A SIMPLE PROOF OF INEQUALITIES RELATED TO MEANS

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Abstract. The purpose of this paper is to give a fairly elementary method to prove that the function
\[u(\alpha) = \left(\frac{x^\alpha - y^\alpha}{\alpha(x-y)}\right)^{1/\alpha}, \quad \alpha \neq 0, 1, u(0) = I, u(1) = L,\]
is strictly increasing, and to give a simple proof of the inequalities
\[x < H < \frac{xy}{I} < \frac{xy}{L} < G < L < I < A < y,\]
where \(0 < x < y\) and \(A, G, H, I, L\) are the arithmetic, the geometric, the harmonic, the identric and logarithmic means of \(x\) and \(y\), respectively.

1. Introduction

Given two positive real numbers \(x\) and \(y\), the arithmetic mean \(A(x, y)\), the geometric mean \(G(x, y)\), the logarithmic mean \(L(x, y)\), the identric mean \(I(x, y)\) and the harmonic mean \(H(x, y)\) of \(x\) and \(y\) are defined, respectively, by
\[A = A(x, y) = \frac{x + y}{2}, \quad G = G(x, y) = \sqrt{xy}, \quad H = H(x, y) = \frac{2xy}{x + y},\]
\[L = L(x, y) = \begin{cases} \frac{x - y}{\ln x - \ln y}, & x \neq y, \\ x, & x = y. \end{cases} \quad I = I(x, y) = \begin{cases} e^{-1}\left(\frac{x^x}{y^y}\right)^{1/\alpha}, & x \neq y, \\ x, & x = y. \end{cases}\]

It is known that (see[1], p.130), if \(0 < x < y\), then
\[x < H < \frac{xy}{I} < \frac{xy}{L} < G < L < I < A < y.\] (1)

Throughout, we assume \(0 < x < y\).

In [5], K. B. Stolarsky, defined the function
\[u(\alpha) = \begin{cases} \left(\frac{x^\alpha - y^\alpha}{\alpha(x-y)}\right)^{1/\alpha}, & \alpha \neq 0, 1, \\ L(x, y), & \alpha = 0, \\ I(x, y), & \alpha = 1. \end{cases}\]

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and proved that $u(\alpha)$ satisfies the following properties:

(i) $0 < x < u(\alpha) < y$, $\forall \alpha \in \mathbb{R}$.

(ii) $u(\alpha)$ is strictly increasing in $\alpha$, and $u(\alpha)$ approaches to $y$ (or $x$) as $\alpha$ approaches to $\infty$ (or $-\infty$).

(iii) $u(\alpha)$ is continuous in $\alpha$, and

$$u(-1) = G(x,y), \quad u(0) = L(x,y), \quad u(1) = I(x,y), \quad u(2) = A(x,y).$$

The purpose of this paper is to give a fairly elementary method to prove that the function $u(\alpha)$ is strictly increasing and to give a simple proof of the inequalities (1).

2. A simple proof of the monotone of $u(\alpha)$.

To prove that $u(\alpha)$ is strictly increasing, we need the following Lemmas:

**Lemma 1.** Let $g(t) = t(\ln t)^2 - (t - 1)^2$, $t > 0$. Then $g(t) < 0$, $\forall t \in (0,1) \cup (1,\infty)$.

**Proof.** For $t > 0$, we have

$$g'(t) = 2 \ln t + (\ln t)^2 - 2(t - 1),$$

$$g''(t) = \frac{2}{t} + \frac{2 \ln t}{t} - 2,$$

$$g'''(t) = \frac{-2 \ln t}{t^2},$$

it follows that, if $t > 1$, then $g'''(t) < 0$, so that $g''(t) < g''(1) = 0$, which implies that $g'(t)$ is strictly decreasing on $(1,\infty)$, and hence $g'(t) < g'(1) = 0$, which, again, implies that $g(t)$ is strictly decreasing on $(1,\infty)$. Therefore $g(t) < g(1) = 0$.

Next, if $0 < t < 1$, then $g'''(t) > 0$, so that $g''(t) < g''(1) = 0$, which implies that $g'(t)$ is strictly decreasing on $(0,1)$, and then $g'(t) > g'(1) = 0$.

