TAMKANG JOURNAL OF MATHEMATICS
Volume 38, Number 4, 291-299, Winter 2007

POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR p-LAPLACIAN FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. In this paper, the author studies the boundary value problems of p-Laplacian functional difference equation. By using a fixed point theorem in cones, sufficient conditions are established for the existence of the positive solutions.

1. Introduction

For notation, given $a < b$ in Z, we employ intervals to denote discrete sets such as $[a, b] = \{a, a+1, \ldots, b\}$, $[a, b) = \{a, a+1, \ldots b-1\}$, $[a, \infty) = \{a, a+1, \ldots\}$, etc. Let $\tau, T \in Z$ and $0 \le \tau \le T$. In this paper, we are concerned with the following p-Laplacian difference equation:

$$\begin{aligned} &\Delta\phi_p(\Delta x(t)) + r(t)f(x_t) = 0,\ t \in [0, T], \\ &x_0 = \psi \in C^+,\ x(0) - B_0(\Delta x(0)) = 0,\ \Delta x(t+1) = 0, \end{aligned} \tag{1.1}$$

where $\phi_p(u)$ is the p-Laplacian operator, i.e., $\phi_p(u) = |u|^{p-2}u$, $p > 1$, $(\phi_p)^{-1}(u) = \phi_q(u)$, $\frac{1}{p} + \frac{1}{q} = 1$. $\forall t \in Z$, let $x_t = x_t(k) = x(t+k)$, $k \in [-\tau, -1]$, then $x_t \in C$, where $C = C([-\tau, -1], R)$ is a Banach space with the norm $\|\varphi\|_C = \max_{k\in[-\tau,-1]} |\varphi|$. Let $C^+ = \{\varphi \in C : \varphi(k) \ge 0, k \in [-\tau, -1]\}$. As usual, Δ denotes the forward difference operator defined by $\Delta x(t) = x(t+1) - x(t)$.

We give the following assumptions:

(H_0) $f(\varphi)$ is a nonnegative continuous functional defined on C^+;

(H_1) $r(t)$ is a nonnegative function defined on $[0, T]$ and $\sum_{t=\tau}^{T} r(t) > 0$;

(H_2) $B_0 : R \to R$ is continuous and satisfies that there are $\beta \ge \alpha \ge 0$ such that $\alpha s \le B_0(s) \le \beta s$ for $s \in R^+$, where R^+ denotes the set of nonnegative real numbers.

Received February 22, 2006.
2000 *Mathematics Subject Classification*. 39A05.
Key words and phrases. p-Laplacian boundary value problem, functional difference equations, positive solution, fixed point theorem in cones.
Supported by Natural Science Fund of Guangdong Province, China.

The motivations for the present work stem from many recent investigations in [1-5]. For the continuous or functional case, boundary value prolems analogous to (1.1) are studied by many authors, see, for example [6-11].

The following lemma will be play an important role in the proof of our results and can be found in [12].

Lemma 1.1. *Assume that X is a Banach space and $K \subset X$ is a cone in X; Ω_1, Ω_2 are open subsets of X, and $0 \in \overline{\Omega}_1 \subset \Omega_2$. Furthermore, let $\Phi : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator satisfying one of the following conditions:*
(i) $\|\Phi(x)\| \le \|x\|, \forall\, x \in K \cap \partial\Omega_1$; $\|\Phi(x)\| \ge \|x\|, \forall\, x \in K \cap \partial\Omega_2$;
(ii) $\|\Phi(x)\| \le \|x\|, \forall\, x \in K \cap \partial\Omega_2$; $\|\Phi(x)\| \ge \|x\|, \forall\, x \in K \cap \partial\Omega_1$;
Then there is a fixed point of Φ in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Main results

We note that $x(t)$ is a solution of (1.1) if and only if

$$x(t) = \begin{cases} B_0\Big(\phi_q\Big(\sum_{n=0}^{T} r(n)f(x_n)\Big)\Big) + \sum_{m=0}^{t-1}\phi_q\Big(\sum_{n=m}^{T} r(n)f(x_n)\Big), & t \in [0, T+2], \\ \psi, & t \in [-\tau, -1]. \end{cases} \tag{2.1}$$

Furthermore, a solution $x(t)$ of (1.1) is called a positive solution, if $x(t) > 0$, for $t \in [0, T]$.

