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NEIGHBORHOODS AND PARTIAL SUMS OF CERTAIN MEROMORPHICALLY MULTIVALENT FUNCTIONS

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Abstract. By making use of the familiar concept of neighborhoods of analytic functions, the author proves an inclusion relations associated with the $(n,\delta)-$neighborhoods of a subclass $Q_k[p,\alpha;A,B]$ which was introduced by Srivastava, Hossen and Aouf. The partial sums of the functions in $Q_k[p,\alpha;A,B]$ are also considered.

1. Introduction

Let $\sum_{p,k}$ be the class of functions of the form

$$f(z)=z^{-p}+\sum_{n=k}^{\infty}a_{n+p-1}z^{n+p-1},\quad (p,k\in N=\{1,2,3,\ldots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk

$$U^*=\{z:z\in C \text{ and } 0<|z|<1\}=U\setminus\{0\}.$$

In recent years, many important properties and characteristics of various interesting sbuclasses of the class $\sum_{p,k}$ of meromorphically p-valent functions were investigated extensively by Srivastava et al. [7], Owa et al. [5], Yang [9], Liu and Srivastava [3, 4] and other ([2, 8]). In [7], Srivastava, Hossen and Aouf introduced and studied a subclass $Q_k[p,\alpha;A,B]$ of $\sum_{p,k}$ as following.

Definition. For fixed parameters A and B, with $-1\le A<B\le 1$, $A+B\ge 0$, and $0<B\le 1$. A function $f\in\sum_{p,k}$ is said to be in the class $Q_k[p,\alpha;A,B]$ of meromorphically p-valent functions in U if and only if

$$\left|\frac{(zf'(z)/f(z))+p}{B(zf'(z)/f(z)+[pB+(A-B)(p-\alpha)]}\right|<1\qquad (z\in U^*;\quad 0\le\alpha<p). \tag{1.2}$$

Many interesting properties of the class $Q_k[p,\alpha;A,B]$ were obtained by Srivastava, Hossen and Aouf [7].

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In the present paper, we shall discuss the properties of Neighborhoods and partial sums of the subclass $Q_k[p, \alpha; A, B]$

2. Main results

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [1] and Ruscheweyh [6], we begin by introducing here the δ-neighborhood of a function $f \in \sum_{p,k}$ of the from (1.1) by means of the definition

$$N_\delta(f) = \Bigg\{ g \in \sum_p : g(z) = z^{-p} + \sum_{n=k}^{\infty} b_{n+p-1} z^{n+p-1} \quad \text{and}$$
$$\sum_{n=k}^{\infty} \frac{(1+B)(n+2p-1) + (p-\alpha)(B-A)}{(p-\alpha)(B-A)} |a_{n+p-1} - b_{n+p-1}| \le \delta$$
$$(-1 \le A < B \le 1; \quad \delta \ge 0) \Bigg\} \tag{2.1}$$

Making use of definition (2.1), we now prove the following result.

Theorem 1. *Let* $f \in Q_k[p, \alpha; A, B]$ *be given by* (1.1). If f satisfies the inclusing condition

$$(f(z) + \varepsilon z^{-p})(1+\varepsilon)^{-1} \in Q_k[p, \alpha; A, B] \quad (\varepsilon \in C; |\varepsilon| < \delta; \; \delta > 0), \tag{2.2}$$

then

$$N_\delta(f) \subset Q_k[p, \alpha; A, B]. \tag{2.3}$$

Proof. It is easily seen from (1.2) that a function $f \in Q_k[p, \alpha; A, B]$ if and only if

$$\frac{zf'(z) + pf(z)}{Bzf'(z) + [pB + (A-B)(p-\alpha)]f(z)} \ne \sigma \quad (z \in U; \sigma \in C, |\sigma| = 1) \tag{2.4}$$

which is equivalent to

$$\frac{(f \times h)(z)}{z^{-p}} \ne 0 \qquad (z \in U), \tag{2.5}$$

where, for convenience

$$h(z) = z^{-p} + \sum_{n=k}^{\infty} \frac{(B\sigma - 1)(n+2p-1) + \sigma(p-\alpha)(A-B)}{\sigma(p-\alpha)(A-B)} z^{n+p-1}$$
$$= z^{-p} + \sum_{n=k}^{\infty} c_{n+p-1} z^{n+p-1}. \tag{2.6}$$

We easily find from (2.6) that

$$|c_{n+p-1}| = \Big|\frac{(B\sigma-1)(n+2p-1)+\sigma(p-\alpha)(A-B)}{\sigma(p-\alpha)(A-B)}\Big| \leq \frac{(1+B)(n+2p-1)+(p-\alpha)(B-A)}{(p-\alpha)(B-A)}.$$

Furthermore, under the hypothesis of the theorem, (2.5) yields the following inequality

$$\Big|\frac{(f\times h)(z)}{z^{-p}}\Big| \geq \delta \qquad (z\in U;\delta>0). \tag{2.7}$$

Now, we let

$$\varphi(z) = z^{-p} + \sum_{n=k}^{\infty} b_{n+p-1}z^{n+p-1} \in N_\delta(f),$$

then

$$\begin{aligned}&\Big|\frac{[f(z)-\varphi(z)]\times h(z)}{z^{-p}}\Big|\\ &= \Big|\sum_{n=k}^{\infty}(a_{n+p-1}-b_{n+p-1})c_{n+p-1}z^{n+2p-1}\Big|\\ &\leq \sum_{n=k}^{\infty}\frac{(1+B)(n+2p-1)+(p-\alpha)(B-A)}{(p-\alpha)(B-A)}\big|a_{n+p-1}-b_{n+p-1}\big|\cdot|z^{n+2p-1}|\\ &< \delta \qquad (z\in U;\delta>0).\end{aligned} \tag{2.8}$$

Thus, for any complex number σ such that $|\sigma|=1$, we have

$$\frac{(\varphi\times h)(z)}{z^{-p}} \neq 0 \qquad (z\in U), \tag{2.9}$$

which implies that $\varphi \in Q_k[p,\alpha;A,B]$. The proof of the theorem is thus completed.

