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HOMOLOGY GROUP ON MANIFOLDS AND THEIR FOLDINGS

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Abstract. In this paper, we introduce the definition of the induced unfolding on the homology group. Some types of conditional foldings restricted on the elements of the homology groups are deduced. The effect of retraction on the homology group of a manifold is dicussed. The unfolding of variation curvature of manifolds on their homology group are represented. The relations between homology group of the manifold and its folding are deduced.

1. Introduction

The folding of a manifold introduced by S. A. Robertson 1977 [13]. More studies of the folding of manifolds are studied in [4, 5, 7, 11, 14, 15]. Various folding problems arising in the physics of membrane and polymers reviewed by Francesco [2]. The unfolding of a manifold introduced in [3]. The retraction of a manifold defined and discussed in [6, 12]. The homology groups of some types of a manifold are discussed in [1, 8, 9, 10, 16].

2. Definitions and background

In this section, we give the definitions which are needed especially in this paper.

Definition 1. Consider the sequence $C_{p+1}(k) \xrightarrow{\partial_{p+1}} C_p(k) \xrightarrow{\partial_p} C_{p-1}(k)$, ker ∂_p is denoted by $Z_p(K)$, and the elements of $Z_p(K)$ are called p-cycles, $Im\partial_{p+1}$ is denoted by $B_p(K)$, and the elements of $B_p(K)$ are called p-boundaries. The quotient group $H_p = Z_p(K)/B_p(K)$ is called the P^{th} homology group of K [8].

Definition 2. Let M and N be two manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$, the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ [13]. If f does not preserve length, it is called a topological folding [11].

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Definition 3. Let M and N be two manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M into N if, for every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ [3].

Definition 4. A subset A of a topological space X is called a retract of X if there exists a continuous map $r : X \to A$ (called a retraction) such that $r(a) = a\ \forall a \in A$ [12].

Definition 5. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained by identifying x_0 and y_0 to a single point [9].

3. The main results

Aiming to our study, we will introduce the following:

Definition 6. Let M and $\acute{M}$ be two manifolds of the same dimensions and $unf : M \to \acute{M}$ *be* any unfoding of M into$\acute{M}$. Then, a map $\overline{unf} : H_n(M) \to H_n(\acute{M})$ is said to be an induced unfolding of $H_n(M)$ into $H_n(\acute{M})$ if $\overline{unf}(H_n(M)) = H_n(unf(M))$.

Lemma 1. *Let C_1, C_2 be two disjoint circles in R^2. Then there is unfolding*
$unf : C_1 \cup C_2 \to \acute{C}_1 \cup \acute{C}_2$ *which induces unfolding*
$\overline{unf} : H_n(C_1 \cup C_2) \to H_n(\acute{C}_1 \cup \acute{C}_2)$ *such that*
(1) $\overline{unf}\ (H_0(C_1 \cup C_2)) \approx Z$
(2) $\overline{unf}\ (H_1(C_1 \cup C_2)) \approx Z \oplus Z$
(3) $\overline{unf}\ (H_nC_1 \cup C_2)) \approx 0$, *for* $n \geq 2$.

Proof. Let $unf : C_1 \cup C_2 \to \acute{C}_1 \cup \acute{C}_2$ be unfolding such that
$unf(C_1 \cup C_2) = unf(C_1) \vee unf(C_2)$ as in Figure 1, thus we get the induced unfolding $\overline{unf} : H_n(C_1 \cup C_2) \to H_n(\acute{C}_1 \cup \acute{C}_2)$ such that $H_n(unf(C_1 \cup C_2)) = H_n(unf(C_1) \vee unf(C_2))$.

Now, for $n = 0$, $\overline{unf}\ (H_0(C_1 \cup C_2)) = H_0(unf(C_1 \cup C_2)) \approx Z$. Also, if $n = 1$, $\overline{unf}\ (H_1(C_1 \cup C_2)) = H_1(unf(C_1 \cup C_2)) \approx H_1(unf(C_1)) \oplus H_1(unf(C_2))$. Since $H_1(unf(C_i)) \approx Z$, $i = 0, 1$ it follows that $\overline{unf}\ (H_1(C_1 \cup C_2)) \approx Z \oplus Z$.