Hence $g(t)$ is strictly increasing on $(0,1)$. Therefore $g(t) < g(1) = 0$.

This completes the proof.

**Lemma 2.** If $b > 1$, let

$$f(\alpha) = \begin{cases} 
\frac{(\alpha - 1)b^\alpha \ln b}{b^\alpha - 1} - 1 + \frac{1}{\alpha} - \ln \frac{b^\alpha - 1}{\alpha(b - 1)}, & \alpha \neq 0, \\
\ln \frac{b - 1}{\ln b} - \ln \sqrt{b}, & \alpha = 0.
\end{cases}$$

Then $f(\alpha) > 0$, $\forall \alpha \neq 1$. 
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Proof. For \( \alpha \neq 0 \), we have
\[
f'(\alpha) = \left( \frac{(\alpha^\alpha - 1)\ln b[(\alpha - 1)\ln b + 1] - (\alpha - 1)b^{2\alpha}(\ln b)^2}{(\alpha^\alpha - 1)^2} \right) - \frac{1}{\alpha^2} - \frac{\alpha b^\alpha \ln b - b^\alpha + 1}{\alpha(b^\alpha - 1)}
\]

\[
= \frac{1 - \alpha}{\alpha^2(b^\alpha - 1)^2} \left[ b^\alpha(\ln b^\alpha)^2 - (b^\alpha - 1)^2 \right]
\]

\[
= \frac{1 - \alpha}{\alpha^2(b^\alpha - 1)^2} g(b^\alpha).
\]

where the function \( g \) is defined as in Lemma 1.

If \( \alpha > 1 \), it follows from (2) and Lemma 1, that \( f'(\alpha) > 0 \), so that \( f(\alpha) \) is strictly increasing on \((1, \infty)\). Hence \( f(\alpha) > f(1) = 0 \).

If \( 0 < \alpha < 1 \), it follows from (2) and Lemma 1, that \( f'(\alpha) < 0 \), so that \( f(\alpha) \) is strictly decreasing on \((0, 1)\). Hence \( f(\alpha) > f(1) = 0 \).

If \( \alpha < 0 \), it follows from (2) and Lemma 1, that \( f'(\alpha) < 0 \), so that \( f(\alpha) > f(0) > 0 \).

To show \( f(0) > 0 \), we consider the function \( F(x) = \sqrt{x} - \frac{1}{\sqrt{x}} - \ln x, \, x > 1 \). Differentiating gives \( F'(x) = \left( \frac{x - 1}{2x\sqrt{x}} \right) > 0 \), so that \( F(x) > F(1) = 0 \). This implies that \( f(0) > 0 \).

Consequently, \( f(\alpha) > 0, \forall \alpha \neq 1 \).

Now we are ready to prove that the function \( u(\alpha) \) is strictly increasing in \( \alpha \).

Proof. Let \( b = \frac{y}{x} > 1 \), and \( v(\alpha) = \frac{u(\alpha)}{x} \). Then
\[
v(\alpha) = \begin{cases} 
\left[ \frac{\alpha(b - 1)}{\alpha(b - 1)} \right]^\alpha, & \alpha \neq 0, 1, \\
\frac{L(x, y)}{x}, & \alpha = 0, \\
\frac{I(x, y)}{x}, & \alpha = 1.
\end{cases}
\]

It suffices to show that \( v(\alpha) \) is strictly increasing in \( \alpha \).

Now, if \( \alpha \neq 0, 1 \), we have
\[
v'(\alpha) = v(\alpha) \frac{(\alpha - 1) \left[ \frac{\alpha(b - 1) \alpha b^\alpha \ln b - (b^\alpha - 1)}{\alpha^2(b - 1)} \right] - \ln \frac{b^\alpha - 1}{\alpha(b - 1)}}{(\alpha - 1)^2}
\]

\[
= \frac{v(\alpha)}{(\alpha - 1)^2} \left[ \frac{(\alpha - 1)(\alpha b^\alpha \ln b - b^\alpha + 1)}{\alpha(b^\alpha - 1)} - \ln \frac{b^\alpha - 1}{\alpha(b - 1)} \right]
\]

\[
= \frac{v(\alpha)}{(\alpha - 1)^2} \left[ \frac{(\alpha - 1)b^\alpha \ln b - 1 + \frac{1}{\alpha} - \ln \frac{b^\alpha - 1}{\alpha(b - 1)}}{b^\alpha - 1} \right]
\]

\[
= \frac{v(\alpha)}{(\alpha - 1)^2} f(\alpha),
\]

(3)
where $f(\alpha)$ is defined as in Lemma 2.

