circ \nabla u)(t)\\
&-\frac{b}{2}\int_\Omega \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} |a_2(\kappa)|(\xi^2 + \eta)|\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx \leq 0, \qquad (2.17)
\end{aligned}$$

where

$$C_0 = a_1 - 2bA_0 \int_{t_1}^{t_2} |a_2(\kappa)| d\kappa > 0.$$

Remark 1. In the case of equality in the hypothesis (2.10) then the number $(C_0 = 0)$ but does not affect the sign of the derivative of the energy function and thus remains decreasing.

Proof. Firstly, multiplying the equation $(2.15)_1$ by u_t, using integration by parts over Ω, we get

$$\int_\Omega u_{tt}u_t - \Im \int_\Omega \Delta u u_t dx + a_1\|u_t\|_2^2 + b\int_\Omega u_t \int_{t_1}^{t_2} a_2(\kappa) \int_{-\infty}^{+\infty} \mu(\xi)\phi(x,\xi,\kappa,t) d\xi d\kappa dx$$

$$-\int_\Omega u_t(t)\int_0^\infty h(s)\Delta\varphi^t(s)dsdx=\int_\Omega \mathbb{F}(u)\ u_t\ dx.$$

By (2.14) and integration by parts, we have

$$-\int_\Omega u_t(t)\int_0^\infty h(s)\Delta\varphi^t(s)dsdx \quad = \quad \frac{1}{2}\frac{d}{dt}(h\circ\nabla u)(t)-\frac{1}{2}(h'\circ\nabla u)(t). \tag{2.18}$$

Therefore

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2+\frac{\Im}{2}\|\nabla u\|_2^2+\frac{1}{2}(h\circ\nabla u)(t)-\frac{1}{2p}\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right]\\&+a_1\|u_t\|_2^2-\frac{1}{2}(h'\circ\nabla u)(t)\\&+b\int_\Omega u_t\int_{t_1}^{t_2}a_2(\kappa)\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,\kappa,t)d\xi d\kappa dx=0.\end{aligned}\tag{2.19}$$

Next, multiplying the equation $(2.15)_2$ by $b|a_2(\kappa)|\phi$ and integrating over $\Omega\times(t_1,t_2)\times(-\infty,+\infty)$, we find

$$\begin{aligned}&\frac{b}{2}\frac{d}{dt}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx\\&+b\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx\\&-b\int_\Omega\int_{t_1}^{t_2}|a_2(\kappa)|z(x,1,\kappa,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,\kappa,t)d\xi d\kappa dx=0.\end{aligned}\tag{2.20}$$

After that, multiplying the equation $(2.15)_3$ by $z|a_2(\kappa)|$, integrating over $\Omega\times(0,1)\times(t_1,t_2)$ and using $(2.13)_2$, we get

$$\begin{aligned}&\frac{d}{dt}bA_0\int_\Omega\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)|z^2(x,\varpi,\kappa,t)d\kappa d\varpi dx\\&=\quad bA_0\int_{t_1}^{t_2}|a_2(\kappa)|d\kappa\|u_t\|_2^2-bA_0\int_{t_1}^{t_2}|a_2(\kappa)|\|z(x,1,\kappa,t)\|_2^2d\kappa.\end{aligned}\tag{2.21}$$

Now, by using Cauchy-Schwarz inequality, we find

$$\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,\kappa,t)d\xi\le\left(\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta}d\xi\right)^{1/2}\left(\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2d\xi\right)^{1/2}$$

After that, Young's inequality, gives

$$\begin{aligned}&b\int_\Omega\int_{t_1}^{t_2}|a_2(\kappa)|z(x,1,\kappa,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,\kappa,t)d\xi d\kappa dx\\&\le\quad bA_0\int_\Omega\int_{t_1}^{t_2}|a_2(\kappa)||z(x,1,\kappa,t)|^2d\kappa dx\\&\quad+\frac{b}{4}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx.\end{aligned}\tag{2.22}$$

On the other hand, by using Young's inequality, we have

$$b\int_\Omega\int_{t_1}^{t_2}|a_2(\kappa)u_t\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,\kappa,t)d\xi d\kappa dx\le bA_0(\int_{t_1}^{t_2}|a_2(\kappa)|d\kappa)\|u_t\|_2^2$$

$$+\frac{b}{4}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx. \tag{2.23}$$