We assume that $\overline{x}(t)$ is the solution of BVP (1.1) with $f \equiv 0$. Clearly, it can be expressed as

$$\overline{x}(t) = \begin{cases} 0, & t \in [0, T+2], \\ \psi, & t \in [-\tau, -1]. \end{cases} \tag{2.2}$$

Let $x(t)$ be a solution of BVP (1.1) and $y(t) = x(t) - \overline{x}(t)$, noting that $y(t) = x(t)$ for $t \in [0, T+2]$, then we have from (2.1) that

$$y(t) = \begin{cases} B_0\Big(\phi_q\Big(\sum_{n=0}^{T} r(n)f(y_n+\overline{x}_n)\Big)\Big) + \sum_{m=0}^{t-1}\phi_q\Big(\sum_{n=m}^{T} r(n)f(y_n+\overline{x}_n)\Big), & t \in [0, T+2], \\ 0, & t \in [-\tau, -1]. \end{cases} \tag{2.3}$$

Let $E = \{y : [-\tau, T+2] \to R\}$ be endowed with the norm $\|y\| = \max_{t\in[-\tau,T+2]} |y(t)|$ and $K = \{y \in E : y(t) = 0 \text{ for } t \in [-\tau, -1];\ y(t) \ge \frac{1}{T+2+\beta}\|y\| \text{ for } t \in [0, T+2]\}$.

Clearly, E is a Banach space with the norm $\|y\|$ and K is a cone in E. If $y(t) \in K$, then $\|y\| = \|y\|_{[0,T+2]}$, where $\|y\|_{[0,T+2]} = \max_{t\in[0,T+2]} |y(t)|$.

Definie $\Phi : K \to E$ as

$$(\Phi y)(t) = \begin{cases} B_0\Big(\phi_q\Big(\sum_{n=0}^{T} r(n)f(y_n+\overline{x}_n)\Big)\Big) + \sum_{m=0}^{t-1}\phi_q\Big(\sum_{n=m}^{T} r(n)f(y_n+\overline{x}_n)\Big), & t \in [0, T+2], \\ 0, & t \in [-\tau, -1]. \end{cases} \tag{2.4}$$

It following from (2.4) that

$$\begin{aligned}\|\Phi y\| &= \|\Phi y\|_{[0,T+2]} = (\Phi y)(T+2)\\ &= B_0\Big(\phi_q\Big(\sum_{n=0}^{T} r(n)f(y_n+\overline{x}_n)\Big)\Big) + \sum_{m=0}^{t-1}\phi_q\Big(\sum_{n=m}^{T} r(n)f(y_n+\overline{x}_n)\Big)\\ &\le (T+2+\beta)\phi_q\Big(\sum_{n=0}^{T} r(n)f(y_n+\overline{x}_n)\Big). \qquad (2.5)\end{aligned}$$

Lemma 2.1. $\Phi(K)\subset K$.

Proof. For $t\in[-\tau,-1]$, $(\Phi y)(t)=0$, and for $t\in[0,T+2]$, we have from (2.4)−(2.5)

$$\begin{aligned}(\Phi y)(t) &\ge \phi_q\Big(\sum_{n=0}^{T} r(n)f(y_n+\overline{x}_n)\Big)\\ &\ge \frac{1}{T+2+\beta}\|\Phi y\|_{[0,T+2]} = \frac{1}{T+2+\beta}\|\Phi y\|, \qquad (2.6)\end{aligned}$$

which implies $\Phi(K)\subset K$.