Since $v(\alpha) > 0$, it follows from (3) and Lemma 2, that $v'(\alpha) > 0$, so that $v(\alpha)$ is strictly increasing for $\alpha \neq 0, 1$.

Observe that

$$v'(0) = \lim_{\alpha \to 0} \frac{v(\alpha) - v(0)}{\alpha - 0} = \lim_{\alpha \to 0} \frac{v(\alpha)}{(\alpha - 1)^2} f(\alpha) = \frac{L(x, y)}{y} \lim_{\alpha \to 0} f(\alpha) = \frac{L(x, y)}{y} \left( \ln \frac{b - 1}{\ln b} - \ln \sqrt{b} \right) > 0.$$  

$$v'(1) = \lim_{\alpha \to 1} \frac{v(\alpha) - v(1)}{\alpha - 1} = \lim_{\alpha \to 1} \frac{v(\alpha)}{(\alpha - 1)^2} f(\alpha) = \left[ \lim_{\alpha \to 1} v(\alpha) \right] \left[ \lim_{\alpha \to 1} f'(\alpha) \right] = \frac{I(x, y)}{y} \left[ \lim_{\alpha \to 1} \frac{1}{2(\alpha - 1)} \frac{1 - \alpha}{2(\alpha - 1) \alpha^2 (b^\alpha - 1)^2} g(b^\alpha) \right] = \frac{I(x, y)g(b)}{-2y(b - 1)^2} > 0.$$  

Consequently, $v'(\alpha) > 0$ for all $\alpha$.

3. A simple proof of the inequalities (1)

Let $w(\alpha) = \frac{xy}{u(\alpha)}$. Then

$$w(\alpha) = \begin{cases} 
\frac{xy}{L(x, y)} & \alpha = 0, \\
\frac{xy}{I(x, y)} & \alpha = 1.
\end{cases}$$

Since $u(\alpha)$ is strictly increasing in $\alpha$, so that $w(\alpha)$ is strictly decreasing in $\alpha$.

Now

$$\lim_{\alpha \to \infty} w(\alpha) = x, \quad w(2) = H(x, y), \quad w(1) = \frac{xy}{I(x, y)}, \quad w(0) = \frac{xy}{L(x, y)},$$

$$w(0) = \frac{xy}{L(x, y)},$$
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\[ w(-1) = G(x, y), \quad w(-2) = (AG^2)^\frac{1}{2}, \quad \lim_{\alpha \to -\infty} w(\alpha) = y. \]

We have

\[ x < H(x, y) < \frac{xy}{I(x, y)} < \frac{xy}{L(x, y)} < G(x, y) < (AG^2)^\frac{1}{2} < y \]  \hspace{1cm} (4)

To show that

\[ (AG^2)^\frac{1}{2} < L(x, y) < I(x, y) < A(x, y), \]  \hspace{1cm} (5)

we consider the function \( h(\alpha) = A^\alpha G^{1-\alpha}, \alpha \in [0, 1] \), we see that \( h'(\alpha) = A^\alpha G^{1-\alpha} \ln A \), hence \( h(\alpha) \) is strictly increasing in \( \alpha \).

For \( c = \sqrt{\frac{2}{x}} > 1 \), let

\[ \alpha_1 = \frac{\ln \frac{c^2-1}{2c \ln c}}{\ln \frac{c^2-1}{2c}}, \quad \alpha_2 = \frac{-1 + \frac{c^2+1}{c^2-1} \ln c}{\ln \frac{c^2-1}{2c}}. \]

Then \( h(\alpha_1) = L(x, y) \) and \( h(\alpha_2) = I(x, y) \). Observe that \( h(\frac{1}{4}) = (AG^2)^{\frac{1}{3}} \) and \( h(1) = A(x, y) \).