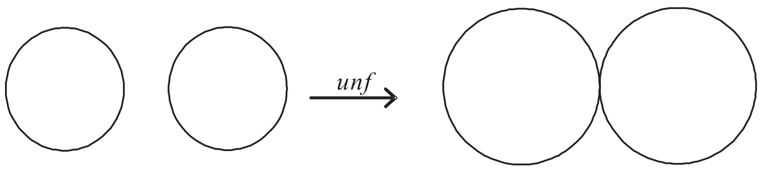

Figure 1:

Moreover, if $n \geq 2$, it follows from$H_n(unf(C_i)) \approx 0$, $i = 0, 1$and from $H_n(unf(C_1 \cup C_2)) \approx H_n(unf(C_1)) \oplus H_n(unf(C_2))$ that $\overline{unf}\ (H_n(C_1 \cup C_2)) = H_n(unf(C_1 \cup C_2)) \approx 0$, for $n \geq 2$.

Lemma 2. *Let I be the closed interval $[0, 1]$ and let $\bar{I}$ be the closed interval $[-1, 0]$. Then there are unfoldings $unf_m : I \cup \bar{I} \to J \cup \overline{J}$ $m = 1, 2, \ldots$ with variation curvature which induces unfoldings $\overline{unf_m} : H_n(I \cup \bar{I}) \to H_n(J \cup \overline{J})$, $m = 1, 2, \ldots$ such that*

(1) $\lim\limits_{m\to\infty}(\overline{unf_m}(H_0(I \cup \bar{I}))) \approx Z$

(2) $\lim\limits_{m\to\infty}(\overline{unf_m}(H_1(I \cup \bar{I}))) \approx Z \oplus Z$

(3) $\lim\limits_{m\to\infty}(\overline{unf_m}(H_n(I \cup \bar{I})))\ \approx 0$, *for $n \geq 2$.*

Proof. Consider the sequence of unfoldings with variation curvature such that $unf_1 : I \cup \bar{I} \to X_1, unf_2 : X_1 \to X_2, \ldots, unf_m : X_{m-1} \to X_m$ such that $\lim\limits_{m\to\infty}(unf_m(I \cup \bar{I})) = S_1^1 \vee S_2^1$ as in Figure 2, thus we get the induced unfoldings $\overline{unf_m} : H_n(I \cup \bar{I}) \to H_n(J \cup \overline{J}\ ), m = 1, 2, \ldots$ such that $\lim\limits_{m\to\infty}(\overline{unf_m}(H_n(I \cup \bar{I}))) = H_n(\lim\limits_{m\to\infty}(unf_m(I \cup \bar{I})))$. Now, for $n = 0$, $\lim\limits_{m\to\infty}(\overline{unf_m}(H_0(I \cup \bar{I}))) = H_0(\lim\limits_{m\to\infty}(unf_m(I \cup \bar{I}))) = H_0(S_1^1 \vee S_2^1) \approx Z$. Also, if $n = 1$, it follows from $H_1(S_1^1 \vee S_2^1) \approx H_1(S_1^1) \oplus H_1(S_2^1)$, and from $H_1(S_i) \approx Z, i = 1, 2$ that $\lim\limits_{m\to\infty}(\overline{unf_m}(H_1(I \cup \bar{I}))) = H_1(\lim\limits_{m\to\infty}(unf_m(I \cup \bar{I}))) \approx Z \oplus Z$. Moreover, if $n \geq 2$, it follows from $H_n(S_1^1 \vee S_2^1) \approx H_n(S_1^1) \oplus H_n(S_2^1)$, and from $H_n(S_i) \approx 0$, $i = 1, 2$, that $\lim\limits_{m\to\infty}(\overline{unf_m}(H_n(I \cup \bar{I}))) = H_n(\lim\limits_{m\to\infty}(unf_m(I \cup \bar{I})))\ \approx 0$. Therefore we get the induced unfoldings $\overline{unf_m} : H_n(I \cup \bar{I}) \to H_n(J \cup \overline{J}\ ), m = 1, 2, \ldots$ such that