Lemma 2.2. $\Phi : K\to K$ *is completely continuous.*

Let

$$l = \frac{1}{(T+2+\beta)\phi_q\Big(\sum\limits_{n=0}^{T} r(n)\Big)}, \quad M = \frac{T+2+\beta}{\phi_q\Big(\sum\limits_{n=\tau}^{T} r(n)\Big)}.$$

Theorem 2.1. *BVP* (1.1) has at least a positive solution if one of the following conditions is satisfied:

(H_3) $\lim\limits_{\|\varphi\|_C\downarrow 0}\sup\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < l^{p-1}$, $\lim\limits_{\|\varphi\|_C\uparrow\infty}\inf\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > M^{p-1}$, $\psi(t)\equiv 0$, $t\in[-\tau,-1]$;

(H_4) $\lim\limits_{\|\varphi\|_C\downarrow 0}\inf\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > M^{p-1}$, $\lim\limits_{\|\varphi\|_C\uparrow\infty}\sup\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < l^{p-1}$.

Proof. Suppose that (H_3) is satisfied. By $\psi(t)\equiv 0$, $t\in[-\tau,-1]$, we know $\overline{x}_n\equiv 0$, $n\in[0,T+2]$. Since $\lim\limits_{\|\varphi\|_C\downarrow 0}\frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < l^{p-1}$, there is a $\rho_1>0$ such that

$$f(\varphi)\le (l\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C\le\rho_1.$$

For any $y\in K$ with $\|y\|=\rho_1$, we deduce that $\|y_n\|_C\le\rho_1$ for $n\in[0,T+2]$ and have from (2.5)

$$\begin{aligned}\|\Phi y\| &\le (T+2+\beta)\phi_q\Big(\sum_{n=0}^{T} r(n)f(y_n)\Big)\\ &\le l(T+2+\beta)\|y\|\phi_q\Big(\sum_{n=0}^{T} r(n)\Big)\end{aligned}$$

$$= \|y\| \text{ and } y \in K \cap \partial\Omega_{\rho_1}, \tag{2.7}$$

where $\Omega_{\rho_1} = \{y \in K : \|y\| < \rho_1\}$.

On the other hand, since $\lim_{\|\varphi\|_C \uparrow \infty} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > M^{p-1}$, there exists a $\rho_2 > \rho_1$ such that

$$f(\varphi) \geq (M\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C \geq \frac{\rho_2}{T+2+\beta}.$$

Define $\Omega_{\rho_2} = \{y \in K : \|y\| < \rho_2\}$. For $y \in K$ with $\|y\| = \rho_2$, we have

$$y(t) \geq \frac{1}{T+2+\beta}\|y\|, \quad t \in [0, T+2],$$

and

$$\|y_n\|_C \geq \frac{1}{T+2+\beta}\|y\|. \tag{2.8}$$

Thus, we have from (2.5)−(2.8)

$$\begin{aligned}
\|\Phi y\| &= \|\Phi y\|_{[0,T+2]} = (\Phi y)(T+2) \\
&\geq \sum_{m=0}^{T+1} \phi_q\Big(\sum_{n=m}^{T} r(n) f(y_n)\Big) \\
&\geq \sum_{m=\tau}^{T+1} \phi_q\Big(\sum_{n=m}^{T} r(n)(M\|y_n\|_C)^{p-1}\Big) \\
&\geq \frac{M}{T+2+\beta}\|y\|\phi_q\Big(\sum_{n=\tau}^{T} r(n)\Big) \\
&= \|y\| \text{ for } y \in K \cap \partial\Omega_{\rho_2}.
\end{aligned} \tag{2.9}$$

According to the first part of Lemma 1.1, it follows that Φ has a fixed point $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that

$$0 < \rho_1 \leq \|y\| = \|y\|_{[0,T+2]} \leq \rho_2.$$

Now, suppose that (H_4) is satisfied. Since $\lim_{\|\varphi\|_C \downarrow 0} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > M^{p-1}$, there exists a $\rho_1 > 0$ such that

$$f(\varphi) \geq (M\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C \leq \rho_1.$$

