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ON NEW INEQUALITIES OF SIMPSON'S TYPE FOR QUASI-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we introduce some inequalities of Simpson's type based on quasi-convexity. Some applications for special means of real numbers are also given.

1. Introduction

The following inequality is well known in the literature as Simpson's inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\|f^{(4)}\|_\infty (b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see ([1],[2],[4]).

In [2], Dragomir, Agarwal and Cerone proved the following some recent developments on Simpson's inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 2. *Suppose $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L[a,b]$. Then the following inequality*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{3}\|f'\|_1 \tag{1.1}$$

holds, where $\|f'\|_1 = \int_a^b |f'(x)|\,dx$.

The bound of (1.1) for L-Lipschitzian mapping was given in [2] by $\frac{5}{36}L(b-a)$.

Also, the following inequality was obtained in [2].

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Theorem 3. *Suppose $f:[a,b]\to\mathbb{R}$ is an absolutely continuous mapping on $[a,b]$ whose derivative belongs to $L_p[a,b]$. Then the following inequality holds,*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{6}\left[\frac{2^{q+1}+1}{3(q+1)}\right]^{\frac{1}{q}}(b-a)^{\frac{1}{q}}\left\|f'\right\|_p \tag{1.2}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

We recall that the notion of quasi-convex functions generalized the notion of convex functions. More precisely, a function $f:[a,b]\to\mathbb{R}$ is said to be *quasi-convex* on $[a,b]$ if

$$f\left(tx+(1-t)y\right)\le\max\left\{f(x),f(y)\right\},\quad \forall x,y\in[a,b].$$

Any convex function is a quasi-convex function but the reverse are not true. Because there exist quasi-convex functions which are not convex, (see for example [3])

The main aim of this paper is to establish new Simpson's type inequalities for the class of functions whose derivatives in absolute value at certain powers are quasi-convex functions.

2. Simpson's Type Inequalities for Quasi-Convex

In order to prove our main theorems, we need the following lemma, see [1].

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be an absolutely continuous mapping on I° where $a,b\in I$ with $a<b$. Then the following equality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| = (b-a)\int_0^1 p(t)\,f'(tb+(1-t)a)\,dt \tag{2.1}$$

where

$$p(t)=\begin{cases} t-\frac{1}{6}, & t\in\left[0,\frac{1}{2}\right), \\ t-\frac{5}{6}, & t\in\left[\frac{1}{2},1\right]. \end{cases}$$

A simple proof of this equality can be also done by integrating by parts in the right hand side. The details are left to the interested reader.

The next theorem gives a new result of the Simpson inequality for quasi-convex functions.

Theorem 4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is quasi-convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{36}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}. \tag{2.2}$$

Proof. From Lemma 1, and since $|f'|$ is quasi-convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&= (b-a)\left|\int_0^1 p(t)\, f'(tb+(1-t)a)\,dt\right| \\
&\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt \\
&\quad + (b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt \\
&\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
&\quad + (b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right|\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
&= (b-a)\int_0^{1/6}\left(\frac{1}{6}-t\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
&\quad + (b-a)\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
&\quad + (b-a)\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
&\quad + (b-a)\int_{5/6}^{1}\left(t-\frac{5}{6}\right)\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
&= \frac{5(b-a)}{36}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}
\end{aligned}$$

which completes the proof. □

Corollary 1. *In Theorem 4, if $f(a) = f(\frac{a+b}{2}) = f(b)$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{5(b-a)}{36}\max\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}.$$

A similar results is embodied in the following theorem.

Theorem 5. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is quasi-convex on $[a,b]$ and $q > 1$, then the following inequality* *holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{6}(b-a)\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \tag{2.3}$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof. From Lemma 1, using the well known Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&= (b-a)\left|\int_0^1 p(t)\, f'(tb+(1-t)a)\,dt\right| \\
&\leq (b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt \\
&\quad +(b-a)\int_{1/2}^1\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt \\
&\leq (b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1/2}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad +(b-a)\left(\int_{1/2}^1\left|t-\frac{5}{6}\right|^p dt\right)^{\frac{1}{p}}\left(\int_{1/2}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&= (b-a)\left(\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_0^{1/2}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad +(b-a)\left(\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)^p dt\right)^{\frac{1}{p}} \\
&\quad \times\left(\int_{1/2}^1\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f'\right|^q$ is quasi-convex, we have

$$\left|f'(tb+(1-t)a)\right|^q\leq\max\left\{\left|f'(b)\right|^q,\left|f'(a)\right|^q\right\}$$

hence

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq 2.(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\frac{\max\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}}{2}\right)^{\frac{1}{q}}$$

$$\leq 2^{\frac{1}{p}}(b-a)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}$$

where we use the fact that

$$\int_0^{1/6}\left(\frac{1}{6}-t\right)^p dt+\int_{1/6}^{1/2}\left(t-\frac{1}{6}\right)^p dt=\int_{1/2}^{5/6}\left(\frac{5}{6}-t\right)^p dt+\int_{5/6}^{1}\left(t-\frac{5}{6}\right)^p dt$$
$$=\frac{1+2^{p+1}}{6^{p+1}(p+1)}$$

which completes the proof. □

Corollary 2. *In Theorem 5, if* $f(a)=f(\frac{a+b}{2})=f(b)$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{6}(b-a)\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

Corollary 3. *In Theorem 5, if* $f(a)=f(\frac{a+b}{2})=f(b)$ *and* $p=2$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)}{6}\sqrt{\max\left\{\left|f'(a)\right|^{2},\left|f'(b)\right|^{2}\right\}}.$$

A more general inequality is given using Lemma 1, as follows.

