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A SIMPLE PROOF OF INEQUALITIES RELATED TO MEANS

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Abstract. The purpose of this paper is to give a fairly elementary method to prove that the function $u(\alpha)=[\frac{x^\alpha-y^\alpha}{\alpha(x-y)}]^{\frac{1}{\alpha-1}}$, $\alpha\neq 0,1, u(0)=I, u(1)=L$, is strictly increasing, and to give a simple proof of the inequalities $x<H<\frac{xy}{I}<\frac{xy}{L}<G<L<I<A<y$, where $0<x<y$ and A,G,H,I,L are the arithmetic, the geometric, the harmonic, the identric and logarithmic means of x and y, respectively.

1. Introduction

Given two positive real numbers x and y, the arithmetic mean $A(x,y)$, the geometric mean $G(x,y)$, the logarithmic mean $L(x,y)$, the identric mean $I(x,y)$ and the harmonic mean $H(x,y)$ of x and y are defined, respectively, by

$$A=A(x,y)=\frac{x+y}{2},\qquad G=G(x,y)=\sqrt{xy},\qquad H=H(x,y)=\frac{2xy}{x+y},$$

$$L=L(x,y)=\begin{cases}\dfrac{x-y}{\ln x-\ln y} & ,x\neq y\ ,\\ x & ,x=y\ .\end{cases}\qquad I=I(x,y)=\begin{cases}e^{-1}\left(\dfrac{x^x}{y^y}\right)^{\frac{1}{x-y}} & ,x\neq y\ ,\\ x & ,x=y\ .\end{cases}$$

It is known that (see[1], p.130), if $0<x<y$, then

$$x<H<\frac{xy}{I}<\frac{xy}{L}<G<L<I<A<y. \tag{1}$$

Throughout, we assume $0<x<y$.

In [5], K. B. Stolarsky, defined the function

$$u(\alpha)=\begin{cases}\left[\dfrac{x^\alpha-y^\alpha}{\alpha(x-y)}\right]^{\frac{1}{\alpha-1}} & ,\alpha\neq 0,1,\\ L(x,y) & ,\alpha=0\,,\\ I(x,y) & ,\alpha=1\ .\end{cases}$$

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and proved that $u(\alpha)$ satisfies the following properties:

(i) $0 < x < u(\alpha) < y,\ \forall \alpha \in \mathbb{R}$.

(ii) $u(\alpha)$ is strictly increasing in α, and $u(\alpha)$ approaches to y (or x) as α approaches to ∞ $(or -\infty)$.

(iii) $u(\alpha)$ is continuous in α, and
$u(-1) = G(x,y),\quad u(0) = L(x,y),\quad u(1) = I(x,y),\quad u(2) = A(x,y)$.

The purpose of this paper is to give a fairly elementary method to prove that the function $u(\alpha)$ is strictly increasing and to give a simple proof of the inequalities (1).

2. A simple proof of the monotone of $u(\alpha)$.

To prove that $u(\alpha)$ is strictly increasing, we need the following Lemmas:

Lemma 1. *Let* $g(t) = t(\ln t)^2 - (t-1)^2,\ t > 0$. *Then* $g(t) < 0,\ \forall t \in (0,1) \cup (1,\infty)$.

Proof. For $t > 0$, we have

$$\begin{aligned} g'(t) &= 2\ln t + (\ln t)^2 - 2(t-1), \\ g''(t) &= \frac{2}{t} + \frac{2\ln t}{t} - 2, \\ g'''(t) &= \frac{-2\ln t}{t^2}, \end{aligned}$$

it follows that, if $t > 1$, then $g'''(t) < 0$, so that $g''(t) < g''(1) = 0$, which implies that $g'(t)$ is strictly decreasing on$(1,\infty)$, and hence $g'(t) < g'(1) = 0$, which, again, implies that g(t) is strictly decreasing on$(1,\infty)$. Therefore $g(t) < g(1) = 0$.

Next, if $0 < t < 1$, then $g'''(t) > 0$, so that $g''(t) < g''(1) = 0$, which implies that $g'(t)$ is strictly decreasing on (0,1), and then $g'(t) > g'(1) = 0$.

Hence $g(t)$ is strictly increasing on (0,1). Therefore $g(t) < g(1) = 0$.