For $y \in K$ with $\|y\| = \rho_1$, we have $\|y_n\|_C \leq \rho_1$ for $n \in [\tau, T+2]$. Furthermore, by asimilar argument as (2.8), we have

$$\|y\| \geq \|y_n\|_C \geq \frac{1}{T+2+\beta}\|y\|, \quad n \in [\tau, T+2].$$

For $n \in [\tau, T+2]$, we have $\overline{x}_n = 0$. Thus, we obtain

$$\begin{aligned}\|\Phi y\| &= \|\Phi y\|_{[0,T+2]} = (\Phi y)(T+2)\\ &\geq \sum_{m=0}^{T+1} \phi_q\Big(\sum_{n=m}^{T} r(n) f(y_n)\Big)\\ &\geq \sum_{m=\tau}^{T+1} \phi_q\Big(\sum_{n=m}^{T} r(n)(M\|y_n\|_C)^{p-1}\Big)\\ &\geq \frac{M}{T+2+\beta}\|y\|\phi_q\Big(\sum_{n=\tau}^{T} r(n)\Big)\\ &= \|y\| \text{ for } y \in K \cap \partial\Omega_{\rho_1}. \qquad (2.10)\end{aligned}$$

On the other hand, since $\lim\limits_{\|\varphi\|_C\uparrow\infty} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < l^{p-1}$, there exists $N > \max\{\rho_1, \max\limits_{k\in[-\tau,-1]} |\psi(k)|\}$ such that

$$f(\varphi) < (l\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C > N.$$

Choose a positive constant ρ_2 such that

$$\rho_2 > \rho_1 + l^{-1}\max\{f^{q-1}(\varphi) : 0 \leq \|\varphi\|_C \leq N + \|\overline{X}\|\}.$$

For $y \in K$, $\|y\| = \rho_2$, we have from the facts: $\overline{x}(t) \geq 0$, $y(t) \geq 0$ for $t \in [-\tau, T+2]$, that for $n \in [0, T]$

$$\begin{aligned}&\|y_n + \overline{x}_n\|_C \geq \|y_n\|_C > N, \quad \|y_n\|_C > N,\\ &\|y_n + \overline{x}_n\|_C \leq \|y_n\|_C + \|\overline{x}_n\| \leq N + \|\overline{x}\|, \quad \|y_n\|_C \leq N.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|\Phi y\| &= \|\Phi y\|_{[0,T+2]} = (\Phi y)(T+2)\\ &= B_0\Big(\phi_q\Big(\sum_{n=0}^{T} r(n) f(y_n + \overline{x}_n)\Big)\Big) + \sum_{m=0}^{T+1} \phi_q\Big(\sum_{n=m}^{T} r(n) f(y_n + \overline{x}_n)\Big)\\ &\leq (T+2+\beta)\phi_q\Big(\sum_{n=0}^{T} r(n) f(y_n + \overline{x}_n)\Big)\\ &= (T+2+\beta)\phi_q\Big(\sum_{\|y_n\|_C > N} r(n) f(y_n + \overline{x}_n) + \sum_{\|y_n\|_C < N} r(n) f(y_n + \overline{x}_n)\Big)\\ &\leq (T+2+\beta)\max\Big\{l\|y\|, \max\{f^{q-1}(\varphi) : 0 \leq \|\varphi\|_C \leq N + \|\overline{x}|\}\Big\}\phi_q\Big(\sum_{n=0}^{T} r(n)\Big)\\ &= l^{-1}\max\Big\{l\|y\|, \max\{f^{q-1}(\varphi) : 0 \leq \|\varphi\|_C \leq N + \|\overline{x}\|\}\Big\}\\ &\leq \rho_2 = \|y\| \text{ for } y \in K \cap \partial\Omega_{\rho_2}.\end{aligned}$$

By the second prat of Lemma 1.1, it follows that Φ has a fixed point $y \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that

$$0 < \rho_1 < \|y\| = \|y\|_{[0,T+2]} \le \rho_2.$$

Suppose that y is the fixed point of Φ in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, then $x = y + \overline{x}$ is a positive solution of BVP (1.1).

In what follows, we shall consider the existence of twin positive solutions for BVP (1.1).

