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INTEGRAL OPERATORS AND UNIVALENT FUNCTIONS

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Abstract. In this paper, we define two new integral operators L^k and L_k which are iterative in nature. We show that for $f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$ with radius of convergence larger than one, $L^k f(z)$ and $L_k f(z)$ when restricted on $E = \{z : |z| < 1\}$ will eventually be univalent for large enough k. We then show that these are the best possible results by demonstrating that there exists a holomorphic function $T(z)$ in normalized form and with radius of convergence equal to one such that $L^k T(z)$ and $L_k T(z)$ fail to be univalent when restricted to E for every $k \in \mathbb{N}$.

1. Introduction

Let E be the unit disk $\{z : |z| < 1\}$ and S be the set of univalent functions in E which can be normalized to the conditions $f(0) = 0$ and $f'(0) = 1$. In 1960 Biernacki [1] falsely claimed that $F(z) = \int_0^z \frac{f(\zeta)}{\zeta} \, d\zeta$ is in S whenever $f \in S$. Three years later this error was notice by Krzyz and Lewandowski [3, p.149]. Nevertheless, Biernacki's consideration of the integral transform gave rise to the study of the following problem:

for what choices of the parameter α is $F_\alpha(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta} \right)^\alpha \, d\zeta$ in S whenever $f \in S$.

$F_\alpha(z)$ defined above is known as integrals of the *first type*. Many results were obtained by Merkes and Wright [4] in the study of this integral. Other types of integral operators are also studied by researchers in this area. For example $G_\alpha(z) = \int_0^z \left[f'(\zeta) \right]^\alpha \, d\zeta$ is known as integrals of the *second type* and theorems similar to $F_\alpha(z)$ were proved by Pfaltzgraff [5], Merkes and Wright [4]. What we consider in this paper are slightly different, we are interested in iterative integral operators that take normalized functions which are not necessarily univalent into the class S when restricted to E.

2. Integral Operators L^k and L_k

We define $S_{R\geq 1}$ as the class of holomorphic functions f which are normalized and have radius of convergence R, where R is larger or equal to one. Thus for each $f \in S_{R\geq 1}$, f has a Taylor series expansion of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad |z| < R, \ for \ R \geq 1.$$

We define $S_{R>1}$ as the class of holomorphic functions f which are normalized and have radius of convergence R, where R is strictly greater than one.

For $f \in S_{R\geq 1}$ or $f \in S_{R>1}$, f has a Taylor series expression of the form $f(z) = z + a_2 z^2 + \cdots$, hence $z = 0$ is a removable singularity of $f(z)/z$. If we define $f(z)/z = 1$ when $z = 0$, then $f(z)/z$ is holomorphic and the complex integral is independent of path.

We then introduce the following two operators which act on $f \in S_{R\geq 1}$ and their definitions are given below.

Definition 2.1. For $f(z) \in S_{R\geq 1}$ let

$$Lf(z) = \int_0^z \frac{1}{z_1} f(z_1) \, dz_1$$

$$= z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n.$$

$$L^2 f(z) = L(Lf(z))$$

$$= z + \sum_{n=2}^{\infty} \frac{1}{n^2} a_n z^n.$$

In general, for positive integer k, we have

$$L^k f(z) = L(L^{k-1} f(z))$$

$$= z + \sum_{n=2}^{\infty} \frac{1}{n^k} a_n z^n.$$

Definition 2.2. For $f \in S_{R\geq 1}$, let

$$L_k f(z) = \frac{k!}{z^{k-1}} \int_0^z \int_0^{z_k} \int_0^{z_{k-1}} \cdots \int_0^{z_2} \frac{1}{z_1} f(z_1) \, dz_1 dz_2 dz_3 \cdots dz_k$$

$$= z + \sum_{n=2}^{\infty} \frac{1}{\binom{n+k-1}{n-1}} a_n z^n.$$

Notice that in the Definition 2.1 and 2.2, $L^1 f(z)$ and $L_1 f(z)$ are both equal to $F(z) = \int_0^z \frac{f(\zeta)}{\zeta} \, d\zeta$ which is the integral consider by Biernacki in his 1960 paper. We next show that

for $f \in S_{R>1}$, there exists a positive integer N such that if $k \geq N$, then $L^k f(z)$ and $L_k f(z)$ restricted to E are univalent and belongs to S.

We will need the following results for the proof in the next section. The first is the celebrated Bieberbach's theorem proved by Louis de Branges in 1985. For the proof of this theorem, please refer to [2].