To complete the proof of the inequalities (5), it suffices to prove that \( \frac{1}{4} < \alpha_1 < \alpha_2 < 1 \).

In order to prove that \( \alpha_1 > \frac{1}{4} \), we need the following Lemma:

**Lemma 3.** Let \( k(x) = 2(x^4 + 4x^2 + 1) \ln x - 3(x^4 - 1), x > 0 \). Then \( k(x) > 0, \forall x > 1 \).

**Proof.** Let

\[ p(x) = (4x^4 + 8x^2) \ln x - 5x^4 + 4x^2 + 1, \]
\[ q(x) = (x^2 + 1) \ln x - (x^2 - 1), \]
\[ r(x) = 2x^2 \ln x - x^2 + 1. \]

Then

\[ k'(x) = 2(4x^3 + 8x) \ln x + \frac{-10x^4 + 8x^2 + 2}{x} \]
\[ = \frac{2}{x} p(x), \]
\[ p'(x) = 16x(x^2 + 1) \ln x - 16x(x^2 - 1) \]
\[ = (16x) q(x), \]
\[ q'(x) = 2x \ln x - x + \frac{1}{x} \]
\[ = \frac{1}{x} r(x), \]
\[ r'(x) = 4x \ln x. \]

If \( x > 1 \), then \( r'(x) > 0 \), so that \( r(x) > r(1) = 0 \), which implies that \( q(x) \) is strictly increasing on \((1, \infty)\), then \( q(x) > q(1) = 0 \), which implies that \( p(x) \) is strictly increasing on \((1, \infty)\), then \( p(x) > p(1) = 0 \), which again, implies that \( k(x) \) is strictly increasing on \((1, \infty)\).
Consequently, $k(x) > k(1) = 0$, $\forall x > 1$.
This completes the proof of the Lemma.

To prove that $\alpha_1 > \frac{1}{3}$, let

$$s(x) = \begin{cases} 
\ln \frac{x^2 - 1}{2x \ln x} - \ln \frac{x^2 + 1}{2x} & , \ x > 1, \\
0 & , \ x = 1.
\end{cases}$$

Then for $x > 1$, we have

$$s'(x) = \frac{2x \ln x}{x^2 - 1} - \frac{2x(2x \ln x) - (x^2 - 1)(2 + 2 \ln x)}{4x^2(\ln x)^2} - \frac{2x}{3(x^2 + 1)} = \frac{x^2 \ln x - x^2 + 1 + \ln x}{x(x^2 - 1) \ln x} - \frac{x^2 - 1}{3x(x^2 + 1)} = \frac{\ln x}{3x(x^4 - 1) \ln x},$$

where $k(x)$ is defined as in Lemma 3.

It follows from (6) and Lemma 3. that $s'(x) > 0$, $\forall x > 1$. Hence $s(x) > s(1) = 0$, $\forall x > 1$, which is equivalent to $\frac{1}{3} < \alpha_1$. Therefore,

$$h\left(\frac{1}{3}\right) = (AG^2)^{\frac{1}{3}} < h(\alpha_1) = L(x, y)$$

(see [2], [3], [4]).

In order to prove that $\alpha_2 > \alpha_1$, we need the following Lemma:

**Lemma 4.** Let $\ell(x) = (x^2 - 1)^2 - (2x \ln x)^2$, $x > 1$. Then $\ell(x) > 0$, $\forall x > 1$.

**Proof.** Let $\ell_1(x) = x^2 - 1 - 2x \ln x$, $x > 1$.

It suffices to show $\ell_1(x) > 0$, $\forall x > 1$. Differentiating gives

$$\ell_1'(x) = 2x - 2 - 2 \ln x,$$

$$\ell_1''(x) = 2 - \frac{2}{x}.$$ 

If $x > 1$, then $\ell_1''(x) > 0$, so that $\ell_1(x)$ is strictly increasing on $(1, \infty)$, and then $\ell_1(x) > \ell_1(1) = 0$, which implies that $\ell_1(x)$ is strictly increasing on $(1, \infty)$.

Consequently, $\ell_1(x) > \ell_1(1) = 0$, $\forall x > 1$.

