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INVERSE NODAL PROBLEMS FOR DIFFERENTIAL OPERATORS ON GRAPHS WITH A CYCLE

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Abstract. Inverse nodal problems are studied for second-order differential operators on graphs with a cycle and with standard matching conditions in the internal vertex. Uniqueness theorems are proved, and a constructive procedure for the solution is provided.

1. Introduction

The main purpose of this note is to study inverse nodal problems for Sturm-Liouville differential operators on graphs with a cycle. The inverse nodal problems under consideration consist in recovering operators from the given nodes (zeros) of eigenfunctions. These problems are related to some questions in mechanics and mathematical physics (see, for example, [1]). Moreover, there are close connections of this area with inverse spectral problems. Inverse nodal problems for Sturm-Liouville operators on *an interval* have been studied fairly completely in [1]−[6] and other papers. The main results on inverse spectral problems on an interval are presented in the monographs [7]−[14].

Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [15], [16] and the references therein). In particular, inverse spectral problems of recovering coefficients of differential operators on graphs from their spectral characteristics were investigated in [17]−[23].

In this paper, we give the formulations and the solutions of inverse nodal problems for Sturm-Liouville operators on graphs with a cycle from given subsets of nodal points on a fixed edge or on a certain part of it. We prove the corresponding uniqueness theorems and provide a constructive procedure for the solution. We also show connections of these problems with inverse spectral problems on graphs.

Consider a compact graph G in $\mathbf{R^m}$ with the set of vertices $V = \{v_0, \dots, v_r\}$ and the set of edges $\mathcal{E} = \{e_0, \dots, e_r\}$, where $v_1, \dots, v_r$ are the boundary vertices, v_0 is the internal vertex, $e_j = [v_j, v_0]$, $j = \overline{1, r}$, $\bigcap\limits_{j=0}^{r} e_j = \{v_0\}$, and e_0 is a cycle. Thus, the graph G has one

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cycle e_0 and one internal vertex v_0. We suppose that the length of each edge is equal to 1. Each edge $e_j \in \mathcal{E}$ is parameterized by the parameter $x \in [0,1]$; below we identify the value x of the parameter with the corresponding point on the edge. It is convenient for us to choose the following orientation: for $j = \overline{1,r}$, the vertex v_j corresponds to $x = 0$, and the vertex v_0 corresponds to $x = 1$; for $j = 0$, both ends $x = +0$ and $x = 1 - 0$ correspond to v_0.

An integrable function Y on G may be represented as $Y = \{y_j\}_{j=\overline{0,r}}$, where the function $y_j(x)$, $x \in [0,1]$, is defined on the edge e_j. Let $q = \{q_j\}_{j=\overline{0,r}}$ be an integrable real-valued function on G; q is called the potential. Consider the following differential equation on G:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda y_j(x), \quad j = \overline{0,r}, \tag{1}$$

where λ is the spectral parameter, the functions y_j, y_j', $j = \overline{0,r}$, are absolutely continuous on $[0,1]$ and satisfy the following matching conditions in the internal vertex v_0:

$$\left.\begin{array}{l} y_j(1) = y_0(0), \quad j = \overline{0,r} \quad \text{(continuity condition)}, \\ \displaystyle\sum_{j=0}^{r} y_j'(1) = y_0'(0) \quad \text{(Kirchhoff's condition)}. \end{array}\right\} \tag{2}$$

The matching conditions (2) are called the *standard conditions*. In electrical circuits, (2) expresses Kirchhoff's law; in elastic string networks, it expresses the balance of tension, and so on.

Let us consider the boundary value problem $B = B(q)$ on G for equation (1) with the matching conditions (2) and with Dirichlet boundary conditions at the boundary vertices $v_1, \ldots, v_r$:

$$y_j(0) = 0, \quad j = \overline{1,r}. \tag{3}$$

In section 2 we study the inverse nodal problem of recovering the potential from any dense subset of the nodal points of B. The uniqueness theorem is proved and a constructive procedure for the solution is provided. In section 3 we investigate the so-called incomplete inverse problems of recovering the potential on a fixed edge from a subset of nodal points situated only on a part of the edge. The main results of this section is presented in Theorem 4. In order to prove this theorem we use connections with inverse spectral problems established in Section 3. In particular, we essentially use the results on incomplete inverse spectral problem (presented in Theorem 3) of recovering the potential on a part of a fixed edge from a part of the spectrum of B.