Theorem 2.1 (Bieberbach's Theorem). *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is in S, then $|a_n| \leq n$. *The inequality is sharp with equality occurs iff f is a rotation of the Koebe function.*

Noshiro and Warschawski [6] gave a simple but important criterion for univalence in 1935, this criterion now bears their names.

Theorem 2.2 (Noshiro-Warschawski Theorem). *Suppose that f is holomorphic in a convex domain D and for some real α we have*

$$Re\left\{e^{i\alpha} f'(z)\right\} > 0$$

for all $z \in D$. Then $f(z)$ is univalent in D.

3. Main results

Theorem 3.1. *For $f \in S_{R>1}$, there exists a positive integer N such that if $k \geq N$, then $L^k f(z)$ restricted to E is univalent and belongs to S.*

Proof. For fixed $k \in \mathbb{N}$ and $f \in S_{R>1}$. We have

$$\{L^k f(z)\}' = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} a_n z^{n-1}.$$

$$Re\{L^k f(z)\}' = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} Re\{a_n z^{n-1}\}$$

$$= 1 + H(z).$$

Note that $H(z) = \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} Re\{a_n z^{n-1}\}$, and

$$|H(z)| \leq \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} |Re\{a_n z^{n-1}\}|$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} |a_n z^{n-1}|$$

$$\leq \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} |a_n|, \ for \ |z| < 1.$$

Since $f \in S_{R>1}$, then the radius of convergence R for f, is greater than 1. Hence there exists an $\epsilon > 0$ such that $B = \frac{1}{R} + \epsilon < 1$. From $\limsup |a_n|^{\frac{1}{n}} = \frac{1}{R}$ and property of limit supremum, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|a_n| < B^n$.

Let $M = max\{|a_2|, |a_3|, \cdots, |a_{N_1-1}|\}$, we have

$$|H(z)| < \frac{1}{2^{k-1}}(N_1 - 2)M + \frac{1}{N_1^{k-1}}\left(\frac{B^{N_1}}{1-B}\right).$$

Now, there exists $N \in \mathbb{N}$ such that $|H(z)| < \frac{1}{2}$ whenever $k \geq N$. With this $N \in \mathbb{N}$ we see that

$$Re\{L^k f(z)\}' = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{k-1}} Re\{a_n z^{n-1}\}$$

$$= 1 + H(z)$$

$$> \frac{1}{2}, \; for \; |z| < 1,$$

whenever $k \geq N$.

By Noshiro-Warschawski theorem, we conclude that $L^k f(z)$ is univalent in E, whenever $k \geq N$. $\qquad \square$

We give the following lemma before we prove the next theorem.

Lemma 3.2. *For $j, k \in \mathbb{N}$, we have*

$$\frac{j+1}{\binom{j+k}{j}} \leq \frac{j}{\binom{j+k-1}{j-1}}.$$

Proof. Observe the following trivial inequality $\frac{j+1}{j+k} \leq 1$, multipying both sides of the inequality by $\frac{j!}{(j+k-1)!}$, we obtained $\frac{(j+1)!}{(j+k)!} \leq \frac{j!}{(j+k-1)!}$. The last inequality implies that

$$\frac{j+1}{\frac{(j+k)!}{k!j!}} \leq \frac{j}{\frac{(j+k-1)!}{k!(j-1)!}}.$$

This is what we wanted to show. $\qquad \square$

Theorem 3.3. *For $f \in S_{R>1}$, there exists a positive integer N such that if $k \geq N$, then $L_k f(z)$ restricted to E is univalent and belongs to S.*

Proof. For fixed $k \in \mathbb{N}$ and $f \in S_{R>1}$, we have

$$\{L_k f(z)\}' = 1 + \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} a_n z^{n-1}.$$

$$Re\{L_k f(z)\}' = 1 + \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} Re\{a_n z^{n-1}\}$$

$$= 1 + G(z).$$

Note that $G(z) = \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} Re\{a_n z^{n-1}\}$, and

$$|G(z)| \le \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} |Re\{a_n z^{n-1}\}|$$

$$\le \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} |a_n z^{n-1}|$$

$$\le \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} |a_n|, \; for \; |z| < 1.$$

Since $f \in S_{R>1}$, then the radius of convergence R for f, is greater than 1. Hence there exists an $\epsilon > 0$ such that $A = \frac{1}{R} + \epsilon < 1$. From $\limsup |a_n|^{\frac{1}{n}} = \frac{1}{R}$ and property of limit supremum, there exists $N_1 \in \mathbb{N}$ such that if $n \ge N_1$, then $|a_n| < A^n$.