From (2.19)-(2.23), we obtain (2.16) and

$$\begin{aligned}\frac{d}{dt}E(t) &\leq -\Big(a_1-2bA_0\int_{t_1}^{t_2}|a_2(\kappa)|d\kappa\Big)\|u_t\|_2^2+\frac{1}{2}(h'\circ\nabla u)(t)\\ &-\frac{b}{2}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx\leq 0.\end{aligned}$$

Keeping in mind condition (2.10), we observe that

$$C_0=a_1-2bA_0\int_{t_1}^{t_2}|a_2(\kappa)|d\kappa>0. \tag{2.24}$$

Then, by (2.12) we obtain (2.17). Consequently

$$E(t)\leq E(0). \tag{2.25}$$

□

Now, we need the following lemmas especially in the next sections for proof the our results. The proof of these lemmas is based mainly on the reference [20] with some basic changes.

Lemma 2.5. $\exists\, c(\Omega)>0$, *such that*

$$\Big(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big)^{\frac{\sigma}{2p}}\quad\leq\quad c\Big(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\|\nabla u\|_2^2\Big),$$

for all $2\leq\sigma\leq 2p$, *provided that* $\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\geq 0$.

Lemma 2.6. $\exists\, c(\Omega)>0$, *such that*

$$\|u\|_{2p}^{2p}\quad\leq\quad c\Big(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\|\nabla u\|_2^2\Big),$$

$\forall u\in L^{2p}(\Omega)$ *and provided that* $\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\geq 0$.

Corollary 2.3. $\exists\, c(\Omega)>0$, *such that*

$$\|u\|_2^2\quad\leq\quad c\Big[\Big(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\Big)^{\frac{2}{2p}}+\|\nabla u\|_2^{\frac{4}{2p}}\Big],$$

provided that $\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\geq 0$.

Lemma 2.7. $\exists\, c(\Omega)>0$, *such that*

$$\|u\|_{2p}^{\sigma}\quad\leq\quad c\Big(\|u\|_{2p}^{2p}+\|\nabla u\|_2^2\Big),$$

$\forall u\in L^{2p}(\Omega)$ *and* $2\leq\sigma\leq 2p$.

Before proving the results of the blow-up results, we define the functional

$$\begin{aligned}\mathbb{H}(t)=-E(t) &= -\frac{1}{2}\|u_t\|_2^2-\frac{\Im}{2}\|\nabla u(t)\|_2^2+\frac{1}{2p}\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\\ &-\frac{1}{2}(h\circ\nabla u)(t)-\frac{b}{2}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx\\ &-bA_0\int_\Omega\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)||z\left(x,\varpi,\kappa,t\right)|^2d\kappa d\varpi dx. \end{aligned}\tag{2.26}$$

Hence

$$\begin{aligned}\mathbb{H}'(t) &\geq C_0\|u_t\|_2^2-\frac{1}{2}(h'\circ\nabla u)(t)\\ &+\frac{b}{2}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx.\end{aligned}\tag{2.27}$$

Therfore, we have

$$\begin{aligned}\mathbb{H}'(t) &\geq C_0\|u_t\|_2^2\geq 0\\ \mathbb{H}'(t) &\geq -\frac{1}{2}(h'\circ\nabla u)(t)\geq 0\\ \mathbb{H}'(t) &\geq \frac{b}{2}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx\geq 0.\end{aligned}\tag{2.28}$$

From (2.25), we have

$$0<\mathbb{H}(0)\leq\mathbb{H}(t)\leq\frac{1}{2p}\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy.\tag{2.29}$$

3 Blow up result

In this section, we prove the blow up result of solution of problem (2.15), also with negative initial energy.