We note that in the recent paper [24], inverse nodal problem on graphs was studied in a different formulation. It was proved in [24] that the specification of the *spectrum* and the set of *all nodal points* uniquely determines the potential. In the present paper we study a particular class of graphs with a cycle and show that we do not need to specify the spectrum. We pay our main attention to incomplete inverse spectral and nodal problems when nodal points are specified on a part of the graph. We also note that in [25] similar results were obtained on star-type graphs where the structure of the characteristic function is simpler than for graphs having a cycle; it turns out that the results of [25] also remain valid in the more general situation under consideration.

2. Inverse nodal problems

Denote by $S_j(x,\lambda)$ and $C_j(x,\lambda)$, $j=\overline{0,r}$, the solutions of equation (1) on the edge e_i satisfying the initial conditions

$$S_j(0,\lambda)=C_j'(0,\lambda)=0,\quad S_j'(0,\lambda)=C_j(0,\lambda)=1.$$

For each fixed $x\in[0,1]$, the functions $S_j^{(\nu)}(x,\lambda)$ and $C_j^{(\nu)}(x,\lambda)$, $j=\overline{1,r}$, $\nu=0,1$, are entire in λ of order $1/2$. Moreover, the functions $S_j(x,\lambda)$ and $C_j(x,\lambda)$ are the unique solutions of the integral equations

$$S_j(x,\lambda)=\frac{\sin\rho x}{\rho}+\int_0^x\frac{\sin\rho(x-t)}{\rho}\,q_j(t)S_j(t,\lambda)\,dt, \tag{4}$$

$$C_j(x,\lambda)=\cos\rho x+\int_0^x\frac{\sin\rho(x-t)}{\rho}\,q_j(t)C_j(t,\lambda)\,dt, \tag{5}$$

where $\lambda=\rho^2$. Using (4) and (5) one gets (see, for example, [10, Chap. 1] for details) the following asymptotical formulae as $|\lambda|\to\infty$ uniformly in $x\in[0,1]$:

$$S_j(x,\lambda)=\frac{\sin\rho x}{\rho}-\frac{\cos\rho x}{2\rho^2}\int_0^x q_j(t)dt+\frac{1}{2\rho^2}\int_0^x q_j(t)\cos\rho(x-2t)dt+O\Big(\frac{\exp(|\tau|x)}{\rho^3}\Big), \tag{6}$$

$$S_j'(x,\lambda)=\cos\rho x+\frac{\sin\rho x}{2\rho}\int_0^x q_j(t)\,dt-\frac{1}{2\rho}\int_0^x q_j(t)\sin\rho(x-2t)\,dt+O\Big(\frac{\exp(|\tau|x)}{\rho^2}\Big), \tag{7}$$

$$C_j(x,\lambda)=\cos\rho x+\frac{\sin\rho x}{2\rho}\int_0^x q_j(t)\,dt+\frac{1}{2\rho}\int_0^x q_j(t)\sin\rho(x-2t)\,dt+O\Big(\frac{\exp(|\tau|x)}{\rho^2}\Big). \tag{8}$$

where $\tau=Im\,\rho$. Let us consider the function

$$\Delta(\lambda)=\Big(\prod_{j=1}^r S_j(1,\lambda)\Big)\Big(C_0(1,\lambda)+S_0'(1,\lambda)-2\Big)+\Big(\sum_{j=1}^r S_j'(1,\lambda)\prod_{k=1,\ k\neq j}^r S_k(1,\lambda)\Big)S_0(1,\lambda). \tag{9}$$

The function $\Delta(\lambda)$ is entire in λ of order $1/2$, and its zeros coincide with the eigenvalues of the boundary value problem B. Indeed, let

$$\begin{aligned}&Y(x,\lambda)=\{y_j(x,\lambda)\}_{j=\overline{0,r}},\\ &y_j(x,\lambda)=A_j(\lambda)S_j(x,\lambda),\ j=\overline{1,r},\quad y_0(x,\lambda)=A_0(\lambda)S_0(x,\lambda)+B_0(\lambda)C_0(x,\lambda),\end{aligned} \tag{10}$$

where the functions $A_j(\lambda)$ and $B_0(\lambda)$ do not depend on x. Then the function $Y(x,\lambda)$ satisfies (1) and (3). Substituting (10) into (2) we obtain a linear homogeneous algebraic system s with respect to $A_j(\lambda)$ and $B_0(\lambda)$. The determinant of this system is $\Delta(\lambda)$. If λ_0 is a zero of $\Delta(\lambda)$, then the function $Y(x,\lambda_0)$ of the form (10) is an eigenfunction, and λ_0

is an eigenvalue. Conversely, if λ_0 is an eigenvalue, then the corresponding eigenfunction has the form (10) with $\lambda = \lambda_0$. Since $Y \neq 0$, the algebraic system s has a nontrivial solution, and consequently, $\Delta(\lambda_0) = 0$. The function $\Delta(\lambda)$ is called the *characteristic function* for the boundary value problem B.