This completes the proof.

Lemma 2. *If* $b > 1$*, let*

$$f(\alpha) = \begin{cases} \dfrac{(\alpha-1)b^\alpha \ln b}{b^\alpha - 1} - 1 + \dfrac{1}{\alpha} - \ln \dfrac{b^\alpha - 1}{\alpha(b-1)}\ , & \alpha \neq 0, \\ \ln \dfrac{b-1}{\ln b} - \ln\sqrt{b}\ , & \alpha = 0. \end{cases}$$

Then $f(\alpha) > 0\ ,\ \forall \alpha \neq 1$.

Proof. For $\alpha \neq 0$, we have

$$\begin{aligned} f'(\alpha) &= \frac{(b^\alpha-1)b^\alpha \ln b[(\alpha-1)\ln b+1]-(\alpha-1)b^{2\alpha}(\ln b)^2}{(b^\alpha-1)^2} - \frac{1}{\alpha^2} - \frac{\alpha b^\alpha \ln b - b^\alpha + 1}{\alpha(b^\alpha-1)} \\ &= \frac{1-\alpha}{\alpha^2(b^\alpha-1)^2}\left[b^\alpha(\ln b^\alpha)^2-(b^\alpha-1)^2\right] \\ &= \frac{1-\alpha}{\alpha^2(b^\alpha-1)^2}\, g(b^\alpha). \end{aligned} \tag{2}$$

where the function g is defined as in Lemma 1..

If $\alpha > 1$, it follows from (2) and Lemma 1. that $f'(\alpha) > 0$, so that $f(\alpha)$ is strictly increasing on $(1,\infty)$. Hence $f(\alpha) > f(1) = 0$.

If $0 < \alpha < 1$, it follows from (2) and Lemma 1. that $f'(\alpha) < 0$, so that $f(\alpha)$ is strictly decreasing on $(0,1)$. Hence $f(\alpha) > f(1) = 0$.

If $\alpha < 0$, it follows from (2) and Lemma 1. that $f'(\alpha) < 0$, so that$f(\alpha) > f(0) > 0$.

To show $f(0) > 0$, we consider the function $F(x) = \sqrt{x} - \frac{1}{\sqrt{x}} - \ln x,\ \ x > 1$. Differentiating gives $F'(x) = \frac{(\sqrt{x}-1)^2}{2x\sqrt{x}} > 0$, so that $F(x) > F(1) = 0$. This implies that $f(0) > 0$. Consequently, $f(\alpha) > 0,\ \forall \alpha \neq 1$.

Now we are ready to prove that the function $u(\alpha)$ is strictly increasing in α.

Proof. Let $b = \dfrac{y}{x} > 1$, and $v(\alpha) = \dfrac{u(\alpha)}{x}$. Then

$$v(\alpha) = \begin{cases} \left[\dfrac{b^\alpha - 1}{\alpha(b-1)}\right]^{\frac{1}{\alpha-1}}, & \alpha \neq 0, 1, \\ \dfrac{L(x,y)}{x}, & \alpha = 0, \\ \dfrac{I(x,y)}{x}, & \alpha = 1. \end{cases}$$

It suffices to show that $v(\alpha)$ is strictly increasing in α.

Now, if $\alpha \neq 0, 1$, we have

$$\begin{aligned} v'(\alpha) &= v(\alpha)\frac{(\alpha-1)\left[\dfrac{\alpha(b-1)}{b^\alpha-1}\,\dfrac{\alpha b^\alpha \ln b-(b^\alpha-1)}{\alpha^2(b-1)}\right] - \ln\dfrac{b^\alpha-1}{\alpha(b-1)}}{(\alpha-1)^2} \\ &= \frac{v(\alpha)}{(\alpha-1)^2}\left[\frac{(\alpha-1)(\alpha b^\alpha \ln b - b^\alpha+1)}{\alpha(b^\alpha-1)} - \ln\frac{b^\alpha-1}{\alpha(b-1)}\right] \\ &= \frac{v(\alpha)}{(\alpha-1)^2}\left[\frac{(\alpha-1)b^\alpha \ln b}{b^\alpha-1} - 1 + \frac{1}{\alpha} - \ln\frac{b^\alpha-1}{\alpha(b-1)}\right] \\ &= \frac{v(\alpha)}{(\alpha-1)^2}\, f(\alpha), \end{aligned} \tag{3}$$

where $f(\alpha)$ is defined as in Lemma 2..