This completes the proof of the lemma.

To prove that $\alpha_2 = \frac{1 + e^{\frac{x^2}{e} \ln e}}{\ln e} > \ln \frac{x^2 - 1}{2x \ln x}$, $\alpha_1$, let

$$\ell_2(x) = \begin{cases} 
-1 + 2x \ln x - \ln \frac{x^2 - 1}{2x \ln x} & , \ x > 1, \\
0 & , \ x = 1.
\end{cases}$$
It suffices to show that \( \ell_2(x) > 0 \), \( \forall x > 1 \).

To this end, we observe that for \( x > 1 \), we have
\[
\ell_2'(x) = \frac{2x(x^2 - 1) - 2x(x^2 + 1)}{(x^2 - 1)^2} \ln x + \frac{x^2 + 1}{x(x^2 - 1)} \frac{2x \ln x}{x^2 - 1} - \frac{4x^2 \ln x - (x^2 - 1)(2 + 2 \ln x)}{4x^2(\ln x)^2} \\
= \frac{-4x^2(\ln x)^2 + (x^4 - 1) \ln x + (1 - x^2)(x^2 \ln x - x^2 + \ln x + 1)}{x(x^2 - 1)^2 \ln x} \\
= \frac{\ell(x)}{x(x^2 - 1)^2 \ln x},
\]
where \( \ell(x) \) is defined as in Lemma 4.

It follows from (8) and Lemma 4 that \( \ell_2'(x) > 0 \), \( \forall x > 1 \), so that \( \ell_2(x) \) is strictly increasing on \((1, \infty)\). Hence \( \ell_2(x) > \ell_2(1) = 0 \). Therefore
\[
h(\alpha_2) = I(x, y) > h(\alpha_1) = L(x, y).
\]  
(9)

To prove that \( \alpha_2 < 1 \), we need the following Lemma:

Lemma 5. Let \( m(x) = x^4 \ln x - x^4 + x^2 \ln x + x^2 \), \( x > 0 \). Then \( m(x) > 0 \), \( \forall x > 1 \).

Proof. We have
\[
m'(x) = 4x^3 \ln x - 3x^3 + 2x \ln x + 3x \\
m''(x) = 12x^2 \ln x - 5x^2 + 2 \ln x + 5 \\
m'''(x) = 24x \ln x + 2x + \frac{2}{x}.
\]

If \( x > 1 \), then \( m'''(x) > 0 \), so that \( m''(x) > m''(1) = 0 \), implies \( m'(x) \) is strictly increasing on \((1, \infty)\), and \( m'(x) > m'(1) = 0 \), which again, implies that \( m(x) \) is strictly increasing on \((1, \infty)\).

Consequently, \( m(x) > m(1) = 0 \), \( \forall x > 1 \).

This completes the proof of the lemma.

Now, let
\[
n(x) = \begin{cases} 
\ln \frac{x^2 + 1}{2x} + 1 - \frac{x^2 + 1}{x^2 - 1} \ln x, & x > 1, \\
0, & x = 1.
\end{cases}
\]

Then for \( x > 1 \), we have
\[
n'(x) = \frac{2x}{x^2 + 1} \frac{4x^2 - 2x^2 - 2}{4x^2} - \frac{(x^2 - 1)(2x \ln x + x + \frac{1}{x}) - 2x(x^2 \ln x + \ln x)}{(x^2 - 1)^2} \\
= \frac{x^2 - 1}{x(x^2 + 1)} - \frac{x^3 - 4x \ln x - \frac{1}{x}}{(x^2 - 1)^2} \\
= \frac{4m(x)}{x(x^2 + 1)(x^2 - 1)^2},
\]  
(10)
where \( m(x) \) is defined as in Lemma 5.

It follows from (10) and Lemma 5. that \( n'(x) > 0 \), \( \forall x > 1 \).
Hence \( n(x) > n(1) = 0 \), \( \forall x > 1 \), which is equivalent to \( \alpha_2 < 1 \). Therefore,

\[
h(\alpha_2) = I(x, y) < h(1) = A(x, y).
\]

(11)
The inequalities (1) then follows from (4), (7), (9) and (11).

References


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