Substituting (6)−(8) into (9) we get

$$\Delta(\lambda) = \Delta^0(\lambda) + \Big(\frac{\sin\rho}{\rho}\Big)^{r-1}\frac{F(\rho)}{\rho^2} + o\Big(\frac{\exp((r+1)|\tau|)}{\rho^{r+1}}\Big), \quad |\lambda| \to \infty, \tag{11}$$

where

$$F(\rho) = \Big(\int_0^1 q_0(x)\,dx + \frac{1}{2}\int_0^1 \sum_{j=1}^r q_j(x)\,dx\Big)\sin^2\rho$$
$$-\Big(\frac{r}{2}\int_0^1 q_0(x)\,dx + \frac{r+1}{2}\int_0^1\sum_{j=1}^r q_j(x)dx\Big)\cos^2\rho + \Big(\int_0^1\sum_{j=1}^r q_j(x)dx\Big)\cos\rho, \tag{12}$$

$$\Delta^0(\lambda) = \Big(\frac{\sin\rho}{\rho}\Big)^r\Big((r+2)\cos\rho - 2\Big). \tag{13}$$

Notice that $\Delta^0(\lambda)$ is the characteristic function for the boundary value problem $B^0 := B(0)$ with the zero potential. It follows from (13) that the boundary value problem B^0 has a countable set of eigenvalues $\Lambda^0 = \{\lambda^0_{ns}\}_{n\geq 0,\ s=\overline{0,r}}$ (counting multiplicities), where $\lambda^0_{ns} = (\rho^0_{ns})^2$,

$$\rho^0_{ns} = (n+1)\pi,\ s = \overline{1,r}, \quad \rho^0_{n0} = n\pi + \Big(\frac{\pi}{2} + (-1)^n\Big(\xi_0 - \frac{\pi}{2}\Big)\Big), \tag{14}$$

$$\xi_0 = \arccos\frac{2}{r+2} \in \Big(0, \frac{\pi}{2}\Big). \tag{15}$$

Using (11) by the well-known method (see, for example, [10, Chap. 1]) we obtain that the boundary value problem B has a countable set of eigenvalues $\Lambda = \{\lambda_{ns}\}_{n\geq 0,\ s=\overline{0,r}}$. All eigenvalues are real and have the asymptotics

$$\rho_{ns} := \sqrt{\lambda_{ns}} = \rho^0_{ns} + O\Big(\frac{1}{n}\Big), \quad s = \overline{0,r}, \quad n \to \infty, \tag{16}$$

where ρ^0_{ns} are calculated via (14) and (15).

Denote $\lambda_n := \lambda_{n0}$, $\rho_n := \rho_{n0}$. Substituting (16) into (11) and using (12) and the relation $\Delta(\lambda_n) = 0$, we obtain the following more precise asymptotical formula

$$\rho_n := \sqrt{\lambda_n} = \rho^0_{n0} + \frac{\xi_1}{\pi n} + o\Big(\frac{1}{n}\Big), \quad n \to \infty, \tag{17}$$

where

$$\xi_1 = \frac{1}{r+4}\int_0^1 q_0(x)\,dx + \frac{r+2}{2r(r+4)}\int_0^1\sum_{j=1}^r q_j(x)\,dx. \tag{18}$$

Using (6) and (17) we get the asymptotics for the components of the eigenfunctions as $n \to \infty$, uniformly in $x \in [0,1]$:

$$\rho_n S_i(x,\lambda_n) = \sin \rho_{n0}^0 x - \frac{1}{2\pi n}\Big(\int_0^x q_i(x)\,dx - 2\xi_1 x\Big)\cos \rho_{n0}^0 x + o\Big(\frac{1}{n}\Big), \tag{19}$$

where ρ_{n0}^0 and ξ_1 are calculated by (14) and (18), respectively.

Fix $i = \overline{1,r}$. There exists N_0 such that for all $n > N_0$ the function $S_i(x,\lambda_n)$ has exactly n (simple) zeros inside the interval $(0,1)$, namely: $0 < x_{ni}^1 < \cdots < x_{ni}^n < 1$. The points $X_i := \{x_{ni}^j\}$ are called *nodal points* on the edge e_i related to the eigenvalues $\{\lambda_n\}$.