Theorem 2.2. *If the following conditions are satisfied:*

(H_5) $\lim\limits_{\|\varphi\|_C \downarrow 0} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > M^{p-1}$; $\lim\limits_{\|\varphi\|_C \uparrow \infty} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} > M^{p-1}$;

(H_6) *there exists a* $p_1 > 0$ *such that for* $\forall\ 0 \le \|\varphi\|_C \le p_1 + p_0$, *one has* $f(\varphi) \le (lp_1)^{p-1}$, *where* $p_0 = \max\limits_{k \in [-\tau,-1]} |\psi(k)|$.

Then BVP (1.1) has at least two positive solutions x_1, x_2 such that $0 < \|x_1\|_{[0,T+2]} < p_1 < \|x_2\|_{[0,T+2]}$.

Proof. By (H_5), there exists a $r : 0 < r < p_1$ such that

$$f(\varphi) \ge (M\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C \le r.$$

For $y \in K$, $\|x\| = r$, we have

$$r \ge \|y_n\|_C \ge \frac{1}{T+2+\beta}\|y\| = \frac{r}{T+2+\beta}, \quad n \in [\tau, T+1].$$

Therefore we obtain a analogous inequality:

$$\|\Phi(y)\| \ge \|y\| \text{ for } y \in K \cap \partial\Omega_r,$$

where $\Omega_r = \{y \in K : \|y\| < r\}$.

On the other hand, we have from (H_5) that there exists a $R > p_1$ such that

$$f(\varphi) \ge (M\|\varphi\|_C)^{p-1}, \quad \|\varphi\|_C \ge \frac{R}{T+2+\beta}.$$

For $y \in K$, $\|y\| = R$, we have a analogous result to (2.8):

$$\|y_n\|_C \ge \frac{1}{T+2+\beta}\|y\| = \frac{R}{T+2+\beta} \text{ for } n \in [\tau, T+1].$$

Furthermore, we have $\|\Phi(y)\| \ge \|y\|$ for $y \in K \cap \partial\Omega_R$, where $\Omega_R = \{y \in K : \|y\| < R\}$.

Now, by (H_6), for $\forall\ y \in K$ with $\|y\| = p_1$ one has

$$\begin{aligned}
\|\Phi y\| &= \|\Phi y\|_{[0,T+2]} = (\Phi y)(T+2) \\
&= B_0\Big(\phi_q\Big(\sum_{n=0}^{T} r(n) f(y_n + \overline{x}_n)\Big)\Big) + \sum_{m=0}^{T+1} \phi_q\Big(\sum_{n=m}^{T} r(n) f(y_n + \overline{x}_n)\Big) \\
&\le (T+2+\beta) l p_1 \phi_q\Big(\sum_{n=0}^{T} r(n)\Big) \\
&= p_1 = \|y\|.
\end{aligned}$$

According to Lemma 1.1, it follows that Φ has two fixed points y_1, y_2 such that $y_1 \in K \cap \overline{\Omega}_{p_1} \setminus \Omega_r$, $y_2 \in K \cap \overline{\Omega}_R \setminus \Omega_{p_1}$, where $\Omega_{p_1} = \{y \in K : \|y\| < p_1\}$, that is $0 < \|y_1\| < p_1 < \|y_2\|$. Since $y_i \in K$, we have $y_i(t) > 0$, $\forall\ t \in [0, T+2]$, $i = 1, 2$. Let $x_1 = y_1 + \overline{x}$, $x_2 = y_2 + \overline{x}$, then x_1, x_2 are positive solutions of BVP (1.1) satisfying $0 < \|x_1\|_{[0,T+2]} < p_1 < \|x_2\|_{[0,T+2]}$.

