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NEW INEQUALITIES FOR SOME SPECIAL FUNCTIONS VIA THE CAUCHY-BUNIAKOVSKY-SCHWARZ INEQUALITY

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Abstract. We establish new inequalities involving some special functions, using a form of the Cauchy-Buniakovski-Schwarz inequality. Our new inequalities extend the class of the Turan-type inequalites.

1. Introduction

The well-known Cauchy-Buniakovsky-Schwarz (CBS) inequality states that

$$\left(\int_a^b u(t)\,dt\right)\left(\int_a^b v(t)\,dt\right) \geq \left(\int_a^b u^{1/2}(t)\,v^{1/2}(t)\,dt\right)^2, \tag{1.1}$$

for every functions u, $v : [a,b] \to [0,\infty)$, such that the integrals does exist.

A. Laforgia and P. Natalini [3] used the following form of the CBS inequality (1.1):

$$\left(\int_a^b g(t)\,f^m(t)\,dt\right)\left(\int_a^b g(t)\,f^n(t)\,dt\right) \geq \left(\int_a^b g(t)\,f^{\frac{m+n}{2}}(t)\,dt\right)^2 \tag{1.2}$$

to established some new Turan-type inequalities involving the special functions as gamma, or polygamma functions.

Motivated by this remark, we have the idea to replace $u(t)$ and $v(t)$ in (1.1) by $g(t)\,f^m(t)\,h^x(t)$, respective $g(t)\,f^n(t)\,h^y(t)$, to introduce the following new inequality:

$$\left(\int_a^b g(t)\,h^x(t)\,f^m(t)\,dt\right)\left(\int_a^b g(t)\,h^y(t)\,f^n(t)\,dt\right)$$
$$\geq \left(\int_a^b g(t)\,h^{\frac{x+y}{2}}(t)\,f^{\frac{m+n}{2}}(t)\,dt\right)^2. \tag{1.3}$$

Here, $g, h, f : [a,b] \to [0,\infty)$ are such that the involved integrals does exist. For $h(t) = 1$, or $x = y = 0$ in (1.3), we obtain (1.2).

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The classical Euler gamma function may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \tag{1.4}$$

while its logarithmic derivative, denoted

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

is called the digamma function. Its derivatives $\psi', \psi'', \psi''', \ldots$ are known as polygamma functions.

In the next section, we show how to use the inequality (1.3) to establish new inequalities involving the special functions $\Gamma, \psi, \psi', \psi'', \ldots$.

2. The results

First, by differentiating (1.4), we obtain, for every $n \geq 1$,

$$\Gamma^{(n)}(x) = \int_0^\infty e^{-t} t^{x-1} \log^n t dt.$$

By taking $g(t) = t^{-1} e^{-t}$, $h(t) = t$, $f(t) = \log t$ in (1.3), we obtain

$$\left(\int_0^\infty t^{-1} e^{-t} t^x \log^m t dt \right) \left(\int_0^\infty t^{-1} e^{-t} t^y \log^n t dt \right)$$
$$\geq \left(\int_0^\infty t^{-1} e^{-t} t^{\frac{x+y}{2}} \log^{\frac{m+n}{2}} t dt \right)^2,$$

which can be rewritten as

Theorem 2.1. *For every even integers m, $n \geq 2$, and for every real numbers $x, y \in (0, \infty)$, it holds:*

$$\Gamma^{(m)}(x) \Gamma^{(n)}(y) \geq \left(\Gamma^{\left(\frac{m+n}{2}\right)} \left(\frac{x+y}{2} \right) \right)^2.$$

In particular, for $x = y$, it obtains the Turan-type inequality:

$$\Gamma^{(m)}(x) \Gamma^{(n)}(x) \geq \left(\Gamma^{\left(\frac{m+n}{2}\right)}(x) \right)^2$$

The polygamma functions have the following integral representations:

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-tx} dt,$$

see [1]. By taking $g(t) = 1/\left(1 - e^{-t}\right)$, $h(t) = e^{-t}$, $f(t) = t$ in (1.3), we obtain

$$\left(\int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-tx} \right) \left(\int_0^\infty \frac{t^m}{1 - e^{-t}} e^{-ty} \right) \geq \left(\int_0^\infty \frac{t^{\frac{m+n}{2}}}{1 - e^{-t}} e^{-t\left(\frac{x+y}{2}\right)} \right)^2,$$

which can be arranged as

Theorem 2.2. *For every integers m, $n \geq 1$ such that $m+n$ is even, and for every real numbers x, $y \in (0,\infty)$, it holds:*

$$\psi^{(n)}(x)\,\psi^{(m)}(y) \geq \psi^{\left(\frac{m+n}{2}\right)}\left(\frac{x+y}{2}\right).$$