Fix $i = \overline{1,r}$. We will consider the inverse nodal problem of recovering the potential $q_i(x)$ on the edge e_i from the given set X_i of nodal points or from a certain of it part. Denote

$$\alpha_n^j := \frac{\pi j}{\rho_{n0}^0} \quad n \ge 1, \quad j = \overline{1,n}. \tag{20}$$

Clearly, if $q = 0$, then $x_{ni}^j = \alpha_n^j$ for all $n \ge 1$, $i = \overline{1,r}$, $j = \overline{1,n}$.

Taking (19) into account, we obtain the following asymptotic formula for the nodal points as $n \to \infty$, uniformly in j:

$$x_{ni}^j = \alpha_n^j + \frac{1}{\pi n^2}\Big(\int_0^{\alpha_n^j} q_i(t)\,dt - 2\xi_1 \alpha_n^j\Big) + o\Big(\frac{1}{n^2}\Big), \tag{21}$$

where the numbers α_n^j are defined by (20). We note that for each fixed $i = \overline{1,r}$, the set X_i is dense on $(0,1)$. Without loss of generality we assume that $\xi_1 = 0$ (this can be achieved by the shift: $q_i(x) \to q_i(x) - 2\xi_1,\ \lambda \to \lambda - 2\xi_1$). Using (21) we arrive at the following assertion.

Theorem 1. *Fix $i = \overline{1,r}$ and $x \in [0,1]$. Let $X_i^0 \subset X_i$ be dense on $(0,1)$. Let $\{x_{ni}^{j_{ni}}\} \in X_i^0$ be chosen such that $\lim\limits_{n\to\infty} x_{ni}^{j_{ni}} = x$. Then there exists a finite limit*

$$g_i(x) := \lim_{n\to\infty} \pi n^2 \Big(x_{ni}^{j_{ni}} - \alpha_n^{j_{ni}}\Big), \tag{22}$$

and

$$g_i(x) = \int_0^x q_i(t)\,dt. \tag{23}$$

Let us now formulate an uniqueness theorem and provide a constructive procedure for the solution of the inverse nodal problem. For this purpose, together with B we consider a boundary value problem $\tilde B = B(\tilde q)$ of the same form but with a different potential $\tilde q$. We agree that if a certain symbol α denotes an object related to B, then $\tilde\alpha$ will denote an analogous object related to $\tilde B$.

Theorem 2. *Fix $i = \overline{1,r}$. Let $X_i^0 \subset X_i$ be a subset of the nodal points which is dense on $(0,1)$. Let $X_i^0 = \tilde X_i^0$. Then $q_i(x) = \tilde q_i(x)$ a.e. on $(0,1)$. Thus, the specification of X_i^0 uniquely determines the potential $q_i(x)$ on the edge e_i. The function $q_i(x)$ can be constructed via the formula*

$$q_i(x) = g_i'(x), \tag{24}$$

where $g_i(x)$ is calculated by (22).

Indeed, formula (24) follows from (23). If $X_i^0 = \tilde{X}_i^0$, then (22) yields $g_i(x) \equiv \tilde{g}_i(x)$, $x \in [0,1]$, and consequently, $q_i(x) = \tilde{q}_i(x)$ a.e. on $(0,1)$.

3. Incomplete inverse problems

First we consider the following incomplete inverse *spectral* problem. Fix i, $i = \overline{1,r}$. Suppose that $q_k(x)$ are known a priori for $k = \overline{0,r} \setminus i$, $x \in (0,1)$. Moreover, suppose that $q_i(x)$ is known on a part of the interval, namely, for $x \in (b,1)$. The inverse problem is to construct $q_i(x)$ for $x \in (0,b)$ from a part of the spectrum of the boundary value problem B. Denote by σ_k the spectrum of the boundary value problem

$$-y_k'' + q_k(x)y_k = \lambda y_k, \quad y_k(0) = y_k(1) = 0.$$

Theorem 3. *Fix i, $i = \overline{1,r}$ and $b \in (0,1/2)$. Let $q_k(x) = \tilde{q}_k(x)$ for $k = \overline{0,r} \setminus i$, a.e. on $(0,1)$, and $q_i(x) = \tilde{q}_i(x)$ a.e. on $(b,1)$. Let $M \subset \mathbf{N} \cup \{0\}$ be a subset of the nonnegative integer numbers, and let $\Omega := \{\lambda_n\}_{n \in M}$ be a part of the spectrum of B such that $\sigma_k \cap \Omega = \emptyset$, $k = \overline{0,r} \setminus i$, and the system of functions $\{\cos 2\rho_n x\}_{n \in M}$ is complete in $L_2(0,b)$. If $\Omega = \tilde{\Omega}$, then $q_i(x) = \tilde{q}_i(x)$ a.e. on $(0,1)$.*