Theorem 3.1. *Suppose that (2.10)-(2.12) and $E(0)<0$. Then, the solution of problem (2.15) blow up in finite time.*

Proof. First, we set

$$\mathcal{R}(t)=\mathbb{H}^{1-\alpha}(t)+\varepsilon\int_\Omega uu_tdx+\frac{\varepsilon a_1}{2}\int_\Omega u^2dx,\tag{3.1}$$

where $\varepsilon>0$ to be assigned later and

$$\frac{p-1}{p^2}<\alpha<\frac{p-1}{2p}<\frac{p-1}{p}<1.\tag{3.2}$$

By multiplying $(2.15)_1$ by u and with a derivative of (3.1), we get

$$\mathcal{R}'(t) = (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2+\varepsilon\int_\Omega\mathbb{F}(u)udx-\varepsilon\Im\|\nabla u\|_2^2$$

$$\underbrace{-\varepsilon b\int_\Omega u\int_{t_1}^{t_2} a_2(\kappa)\int_{-\infty}^{+\infty}\mu(\xi)\phi(x,\xi,\kappa,t)d\xi d\kappa dx}_{J_{01}}$$
$$\underbrace{-\varepsilon\int_\Omega \nabla u\int_0^{+\infty} h(s)\nabla\varphi^t(s)dsdx}_{J_{02}}. \tag{3.3}$$

By applying the Cauchy-Schwarz and Young's inequalities, we get for $\delta > 0$

$$\begin{aligned} J_{01} &\leq \varepsilon\delta bA_0\int_{t_1}^{t_2}|a_2(\kappa)|\|u\|_2^2 d\kappa \\ &\quad +\frac{b\varepsilon}{4\delta}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx, \end{aligned} \tag{3.4}$$

and

$$J_{02}\leq \varepsilon\frac{h_0}{2}\|\nabla u\|_2^2+\frac{\varepsilon}{2}\int_\Omega\int_0^{+\infty}h(s)|\nabla\varphi^t(s)|^2 dsdx. \tag{3.5}$$

By substituting (3.5)and (3.4) in (3.3) and recalling (2.10), we find

$$\begin{aligned} \mathcal{R}'(t) &\geq (1-\alpha)\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2+\varepsilon\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \\ &\quad -\varepsilon(\Im+\frac{h_0}{2})\|\nabla u\|_2^2-\varepsilon\delta bA_0\|u\|_2^2-\frac{\varepsilon}{2}\int_\Omega\int_0^{+\infty}h(s)|\nabla\varphi^t(s)|^2 dsdx \\ &\quad -\varepsilon\frac{b}{4\delta}\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)|(\xi^2+\eta)|\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx. \end{aligned} \tag{3.6}$$

In this second case we select δ in another appropriate way as follows:

$$\frac{1}{2\delta}=\vartheta\mathbb{H}^{-\alpha}(t),$$

by (2.28) and substituting in (3.6), we get

$$\begin{aligned} \mathcal{R}'(t) &\geq [(1-\alpha)-\varepsilon\vartheta]\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon\|u_t\|_2^2+\varepsilon\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \\ &\quad -\varepsilon(\Im+\frac{h_0}{2})\|\nabla u\|_2^2-\varepsilon\left(\frac{bA_0H^\alpha(t)}{2\vartheta}\right)\|u\|_2^2-\frac{\varepsilon}{2}(h\circ\nabla u)(t). \end{aligned} \tag{3.7}$$

Now, for $0 < a < 1$ and from (2.26), we have

$$\begin{aligned} &\varepsilon\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy \\ =\ &\varepsilon a\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\varepsilon p(1-a)\|u_t\|_2^2 \\ &+2\varepsilon p(1-a)\mathbb{H}(t)+\varepsilon p(1-a)\Im\|\nabla u\|_2^2+\varepsilon p(1-a)(h\circ\nabla u)(t) \\ &+\varepsilon p(1-a)b\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx \end{aligned}$$

$$+2bA_0p(1-a)\int_\Omega\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)||z\left(x,\varpi,\kappa,t\right)|^2d\kappa d\varpi dx. \tag{3.8}$$

By substituting in (3.7), we get the following estimate

$$\begin{aligned}
\mathcal{R}'(t) &\geq [(1-\alpha)-\varepsilon\vartheta]\mathbb{H}^{-\alpha}\mathbb{H}'(t)-\varepsilon\left(\frac{bA_0H^\alpha(t)}{2\vartheta}\right)\|u\|_2^2\\
&+\varepsilon a\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\varepsilon\Big(p(1-a)+1\Big)\|u_t\|_2^2\\
&+\varepsilon\left(p(1-a)\Im-(\Im+\frac{h_0}{2})\right)\|\nabla u\|_2^2+2\varepsilon p(1-a)\mathbb{H}(t)\\
&+\varepsilon\left(p(1-a)-\frac{1}{2}\right)(h\circ\nabla u)(t)\\
&+\varepsilon p(1-a)b\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2d\xi d\kappa dx\\
&+2bA_0p(1-a)\int_\Omega\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)||z\left(x,\varpi,\kappa,t\right)|^2d\kappa d\varpi dx.
\end{aligned} \tag{3.9}$$