Theorem 2.3. *If the following conditions are satisfied:*

(H_7) $\lim\limits_{\|\varphi\|_C \downarrow 0} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < l^{p-1}$, $\lim\limits_{\|\varphi\|_C \uparrow \infty} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} < l^{p-1}$, $\psi(t) \equiv 0$, $t \in [-\tau, -1]$;

(H_8) *there exists a* $p_2 > 0$ *such that for* $\forall$ $\frac{p_2}{T+2+\beta} \le \|\varphi\|_C \le p_2$, *one has*

$$f(\varphi) \ge \Big(\frac{Mp_2}{T+2+\beta}\Big)^{p-1}.$$

Then BVP (1.1) *has at least two posivive solutions* x_1, x_2 *satisfying* $0 < \|x_1\|_{[0,T+2]} < p_2 < \|x_2\|_{[0,T+2]}$.

The proof of Theorem 2.3 is analogous to Theorem 2.2 and thus is omitted.

The following Corollaries are obvious.

Corollary 2.1. *BVP* (1.1) *has at least a positive solution if one of the following conditions is satisfied:*

(H_3') $\lim\limits_{\|\varphi\|_C \downarrow 0} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = 0$, $\lim\limits_{\|\varphi\|_C \uparrow \infty} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty$, $\phi(t) \equiv 0$, $t \in [-\tau, -1]$;

(H_4') $\lim\limits_{\|\varphi\|_C \downarrow 0} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty$, $\lim\limits_{\|\varphi\|_C \uparrow \infty} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = 0$.

Corollary 2.2. *If the following conditions are satisfied:*

(H_5') $\lim\limits_{\|\varphi\|_C \downarrow 0} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty$; $\lim\limits_{\|\varphi\|_C \uparrow \infty} \inf \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = +\infty$;

(H_6) *there exists a* $p_1 > 0$ *such that for* $\forall$ $0 \le \|\varphi\|_C \le p_1 + p_0$, *one has* $f(\varphi) \le (lp_1)^{p-1}$, *where* $p_o = \max\limits_{k \in [-\tau, -1]} |\psi(k)|$.

Then BVP (1.1) *has got at least two positive solutions* x_1, x_2 *satisfying* $0 < \|x_1\|_{[0,T+2]} < p_1 < \|x_2\|_{[0,T+2]}$.

Corollary 2.3. *If the following conditions are satisfied:*

(H_7') $\lim\limits_{\|\varphi\|_C \downarrow 0} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = 0$, $\lim\limits_{\|\varphi\|_C \uparrow \infty} \sup \frac{f(\varphi)}{\|\varphi\|_C^{p-1}} = 0$, $\phi(t) \equiv 0$, $t \in [-\tau, -1]$;

(H_8) *there exists a* $p_2 > 0$ *such that for* $\forall$ $\frac{p_2}{T+2+\beta} \le \|\varphi\|_C \le p_2$, *one has* $f(\varphi) \ge (\frac{Mp_2}{T+2+\beta})^{p-1}$.

Then BVP (1.1) *has got at least two positive solutions* x_1, x_2 *satisfying* $0 < \|x_1\|_{[0,T+2]} < p_2 < \|x_2\|_{[0,T+2]}$.

3. Example

Example 3.1. Consider BVP:

$$\begin{aligned}&\Delta\phi_p(\Delta x(t))+r(t)x^3(t-1)=0,\quad t\in[0,T],\\&x(-1)=0;\quad x(0)-B_0(\Delta x(0))=0;\quad x(T+1)=X(T+2),\end{aligned}\tag{3.1}$$

where $\tau=1<T$, $1<p<4$, $f(\varphi)=\varphi^3(-1)$, $r(t)$ satisfies (H_1). As $\varphi\in C^+$, $\|\varphi\|_C\to 0$ we have that

$$\frac{f(\varphi)}{\|\varphi\|_C^{p-1}}=\frac{\varphi^3(-1)}{\|\varphi\|_C^{p-1}}=\frac{\|\varphi\|_C^3}{\|\varphi\|_C^{p-1}}=\|\varphi\|_C^{4-p}\to 0.$$

That is to say that $\lim\limits_{\|\varphi\|_C\downarrow 0}\sup\frac{f(\varphi)}{\|\varphi\|_C^{p-1}}=0$ holds.