Proof. Since $\sigma_k \cap \Omega = \emptyset$, $k = \overline{0,r} \setminus i$, one has

$$S_k(1,\lambda_n) \neq 0, \quad k = \overline{0,r} \setminus i, \quad \lambda_n \in \Omega. \tag{25}$$

Let $Y_n(x) = \{y_{jn}(x)\}_{j=\overline{0,r}}$ be an eigenfunction related to the eigenvalue λ_n. Then

$$y_{jn}(x) = A_{jn} S_j(x,\lambda_n),\ j = \overline{1,r}, \quad y_{0n}(x) = A_{0n} S_0(x,\lambda_n) + B_{0n} C_0(x,\lambda_n), \tag{26}$$

where A_{jn} and B_{0n} are constants. Since Y_n is not identically zero and satisfies the matching conditions (2), one has $A_{jn} \neq 0$ for all $j = \overline{1,r}$. Therefore, in view of (25), $y_{jn}(1) \neq 0$ for all $j = \overline{0,r}$, and $y_{0n}(0) \neq 0$. Using the matching conditions (2) again we obtain

$$\frac{y_{0n}'(0)}{y_{0n}(0)} = \frac{y_{0n}'(1)}{y_{0n}(1)} + \sum_{j=1}^{r} \frac{S_j'(1,\lambda_n)}{S_j(1,\lambda_n)}. \tag{27}$$

Without loss of generality we put $y_{0n}(0) = 1$. Then, in view of (2), $y_{jn}(1) = 1$, $j = \overline{0,r}$. It follows from (26) that

$$B_{0n} = 1, \quad A_{0n} = \frac{1 - C_0(1,\lambda_n)}{S_0(1,\lambda_n)}, \quad A_{jn} = \frac{1}{S_j(1,\lambda_n)},\ j = \overline{1,r}.$$

Since $q_k = \tilde{q}_k$ for $k = \overline{0,r} \setminus i$, one has

$$S_k(x,\lambda) \equiv \tilde{S}_k(x,\lambda), \quad C_k(x,\lambda) \equiv \tilde{C}_k(x,\lambda), \quad k = \overline{0,r} \setminus i, \quad x \in [0,1],$$

and consequently,

$$A_{jn} = \tilde{A}_{jn}, \quad j = \overline{0,r} \setminus i.$$

Together with (26) this yields
$$y_{0n}(x) \equiv \tilde{y}_{0n}(x).$$
Taking (27) into account we infer
$$\frac{S_i'(1,\lambda_n)}{S_i(1,\lambda_n)} = \frac{\tilde{S}_i'(1,\lambda_n)}{\tilde{S}_i(1,\lambda_n)}, \quad \lambda_n = \tilde{\lambda}_n \in \Omega. \tag{28}$$
Since
$$-S_i''(x,\lambda) + q_i(x)S_i(x,\lambda) = \lambda S_i(x,\lambda), \quad -\tilde{S}_i''(x,\lambda) + \tilde{q}_i(x)\tilde{S}_i(x,\lambda) = \lambda\tilde{S}_i(x,\lambda),$$
$$S_i(0,\lambda) = \tilde{S}_i(0,\lambda) = 0, \quad S_i'(0,\lambda) = \tilde{S}_i'(0,\lambda) = 1,$$
it follows that
$$\int_0^1 Q_i(x)S_i(x,\lambda)\tilde{S}_i(x,\lambda)\,dx \equiv S_i'(1,\lambda)\tilde{S}_i(1,\lambda) - S_i(1,\lambda)\tilde{S}_i'(1,\lambda), \tag{29}$$
where $Q_i(x) = q_i(x) - \tilde{q}_i(x)$. Using (28) we calculate
$$\int_0^b Q_i(x)S_i(x,\lambda_n)\tilde{S}_i(x,\tilde{\lambda}_n)\,dx = 0, \quad \lambda_n = \tilde{\lambda}_n \in \Omega. \tag{30}$$
It is known (see [7, 8, 10]) that the following representation holds
$$S_i(x,\lambda) = \frac{\sin\rho x}{\rho} + \int_0^x K_i(x,t)\frac{\sin\rho t}{\rho}\,dt, \tag{31}$$
where $K_i(x,t)$ is a smooth function which does not depend on λ. By virtue of (31) one gets
$$2\rho^2 S_i(x,\lambda)\tilde{S}_i(x,\lambda) = 1 - \cos 2\rho x - \int_0^x V_i(x,t)\cos 2\rho t\,dt, \tag{32}$$
where $V_i(x,t)$ is a continuous function which does not depend on λ. Substituting (32) into (30) and taking the relation $\int_0^b Q_i(x)\,dx = 0$ into account, we calculate
$$\int_0^b \Big(Q_i(x) + \int_x^b V_i(t,x)Q_i(t)\,dt\Big)\cos 2\rho_n x\,dx = 0, \quad \lambda_n \in \Omega,$$
and consequently,
$$Q_i(x) + \int_x^b V_i(t,x)Q_i(t)\,dt = 0 \quad \text{a.e. on} \quad (0,b).$$
Since this homogeneous integral equation has only the trivial solution it follows that $Q_i(x) = 0$ a.e. on $(0,b)$, i.e. $q_i(x) = \tilde{q}_i(x)$ a.e. on $(0,b)$.