On the other hand, accoding (2.29), Corollary 2.3 and Young's inequality, gives

$$\begin{aligned}
\mathbb{H}^\alpha(t)\|u\|_2^2 &\leq \left(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right)^\alpha\|u\|_2^2\\
&\leq c\left[\left(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right)^{\alpha+\frac{2}{2p}}+\left(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right)^\alpha\|\nabla u\|_2^{\frac{4}{2p}}\right]\\
&\leq c\left[\left(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right)^{\frac{(\alpha p+1)}{p}}+\left(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right)^{\frac{\alpha p}{(p-1)}}+\|\nabla u\|_2^2\right].
\end{aligned}$$

By (3.2), yields

$$2<2(\alpha p+1)\leq 2p\ \text{ and }\ 2<\frac{2\alpha p^2}{p-1}\leq 2p.$$

Hence, Lemma 2.5 gives

$$\mathbb{H}^\alpha(t)\|u\|_2^2 \leq c\left(\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\|\nabla u\|_2^2\right). \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\begin{aligned}
\mathcal{R}'(t) &\geq \Big\{(1-\alpha)-\varepsilon\vartheta\Big\}\mathbb{H}^{-\alpha}\mathbb{H}'(t)+\varepsilon\Big\{p(1-a)+1\Big\}\|u_t\|_2^2\\
&+\varepsilon\left(a-\frac{bA_0c}{2\vartheta}\right)\int_\Omega\int_\Omega\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+2\varepsilon p(1-a)\mathbb{H}(t)\\
&+\varepsilon\Big\{\underbrace{p(1-a)\Im-(\Im+\frac{h_0}{2})}_{\widehat{\mathfrak{D}}}-\frac{bA_0c}{2\vartheta}\Big\}\|\nabla u\|_2^2\\
&+\varepsilon\left(\frac{2p(1-a)-1}{2}\right)(h\circ\nabla u)(t)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon p(1-a)b\int_{\Omega}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx\\
&+2bA_0p(1-a)\int_{\Omega}\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)||z\left(x,\varpi,\kappa,t\right)|^2 d\kappa d\varpi dx. \qquad (3.11)
\end{aligned}
$$

At this point, we choose $a>0$ small enough so that

$$\mathfrak{D}_1 = p(1-a)-1>0 \Rightarrow 2p(1-a)-1>0.$$

Now, we assume that

$$\int_0^{+\infty} h(\varrho)d\varrho < \frac{p(a-1)-1}{p(a-1)-\frac{1}{2}} = \frac{2\mathfrak{D}_1}{2\mathfrak{D}_1+1}. \qquad (3.12)$$

Here's what he gives us

$$\widehat{\mathfrak{D}}>0.$$

Then we choose ϑ so large that

$$
\begin{aligned}
\mathfrak{D}_2 &= \widehat{\mathfrak{D}} - \frac{bA_0c}{2\vartheta}>0,\\
\mathfrak{D}_3 &= a-\frac{bA_0c}{2\vartheta}>0.
\end{aligned}
$$

Next step, we fixed ϑ, a, and we select ε so small that

$$\mathfrak{D}_4 = (1-\alpha)-\varepsilon\vartheta>0$$

and

$$\mathcal{R}(0)>0.$$

Hence, estimate (3.9) becomes for some $\mathfrak{m}_1>0$

$$
\begin{aligned}
\mathcal{R}'(t) \geq\ & \mathfrak{m}_1\Bigg\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\int_{\Omega}\int_{\Omega}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\\
&+\int_{\Omega}\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx+(h\circ\nabla u)\\
&+\int_{\Omega}\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)||z\left(x,\varpi,\kappa,t\right)|^2 d\kappa d\varpi dx\Bigg\}. \qquad (3.13)
\end{aligned}
$$