Since $v(\alpha) > 0$, it follows from (3) and Lemma 2. that $v'(\alpha) > 0$, so that $v(\alpha)$ is strictly increasing for $\alpha \neq 0, 1$.

Observe that

$$\begin{aligned}
v'(0) &= \lim_{\alpha\to 0} \frac{v(\alpha) - v(0)}{\alpha - 0} \\
&= \lim_{\alpha\to 0} \frac{v(\alpha)}{(\alpha-1)^2} f(\alpha) \\
&= \frac{L(x,y)}{y} \lim_{\alpha\to 0} f(\alpha) \\
&= \frac{L(x,y)}{y} \left(\ln \frac{b-1}{\ln b} - \ln \sqrt{b} \right) > 0. \\
v'(1) &= \lim_{\alpha\to 1} \frac{v(\alpha) - v(1)}{\alpha - 1} \\
&= \lim_{\alpha\to 1} \frac{v(\alpha)}{(\alpha-1)^2} f(\alpha) \\
&= \left[\lim_{\alpha\to 1} v(\alpha) \right] \left[\lim_{\alpha\to 1} \frac{f'(\alpha)}{2(\alpha-1)} \right] \\
&= \frac{I(x,y)}{y} \left[\lim_{\alpha\to 1} \frac{1}{2(\alpha-1)} \frac{1-\alpha}{\alpha^2(b^\alpha-1)^2} g(b^\alpha) \right] \\
&= \frac{I(x,y) g(b)}{-2y(b-1)^2} > 0.
\end{aligned}$$

Consequently, $v'(\alpha) > 0$ for all α.

3. A simple proof of the inequalities (1)

Let $w(\alpha) = \dfrac{xy}{u(\alpha)}$. Then

$$w(\alpha) = \begin{cases} xy\left[\dfrac{\alpha(x-y)}{x^\alpha - y^\alpha}\right]^{\frac{1}{\alpha-1}}, & \alpha \neq 0, 1, \\ \dfrac{xy}{L(x,y)}, & \alpha = 0, \\ \dfrac{xy}{I(x,y)}, & \alpha = 1. \end{cases}$$

Since $u(\alpha)$ is strictly increasing in α, so that $w(\alpha)$ is strictly decreasing in α.

Now

$$\lim_{\alpha\to\infty} w(\alpha) = x, \quad w(2) = H(x,y), \quad w(1) = \frac{xy}{I(x,y)}, \quad w(0) = \frac{xy}{L(x,y)},$$

$w(-1) = G(x,y), \quad w(-2) = (AG^2)^{\frac{1}{3}}, \quad \lim_{\alpha \to -\infty} w(\alpha) = y.$

We have

$$x < H(x,y) < \frac{xy}{I(x,y)} < \frac{xy}{L(x,y)} < G(x,y) < (AG^2)^{\frac{1}{3}} < y \tag{4}$$

To show that

$$(AG^2)^{\frac{1}{3}} < L(x,y) < I(x,y) < A(x,y), \tag{5}$$

we consider the function $h(\alpha) = A^\alpha G^{1-\alpha}, \alpha \in [0,1]$, we see that $h'(\alpha) = A^\alpha G^{1-\alpha} \ln A - A^\alpha G^{1-\alpha} \ln G > 0$. Hence $h(\alpha)$ is strictly increasing in α.

For $c = \sqrt{\frac{y}{x}} > 1$, let

$$\alpha_1 = \frac{\ln \frac{c^2-1}{2c \ln c}}{\ln \frac{c^2+1}{2c}}, \quad \alpha_2 = \frac{-1 + \frac{c^2+1}{c^2-1} \ln c}{\ln \frac{c^2+1}{2c}}.$$

Then $h(\alpha_1) = L(x,y)$ and $h(\alpha_2) = I(x,y)$.

Observe that $h(\frac{1}{3}) = (AG^2)^{\frac{1}{3}}$ and $h(1) = A(x,y)$.