In particular, for $x = y$, it obtains the Turan-type inequality:

$$\psi^{(m)}(x)\,\psi^{(n)}(x) \geq \left(\psi^{\left(\frac{m+n}{2}\right)}(x)\right)^2.$$

Binet's first formula for $\ln\Gamma(x)$ is given by

$$\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + \log\sqrt{2\pi} + \theta(x),$$

for $x > 0$, where the function

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-tx}}{t}\,dt \tag{2.1}$$

is known as the remainder of the Binet first formula for the logarithm of the gamma function. See [1]. By differentiating (2.1), we obtain

$$\theta^{(n)}(x) = (-1)^n \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{n-1} e^{-tx}\,dt.$$

By taking $g(t) = \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)$, $h(t) = e^{-t}$, $f(t) = t$ in (1.3), we obtain

$$\left(\int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{n-1} e^{-tx}\,dt\right)\left(\int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{m-1} e^{-ty}\,dt\right)$$
$$\geq \left(\int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right) t^{\frac{m+n}{2}-1} e^{-t\left(\frac{x+y}{2}\right)}\,dt\right)^2,$$

which can be presented as the following

Theorem 2.3. *For every integers m, $n \geq 1$ such that $m+n$ is even, and for every real numbers x, $y \in (0,\infty)$, it holds:*

$$\theta^{(n)}(x)\,\theta^{(m)}(y) \geq \theta^{\left(\frac{m+n}{2}\right)}\left(\frac{x+y}{2}\right).$$

In particular, for $x = y$, it obtains the Turan-type inequality:

$$\theta^{(m)}(x)\,\theta^{(n)}(x) \geq \left(\theta^{\left(\frac{m+n}{2}\right)}(x)\right)^2.$$

The generalized inverse Gaussian distribution is defined for $t > 0$ as

$$g(t) = \frac{1}{I(\alpha; a, b)} \cdot t^{\alpha-1} e^{-at-bt^{-1}},$$

where $a>0$, $b>0$, $-\infty<\alpha<\infty$. See [2]. The number $I(\alpha;a,b)$ is the normalizing constant,

$$I(\alpha;a,b)=\int_0^\infty t^{\alpha-1}e^{-at-bt^{-\beta}}dt.$$

By taking $g(t)=t^{\alpha-1}$, $h(t)=e^{-t}$, $f(t)=e^{-t^{-1}}$ in (1.3), we obtain

$$\left(\int_0^\infty t^{\alpha-1}e^{-at-bt^{-1}}dt\right)\left(\int_0^\infty t^{\alpha-1}e^{-ct-dt^{-1}}dt\right)$$
$$\geq\left(\int_0^\infty t^{\alpha-1}e^{-\frac{a+c}{2}t-\frac{b+d}{2}t^{-1}}dt\right)^2,$$

which can be written as the following

Theorem 2.4. *For every reals a, b, c, $d>0$ and $\alpha\in\mathbb{R}$, it holds:*

$$I(\alpha;a,b)\,I(\alpha;c,d)\geq I^2\left(\alpha;\frac{a+b}{2},\frac{c+d}{2}\right).$$

Now let us introduce the Abramowitz's function [1], defined by

$$f_m(x)=\int_0^\infty t^m e^{-t^2-xt^{-1}}dt.$$

has been used in many fields of physiscs, as the theory of the field of particle and radiation transform.

By taking $g(t)=e^{-t^2}$, $h(t)=e^{-t^{-1}}$, $f(t)=t$ in (1.3), we obtain

$$\left(\int_0^\infty t^m e^{-t^2-xt^{-1}}dt\right)\left(\int_0^\infty t^n e^{-t^2-yt^{-1}}dt\right)$$
$$\geq\left(\int_0^\infty t^{\frac{m+n}{2}}e^{-t^2-\frac{x+y}{2}t^{-1}}dt\right)^2,$$

which can be written as

Theorem 2.5. *For every positive reals m, n, x, y, it holds:*

$$f_m(x)\,f_n(y)\geq\left(f_{\frac{m+n}{2}}\left(\frac{x+y}{2}\right)\right)^2.$$

For $x=y$, we obtain the following Turan-type inequality:

$$f_m(x)\,f_n(x)\geq\left(f_{\frac{m+n}{2}}(x)\right)^2.$$

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