Next, using Holder's and Young's inequalities, we have

$$\left|\int_{\Omega}uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c\Big[\|u\|_{2p}^{\frac{\theta}{1-\alpha}}+\|u_t\|_2^{\frac{\mu}{1-\alpha}}\Big]. \qquad (3.14)$$

where $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\mu=2(1-\alpha)$, to get

$$\frac{\theta}{1-\alpha}=\frac{2}{2(1-\alpha)-1}\leq 2p,$$

this is achieved according to the relationship (3.2). After that, estimate (3.14) for $\nu=\frac{\theta}{1-\alpha}=\frac{2}{2(1-\alpha)-1}$, gives

$$\left|\int_{\Omega}uu_t dx\right|^{\frac{1}{1-\alpha}} \leq c\Big[\|u\|_{2p}^{\nu}+\|u_t\|_2^2\Big].$$

Then, Lemma 2.7 and Lemma 2.6 yields

$$\begin{aligned}
\left|\int_\Omega u u_t dx\right|^{\frac{1}{1-\alpha}} &\leq c\Big[\|u\|_{2p}^{2p}+\|u_t\|_2^2+\|\nabla u\|_2^2\Big]\\
&\leq c\Big[\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy+\|u_t\|_2^2+\|\nabla u\|_2^2\Big].
\end{aligned}\tag{3.15}$$

Hence,

$$\begin{aligned}
\mathcal{R}^{\frac{1}{1-\alpha}}(t) &= \left(\mathbb{H}^{1-\alpha}(t)+\varepsilon\int_\Omega u u_t dx+\varepsilon\frac{a_1}{2}\int_\Omega u^2 dx\right)^{\frac{1}{1-\alpha}}\\
&\leq c\left(\mathbb{H}(t)+|\int_\Omega u u_t dx|^{\frac{1}{1-\alpha}}+\|u\|_2^{\frac{2}{1-\alpha}}\right)\\
&\leq c\left(\mathbb{H}(t)+|\int_\Omega u u_t dx|^{\frac{1}{1-\alpha}}+\|u\|_{2p}^{\frac{2}{1-\alpha}}\right)\\
&\leq c\left\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\|u\|_{2p}^{2p}+\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right\}\\
&\leq c\left\{\mathbb{H}(t)+\|u_t\|_2^2+\|\nabla u\|_2^2+\int_\Omega\int_\Omega \frac{|u(x)|^p|u(y)|^p}{|x-y|^{n-2}}dxdy\right.\\
&\quad+\int_\Omega\int_{t_1}^{t_2}\int_{-\infty}^{+\infty}|a_2(\kappa)||\phi(x,\xi,\kappa,t)|^2 d\xi d\kappa dx+(h\circ\nabla u)\\
&\quad\left.+\int_\Omega\int_0^1\int_{t_1}^{t_2}\kappa|a_2(\kappa)||z\left(x,\varpi,\kappa,t\right)|^2 d\kappa d\varpi dx\right\}.
\end{aligned}\tag{3.16}$$

From (3.13) and (3.16), gives

$$\mathcal{R}'(t)\geq \mathfrak{B}\mathcal{R}^{\frac{1}{1-\alpha}}(t),\tag{3.17}$$

where $\mathfrak{B}(\mathfrak{m}_1,c)>0$. Finally, integrating (3.17) over $(0,t)$, gives

$$\mathcal{R}^{\frac{\alpha}{1-\alpha}}(t)\geq \frac{1}{\mathcal{R}^{\frac{-\alpha}{1-\alpha}}(0)-\mathfrak{B}\frac{\alpha}{(1-\alpha)}t}.$$

So, $\mathcal{R}(t)$ blows up in time

$$T\leq T^*=\frac{1-\alpha}{\mathfrak{B}\alpha\mathcal{R}^{\alpha/(1-\alpha)}(0)}.$$

The proof is completed. □

4 Conclusion

By integrating distributed delay with fractional conditions in the presence of infinite memory, we obtain new blow-up results for a Hartree-type nonlinear wave equation.

Our analysis, which requires specific assumptions on the delay coefficients and memory kernel, demonstrates the blow-up of solutions with negative initial energy, thereby extending previous studies.

Subsequent research will explore whether similar outcomes can be achieved when additional damping terms are introduced.

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