To complete the proof of the inequalities (5), it suffices to prove that $\frac{1}{3} < \alpha_1 < \alpha_2 < 1$.

In order to prove that $\alpha_1 > \frac{1}{3}$, we need the following Lemma:

Lemma 3. *Let* $k(x) = 2(x^4 + 4x^2 + 1) \ln x - 3(x^4 - 1), x > 0$. *Then* $k(x) > 0, \ \forall x > 1$.

Proof. Let

$$\begin{aligned} p(x) &= (4x^4 + 8x^2) \ln x - 5x^4 + 4x^2 + 1, \\ q(x) &= (x^2 + 1) \ln x - (x^2 - 1), \\ r(x) &= 2x^2 \ln x - x^2 + 1. \end{aligned}$$

Then

$$\begin{aligned} k'(x) &= 2(4x^3 + 8x) \ln x + \frac{-10x^4 + 8x^2 + 2}{x} \\ &= \frac{2}{x} \ p(x), \\ p'(x) &= 16x(x^2 + 1) \ln x - 16x(x^2 - 1) \\ &= (16x) \ q(x), \\ q'(x) &= 2x \ln x - x + \frac{1}{x} \\ &= \frac{1}{x} \ r(x), \\ r'(x) &= 4x \ln x. \end{aligned}$$

If $x > 1$, then $r'(x) > 0$, so that $r(x) > r(1) = 0$, which implies that $q(x)$ is strictly increasing on $(1, \infty)$, then $q(x) > q(1) = 0$, which implies that $p(x)$ is strictly increasing on $(1, \infty)$, then $p(x) > p(1) = 0$, which again, implies that $k(x)$ is strictly increasing on $(1, \infty)$.

Consequently, $k(x) > k(1) = 0$, $\forall x > 1$.
This completes the proof of the Lemma.
To prove that $\alpha_1 > \frac{1}{3}$, let

$$s(x) = \begin{cases} \ln \dfrac{x^2-1}{2x\ln x} - \dfrac{\ln \dfrac{x^2+1}{2x}}{3}, & x > 1, \\ 0, & x = 1. \end{cases}$$

Then for $x > 1$, we have

$$\begin{aligned} s'(x) &= \frac{2x\ln x}{x^2-1}\frac{2x(2x\ln x) - (x^2-1)(2+2\ln x)}{4x^2(\ln x)^2} - \frac{2x}{3(x^2+1)}\frac{4x^2-2(x^2+1)}{4x^2} \\ &= \frac{x^2\ln x - x^2 + 1 + \ln x}{x(x^2-1)\ln x} - \frac{x^2-1}{3x(x^2+1)} \\ &= \frac{k(x)}{3x(x^4-1)\ln x}, \end{aligned} \tag{6}$$

where $k(x)$ is defined as in Lemma 3..

It follows from (6) and Lemma 3. that $s'(x) > 0$, $\forall x > 1$. Hence $s(x) > s(1) = 0$, $\forall x > 1$, which is equivalent to $\frac{1}{3} < \alpha_1$. Therefore,

$$h\left(\frac{1}{3}\right) = (AG^2)^{\frac{1}{3}} < h(\alpha_1) = L(x,y) \tag{7}$$

(see [2], [3], [4]).

In order to prove that $\alpha_2 > \alpha_1$, we need the following Lemma:

Lemma 4. *Let* $\ell(x) = (x^2-1)^2 - (2x\ln x)^2$, $x > 1$. *Then* $\ell(x) > 0$, $\forall x > 1$.

Proof. Let $\ell_1(x) = x^2 - 1 - 2x\ln x$, $x > 1$.

It suffices to show $\ell_1(x) > 0$, $\forall x > 1$. Differentiating gives

$$\ell_1'(x) = 2x - 2 - 2\ln x ,$$
$$\ell_1''(x) = 2 - \frac{2}{x} .$$

If $x > 1$, then $\ell_1''(x) > 0$, so that $\ell_1'(x)$ is strictly increasing on $(1,\infty)$, and then $\ell_1'(x) > \ell_1'(1) = 0$, which, implies that $\ell_1(x)$ is strictly increasing on $(1,\infty)$.

Consequently, $\ell_1(x) > \ell_1(1) = 0$, $\forall x > 1$.