On the other hand, suppose that $\varphi\in C^+$, then $\|\varphi\|_C=\varphi(-1)$, thus, as $\|\varphi\|_C\to\infty$ we get

$$\frac{f(\varphi)}{\|\varphi\|_C^{p-1}}=\frac{\varphi^3(-1)}{\|\varphi\|_C^{p-1}}=\|\varphi\|_C^{4-p}\to+\infty.$$

That is to say that $\lim\limits_{\|\varphi\|_C\downarrow 0}\inf\frac{f(\varphi)}{\|\varphi\|_C^{p-1}}=\infty$ holds.

According to Corollary 2.1, it follows that BVP (3.1) has at least a positive solution $y(t)$.

Example 3.2. Consider BVP:

$$\begin{aligned}&\Delta\phi_p(\Delta x(t))+r[x^{\frac{1}{9}}(t-1)+x^{\frac{1}{3}}(t-1)]=0,\quad t\in[0,T]\\&x(t)=\psi(t);\quad t=-1;\quad x(0)-B_0(\Delta x(0))=0;\quad x(T+1)=X(T+2),\end{aligned}\tag{3.2}$$

where $\tau=1<T$, $r>0$ is a constant. $\psi(t)\geq 0$, $\|\psi\|_C=m_0=|\psi(-1)|>0$, $p=\frac{7}{6}$, $q=7$, $f(\varphi)=\varphi^{\frac{1}{9}}(-1)+\varphi^{\frac{1}{3}}(-1)$. Suppost that $\varphi\in C^+$, then $\|\varphi\|_C=\varphi(-1)$, thus, as $\|\varphi\|_C\to 0$ or $\|\varphi\|_C\to+\infty$ we get

$$\frac{f(\varphi)}{\|\varphi\|_C^{p-1}}=\frac{\varphi^{1/9}(-1)+\varphi^{1/3}(-1)}{\|\varphi\|_C^{p-1}}=\|\varphi\|_C^{\frac{10-9p}{9}}+\|\varphi\|_C^{\frac{4-3p}{3}}\to+\infty.$$

We deduce that

$$l=\frac{1}{(T+2+\beta)\phi_q\Big(\sum\limits_{n=0}^{T}r(n)\Big)}=\frac{1}{(T+2+\beta)\phi_q((T+1)r)}=\frac{1}{(T+2+\beta)(T+1)^6r^6},$$

then for $\forall\ m>0$ and $0\leq\|\varphi\|_C\leq m+m_0$, one has

$$0\leq f(\varphi)\leq(m+m_0)^{\frac{1}{9}}+(m+m_0)^{\frac{1}{3}}=(m+m_0)^{\frac{1}{9}}\Big(m^{1-p}+\frac{(m+m_0)^{\frac{2}{9}}}{m^{p-1}}\Big)m^{p-1}.$$

Define $H(m) = (m + m_0)^{\frac{1}{9}}(m^{1-p} + \frac{(m+m_0)^{\frac{2}{9}}}{m^{p-1}})$, then

$$\lim_{m\to 0} H(m) = +\infty, \quad \lim_{m\to +\infty} H(m) = +\infty. \tag{3.3}$$

Suppose that, r, T and m_0 satisfy

$$(2m_0)^{\frac{1}{9}}(m_0^{-\frac{1}{6}} + 2^{\frac{2}{9}} m_0^{\frac{1}{18}}) < \frac{1}{r(T+1)(T+2+\beta)^{\frac{1}{6}}} = l^{p-1},$$

then $H(m_0) = (2m_0)^{\frac{1}{9}}(m_0^{-\frac{1}{6}} + 2^{\frac{2}{9}} m_0^{\frac{1}{18}}) < l^{p-1}$ holds. By the continuity of $H(m)$ and (3.3), we can found a $m > 0$ (for example $m = m_0$) such that $f(\varphi) < H(m)m^{p-1} < (lm)^{p-1}$ for $0 \le \|\varphi\|_C \le m + m_0$. By the Corollary 2.2, we know that BVP (3.2) has at least two positive solutions.

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