Theorem 6. *Let* $f:I\subset\mathbb{R}\to\mathbb{R}$ *be a differentiable mapping on* I°, *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *with* $a<b$. *If* $\left|f'\right|^q$ *is quasi-convex on* $[a,b]$ *and* $q\geq 1$, *then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.4}$$
$$\leq\frac{5(b-a)}{36}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

Proof. Suppose that $q\geq 1$. From Lemma 1 and using the well known power mean inequality, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$=(b-a)\left|\int_0^1 p(t)f'(tb+(1-t)a)dt\right|$$
$$\leq(b-a)\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|dt$$
$$+(b-a)\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|dt$$
$$\leq(b-a)\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^{1/2}\left|t-\frac{1}{6}\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+(b-a)\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt\right)^{1-\frac{1}{q}}\left(\int_{1/2}^{1}\left|t-\frac{5}{6}\right|\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}.$$

Since $\left|f'\right|^{q}$ is quasi-convex, we have

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$
$$\leq 2(b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left(\frac{5}{72}\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}$$
$$=\frac{5(b-a)}{36}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}$$

Also, we note that

$$\int_{0}^{1/2}\left|t-\frac{1}{6}\right|dt=\int_{1/2}^{1}\left|t-\frac{5}{6}\right|dt=\frac{5}{72}.$$

Therefore, te proof is completed. □

Remark 1. Theorem 6 is equal to Theorem 4 for $q=1$.

Remark 2. In Theorem 5, since

$$\lim_{p\to\infty}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}=2 \quad \text{and} \quad \lim_{p\to1^{+}}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}=\frac{5}{6}$$

we have

$$\frac{5}{6}<\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}<2 \qquad p\in(1,\infty),$$

so for $q>1$, Theorem 6 is an improvement of Theorem 5.

Corollary 4. *In Theorem 6, if $f(a)=f(\frac{a+b}{2})=f(b)$, then we have*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{5(b-a)}{36}\left(\max\left\{\left|f'(a)\right|^{q},\left|f'(b)\right|^{q}\right\}\right)^{\frac{1}{q}}.$$

3. Applications to Special Means

We now consider the applications of above Theorems to the following special means:

(a) The arithmetic mean: $A=A(a,b):=\dfrac{a+b}{2},\ a,b\geq 0,$

(b) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\ a,b>0,$$

(c) The logarithmic mean:

$$L=L(a,b):=\begin{cases} a & if\ a=b \\ \frac{b-a}{\ln b-\ln a} & if\ a\neq b\end{cases},\quad a,b>0,$$

(d) The $p-$logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \quad p \in \mathbb{R} \diagdown \{-1,0\};\ a,b>0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequalities

$$H \leq L \leq A.$$

Now, using the results of Section 2, some new inequalities is derived for the above means.

Proposition 7. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, we have*

$$\left|\frac{1}{3}A\left(a^n, b^n\right) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq n\frac{5(b-a)}{36}\max\left\{a^{n-1}, b^{n-1}\right\}.$$

Proof. The assertion follows from Theorem 4 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a,b]$ and $n \in \mathbb{N}$. □

Proposition 8. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $p > 1$, we have*

$$\left|\frac{1}{3}H^{-1}(a,b) + \frac{2}{3}A^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{1}{6}(b-a)\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{a^{-2q}, b^{-2q}\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5 applied to the quasi-convex mapping $f(x) = 1/x$, $x \in [a,b]$. □

Proposition 9. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}$, $n \geq 2$. Then, we have*

$$\left|\frac{1}{3}A\left(a^n, b^n\right) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq n\frac{(b-a)}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left(\max\left\{a^{q(n-1)}, b^{q(n-1)}\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 5 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a,b]$ and $n \in \mathbb{N}$. □

Proposition 10. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{N}$, $n \geq 2$.Then, for all $q > 1$, we have*

$$\left|\frac{1}{3}A\left(a^n, b^n\right) + \frac{2}{3}A^n(a,b) - L_n^n(a,b)\right| \leq n\frac{5(b-a)}{36}\left(\max\left\{a^{q(n-1)}, b^{q(n-1)}\right\}\right)^{\frac{1}{q}}.$$

Proof. The assertion follows from Theorem 6 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a,b]$ and $n \in \mathbb{N}$. □

Remark 3. Proposition 10 is equal to Proposition 7 for $q = 1$.

Remark 4. Because of Remark 2, Proposition 10 is an improvement of Proposition 9 for $q = 1$.

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