This completes the proof of the lemma.

To prove that $\alpha_2 = \dfrac{-1 + \frac{c^2+1}{c^2-1}\ln c}{\ln \frac{c^2+1}{2c}} > \dfrac{\ln \frac{c^2-1}{2c\ln c}}{\ln \frac{c^2+1}{2c}} = \alpha_1$, let

$$\ell_2(x) = \begin{cases} -1 + \dfrac{x^2+1}{x^2-1}\ln x - \ln \dfrac{x^2-1}{2x\ln x}, & x > 1, \\ 0, & x = 1. \end{cases}$$

It suffices to show that : $\ell_2(x) > 0\ ,\ \forall x > 1$.

To this end, we observe that for $x > 1$, we have

$$\begin{aligned}\ell_2'(x) &= \frac{2x(x^2-1)-2x(x^2+1)}{(x^2-1)^2}\ln x + \frac{x^2+1}{x(x^2-1)} - \frac{2x\ln x}{x^2-1}\frac{4x^2\ln x-(x^2-1)(2+2\ln x)}{4x^2(\ln x)^2}\\ &= \frac{-4x^2(\ln x)^2+(x^4-1)\ln x+(1-x^2)(x^2\ln x-x^2+\ln x+1)}{x(x^2-1)^2\ln x}\\ &= \frac{\ell(x)}{x(x^2-1)^2\ln x}\ , \qquad (8)\end{aligned}$$

where $\ell(x)$ is defined as in Lemma 4..

It follows from (8) and Lemma 4. that $\ell_2'(x) > 0\ ,\ \forall x > 1$, so that $\ell_2(x)$ is strictly increasing on $(1,\infty)$. Hence $\ell_2(x) > \ell_2(1) = 0$. Therefore

$$h(\alpha_2) = I(x,y) > h(\alpha_1) = L(x,y). \qquad (9)$$

To prove that $\alpha_2 < 1$, we need the following Lemma:

Lemma 5. *Let* $m(x) = x^4\ln x - x^4 + x^2\ln x + x^2\ ,\ x > 0$. *Then* $m(x) > 0\ ,\ \forall x > 1$.

Proof. We have

$$\begin{aligned}m'(x) &= 4x^3\ln x - 3x^3 + 2x\ln x + 3x\\ m''(x) &= 12x^2\ln x - 5x^2 + 2\ln x + 5\\ m'''(x) &= 24x\ln x + 2x + \frac{2}{x}.\end{aligned}$$

If $x > 1$, then $m'''(x) > 0$, so that $m''(x) > m''(1) = 0$, implies $m'(x)$ is strictly increasing on $(1,\infty)$, and $m'(x) > m'(1) = 0$, which again, implies that $m(x)$ is strictly increasing on $(1,\infty)$.

Consequently, $m(x) > m(1) = 0,\ \forall x > 1$.

This completes the proof of the lemma.

Now, let

$$n(x) = \begin{cases} \ln\dfrac{x^2+1}{2x} + 1 - \dfrac{x^2+1}{x^2-1}\ln x\ , & x > 1,\\ 0 & ,\ x = 1.\end{cases}$$

Then for $x > 1$, we have

$$\begin{aligned}n'(x) &= \frac{2x}{x^2+1}\frac{4x^2-2x^2-2}{4x^2} - \frac{(x^2-1)(2x\ln x + x + \frac{1}{x}) - 2x(x^2\ln x + \ln x)}{(x^2-1)^2}\\ &= \frac{x^2-1}{x(x^2+1)} - \frac{x^3 - 4x\ln x - \frac{1}{x}}{(x^2-1)^2}\\ &= \frac{4m(x)}{x(x^2+1)(x^2-1)^2}\ , \qquad (10)\end{aligned}$$

where $m(x)$ is defined as in Lemma 5..

It follows from 10 and Lemma 5. that $n^{'}(x) > 0$, $\forall x > 1$.
Hence $n(x) > n(1) = 0$, $\forall x > 1$, which is equivalent to $\alpha_2 < 1$. Therefore,

$$h(\alpha_2) = I(x, y) < h(1) = A(x, y). \tag{11}$$

The inequalities (1) then follows from (4), (7) (9) and (11).

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