Let us go on to the investigation of an *incomplete inverse nodal problem* when nodal points are given only on a part of the interval. First we recall for convenience of the reader an auxiliary assertion (see [5], [6]) including a short proof.

Lemma 1. *Fix* n, j, i. *Let* $x_{ni}^j = \tilde{x}_{ni}^j$, $x_{ni}^{j+1} = \tilde{x}_{ni}^{j+1}$, *and let* $q_i(x) = \tilde{q}_i(x)$ *a.e. on* (x_{ni}^j, x_{ni}^{j+1}). *Then* $\lambda_n = \tilde{\lambda}_n$.

Proof. On the interval $x \in (x_{ni}^j, x_{ni}^{j+1})$ we consider the boundary value problem B_{ni}^j for equation (1) with the boundary conditions

$$y(x_{ni}^j) = y(x_{ni}^{j+1}) = 0.$$

The function $y_{ni}(x) = S_i(x, \lambda_n)$ is the eigenfunction of B_{ni}^j related to the eigenvalue λ_n. Since $y_{ni}(x)$ has no zeros for $x \in (x_{ni}^j, x_{ni}^{j+1})$, it follows from Sturm's oscillation theorem that λ_n is the first eigenvalue of B_{ni}^j, and $y_{ni}(x)$ is the first eigenfunction. Since $q_i(x) = \tilde{q}_i(x)$ a.e. on (x_{ni}^j, x_{ni}^{j+1}), one has $\lambda_n = \tilde{\lambda}_n$.

For $X \subset X_i$ we denote $\Lambda_X := \{n : \exists j \ x_{ni}^j \in X\}$.

Definition 1. Let $X \subset X_i$. The set X is called *twin* if together with each of its points x_{ni}^j the set X contains at least one of the adjacent nodal points x_{ni}^{j-1} or x_{ni}^{j+1}.

Theorem 4. *Fix* $i = \overline{1, r}$ *and* $b \in (0, 1/2)$. *Let* $q_k(x) = \tilde{q}_k(x)$ *for* $k = \overline{1, r} \setminus i$, *a.e. on* $(0, 1)$. *Let* $X \subset X_i \cap (b, 1)$ *be a dense on* $(b, 1)$ *twin subset of nodal points such that* $\sigma_k \cap \Omega_{\Lambda_X} = \emptyset$, $k = \overline{1, r} \setminus i$, *and the system of functions* $\{\cos 2\rho_n x\}_{n \in \Lambda_X}$ *is complete in* $L_2(0, b)$. *If* $X = \tilde{X}$, *then* $q_i(x) = \tilde{q}_i(x)$ *a.e. on* $(0, 1)$.

Thus, the specification of nodal point on a part of the interval uniquely determines $q_i(x)$ on the whole interval $(0, 1)$.

Proof. Since $X = \tilde{X}$, it follows that $g_i(x) \equiv \tilde{g}_i(x)$ for $x \in (b, 1)$. Using (23) we obtain $q_i(x) = \tilde{q}_i(x)$ a.e. on $(b, 1)$. By Lemma 1, $\lambda_n = \tilde{\lambda}_n$ for $n \in \Lambda_X$. Applying Theorem 3 we conclude that $q_i(x) = \tilde{q}_i(x)$ a.e. on $(0, 1)$.

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