Let $M = max\{|a_2|, |a_3|, \cdots, |a_{N_1-1}|\}$, and from Lemma 3.2 we have

$$|G(z)| < \frac{2}{k+1}(N_1 - 2)M + \frac{N_1!k!}{(N_1+k-1)!}\left(\frac{A^{N_1}}{1-A}\right).$$

Now, there exists $N \in \mathbb{N}$ such that $|G(z)| < \frac{1}{2}$ whenever $k \ge N$. With this $N \in \mathbb{N}$ we see that

$$Re\{L_k f(z)\}' = 1 + \sum_{n=2}^{\infty} \frac{n}{\binom{n+k-1}{n-1}} Re\{a_n z^{n-1}\}$$

$$= 1 + G(z)$$

$$> \frac{1}{2}, \; for \; |z| < 1,$$

whenever $k \ge N$.

By Noshiro-Warschawski theorem, we conclude that $L_k f(z)$ is univalent in E, whenever $k \ge N$. $\qquad\square$

Next, we are going to show that Theorem 3.1 and 3.3 are the best possible results. Let

$$T(z) = z + \sum_{m=2}^{\infty} a_m z^m, \; where \; a_m = \begin{cases} s^s, & \text{if } m = 2^s, s \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$T(z)$ has radius of convergence one since $\limsup |a_m|^{\frac{1}{m}} = 1$. Taking $s > 2$ and $m = 2^s$, we have $a_m = s^s > 2^s = m$, hence by the contrapositive of Bieberbach theorem [Theorem 2.3], we conclude that $T(z)$ is not univalent in E. We then show that $L^k T(z)$ and $L_k T(z)$ is not univalent in E for every positive integer k.

Theorem 3.4. $L^k T(z)$ *is not univalent in E for every $k \in \mathbb{N}$.*

Proof. For fixed $k \in \mathbb{N}$.

$$L^k T(z) = z + \sum_{m=2}^{\infty} b_m z^m, \text{ where } b_m = \begin{cases} \frac{s^s}{(2^s)^k}, & \text{if } m = 2^s, s \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

There exists a $s \in \mathbb{N}$ such that $s > 2^{k+1}$. Choose $m = 2^s$, then we have

$$b_m = \frac{s^s}{(2^s)^k} = \left(\frac{s}{2^k}\right)^s > 2^s = m$$

By the contrapositive of Bieberbach theorem, we concluded that $L^k T(z)$ is not univalent in E. Since $k \in \mathbb{N}$ arbitrary, this proves the theorem. $\square$

Theorem 3.5. $L_k T(z)$ *is not univalent in E for every* $k \in \mathbb{N}$.

Proof. For fixed $k \in \mathbb{N}$.

$$L_k T(z) = z + \sum_{m=2}^{\infty} c_m z^m, \text{ where } c_m = \begin{cases} \frac{s^s}{\binom{2^s+k-1}{2^s-1}}, & \text{if } m = 2^s, s \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

Note that there is a $s_1 \in \mathbb{N}$, such that $2^{s_1} > k - 1$, for the fixed k. Because

$$\lim_{s \to \infty} \frac{s}{2^{\frac{1}{s}}} = \infty,$$

there is a $s_2 \in \mathbb{N}$, such that $\frac{s_2}{2^{\frac{1}{s_2}}} > 2^{k+1}$. Let $s = \max\{s_1, s_2\}$, then we have the following inequality.

$$\frac{s^s}{\binom{2^s+k-1}{2^s-1}} = \frac{s^s k!}{(2^s+k-1)(2^s+k-2)\cdots 2^s} \geq \frac{s^s k!}{(2^s+k-1)^k} \geq \frac{k!}{2^k} \frac{s^s}{2^{sk}}.$$

Note that

$$\frac{k!}{2^k} = \frac{k}{2} \frac{k-1}{2} \frac{k-2}{2} \cdots \frac{3}{2} \frac{2}{2} \frac{1}{2} \geq \frac{1}{2},$$

hence,

$$\frac{s^s}{\binom{2^s+k-1}{2^s-1}} \geq \frac{1}{2} \frac{s^s}{2^{sk}}.$$

Since $\frac{s}{2^{\frac{1}{s}}} > 2^{k+1}$, we have

$$\frac{s^s}{\binom{2^s+k-1}{2^s-1}} \geq \frac{1}{2} \frac{s^s}{2^{sk}} > 2^s.$$

By taking $m = 2^s$, we have

$$c_m = \frac{s^s}{\binom{2^s+k-1}{2^s-1}} \geq \frac{1}{2} \frac{s^s}{2^{sk}} > 2^s = m.$$

By the contrapositive of Bieberbach theorem, we concluded that $L_k T(z)$ is not univalent in E. Since $k \in \mathbb{N}$ arbitrary, this proves the theorem. $\square$

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