Next, we prove

Theorem 2. *Let* $f\in\sum_{p,k}$ *be given by* (1.1) and define the partial sums $s_m(z)$ by

$$s_m(z) = \begin{cases} z^{-p} & m=1,2,\dots k-1;\\ z^{-p}+\displaystyle\sum_{n=k}^{m} a_{n+p-1}z^{n+p-1} & m=k,k+1,\dots.\end{cases} \tag{2.10}$$

Suppose also that

$$\sum_{n=k}^{\infty} l_{n+p-1}|a_{n+p-1}| \leq 1$$
$$\Big(\text{where}\quad l_{n+p-1} = \frac{(1+B)(n+2p-1)+(p-\alpha)(B-A)}{(p-\alpha)(B-A)}\Big). \tag{2.11}$$

Then

(i) $f \in Q_k[p, \alpha; A, B]$;

(ii)

$$\mathrm{Re}\Big(\frac{f(z)}{s_m(z)}\Big) > 1 - \frac{1}{l_{m+p}} \quad (z \in U; m = k, k+1, \ldots) \tag{2.12}$$

and

$$\mathrm{Re}\Big(\frac{s_m(z)}{f(z)}\Big) > \frac{l_{m+p}}{1 + l_{m+p}} \quad (z \in U; m = k, k+1, \ldots). \tag{2.13}$$

Each of the bounds in (2.12) and (2.13) is the best possible.

Proof. (i) It is not difficult to see that

$$z^{-p} \in Q_k[p, \alpha; A, B].$$

Thus, from Theorem 1 and hypothesis (2.11) of Theorem 2, we have

$$N_1(z^{-p}) \subset Q_k[p, \alpha; A, B],$$

which shows that $f \in Q_k[p, \alpha; A, B]$.
(ii) Under the hypothesis in part (ii) of Theorem 2, we can see from (2.11) that

$$l_{n+p} > l_{n+p-1} > 1 \qquad (n = k, k+1, \ldots).$$

Therefore, we have

$$\sum_{n=k}^{m} |a_{n+p-1}| + l_{m+p} \sum_{n=m+1}^{\infty} |a_{n+p-1}| \leq \sum_{n=k}^{\infty} l_{n+p-1} |a_{n+p-1}| \tag{2.14}$$

by using hypothesis (2.11) of Theorem 2 again.

Upon setting

$$\begin{aligned} g_1(z) &= l_{m+p}\left[\frac{f(z)}{s_m(z)} - \Big(1 - \frac{1}{l_{m+p}}\Big)\right] \\ &= 1 + \frac{l_{m+p} \displaystyle\sum_{n=m+1}^{\infty} a_{n+p-1} z^{n+2p-1}}{1 + \displaystyle\sum_{n=k}^{m} a_{n+p-1} z^{n+2p-1}} \quad (m \geq k), \end{aligned} \tag{2.15}$$

and applying (2.14), we find that

$$\begin{aligned} \Big|\frac{g_1(z) - 1}{g_1(z) + 1}\Big| &\leq \frac{l_{m+p} \displaystyle\sum_{n=m+1}^{\infty} |a_{n+p-1}|}{2 - 2\displaystyle\sum_{n=k}^{m} |a_{n+p-1}| - l_{m+p} \displaystyle\sum_{n=m+1}^{\infty} |a_{n+p-1}|} \\ &\leq 1 \qquad (z \in U, m \geq k), \end{aligned} \tag{2.16}$$

which readily yields inequality (2.12).

If we take

$$f(z) = z^{-p} - \frac{z^{m+2p-1}}{l_{m+p}} \quad (m \geq k), \tag{2.17}$$

then

$$\frac{f(z)}{s_m(z)} = 1 - \frac{z^{m+3p-1}}{l_{m+p}} \to 1 - \frac{1}{l_{m+p}} \quad (z \to 1-),$$

which shows that the bound in (2.12) is the best possible.

Similarly, if we put

$$\begin{aligned} g_2(z) &= (1 + l_{m+p})\Big(\frac{s_m(z)}{f(z)} - \frac{l_{m+p}}{1 + l_{m+p}}\Big) \\ &= 1 - \frac{(1 + l_{m+p}) \displaystyle\sum_{n=m+1}^{\infty} a_{n+p-1} z^{n+2p-1}}{1 + \displaystyle\sum_{n=k}^{\infty} a_{n+p-1} z^{n+2p-1}} \quad (m \geq k), \end{aligned} \tag{2.18}$$

and make use of (2.14), we can deduce that

$$\begin{aligned} \Big|\frac{g_2(z) - 1}{g_2(z) + 1}\Big| &\leq \frac{(1 + l_{m+p}) \displaystyle\sum_{n=m+1}^{\infty} |a_{n+p-1}|}{2 - 2\displaystyle\sum_{n=k}^{m} |a_{n+p-1}| - (l_{m+p} - 1) \displaystyle\sum_{n=m+1}^{\infty} |a_{n+p-1}|} \\ &\leq 1 \qquad (z \in U, m \geq k), \end{aligned} \tag{2.19}$$

which leads us immediately to assertion (2.13) of the theorem.

The bound in (2.13) is sharp with the extremal function given by (2.17). The proof of Theorem 2 is completed.

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