(1) $\lim\limits_{n\to\infty}(\overline{unf_m}(H_0(I \cup \bar{I}))) \approx Z$

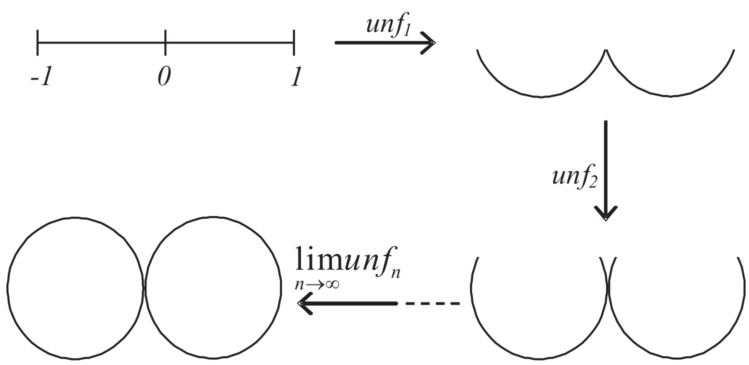

Figure 2:

(2) $\lim_{n\to\infty}(\overline{unf_m}(H_1(I\cup\bar{I})))\approx Z\oplus Z$

(3) $\lim_{n\to\infty}(\overline{unf_m}(H_n(I\cup\bar{I})))\ \approx 0$, for $n\geq 2$.

Theorem 1. *Let C be the circle of radius 1 and centre $(-1,0)$. Then there are infinite number of unfoldings $unf_m : C\vee\overline{C}\to C_1\vee\overline{C}_1, m=1,2,\ldots$ which induces unfoldings $\overline{unf}_m : H_n(C)\oplus H_n(\overline{C})\to H_n(C_1)\oplus H_n(\overline{C}_1), n\geq 1$ such that*

$\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=H_n(unf_m(C))\oplus H_n(unf_m(\overline{C}))$ *or*

$\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=H_n(unf_m(C))\oplus H_n(\overline{C})$ *or*

$\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=H_n(C)\oplus H_n(unf_m(\overline{C}))$.

Proof. Let $unf_m : C\vee\overline{C}\to C_1\vee\overline{C}_1, m=1,2,\ldots$ are unfoldings which are preserving curvature or not preserving curvature such that $unf_m(C\vee\overline{C})=unf_m(C)\vee unf_m(\overline{C})$ where $unf_m(C)$ is the circle of radius m and center $(m,0)$ and $unf_m(\overline{C})$ is the circle of radius m and center $(-m,0)$ as in Figure 3. So we have an induced unfolding $\overline{unf}_m : H_n(C)\oplus H_n(\overline{C})\to H_n(C_1)\oplus H_n(\overline{C}_1), n\geq 1$ such that $\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=\overline{unf}_m(H_n(C))\oplus \overline{unf}_m H_n(\overline{C}))$,thus $\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=H_n(unf_m(C))\oplus H_n(unf_m(\overline{C}))$. Similarly, we can get the induced unfolding $\overline{unf}_m : H_n(C)\oplus H_n(\overline{C})\to H_n(C_1)\oplus H_n(\overline{C}_1)$ such that

$\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=H_n(unf_m(C))\oplus H_n(\overline{C})$ or

$\overline{unf}_m(H_n(C)\oplus H_n(\overline{C}))=H_n(C)\oplus H_n(unf_m(\overline{C}))$.

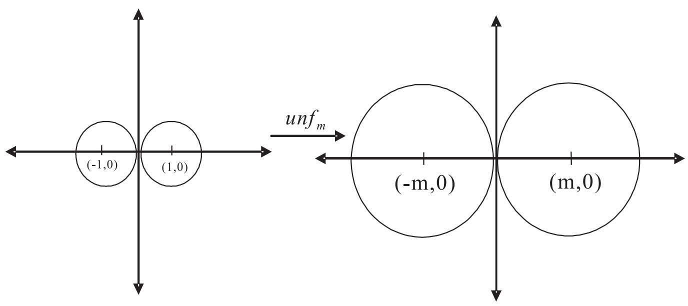

Figure 3:

Theorem 2. *Let X be the subspace of R^2 that is the union of the circles C_m of radii $\frac{1}{m}$ and centres $(\frac{1}{m},0)$ for $m=1,2,\ldots$, then there is unfolding $unf : X\to\overline{X}$ which induced unfolding $\overline{unf} : H_n(X)\to H_n(\overline{X})$ such that*

(1) $\overline{unf}(H_0(X))\approx Z$

(2) $\overline{unf}(H_1(X))$ *is a free abelian group on a countable set of generators.*

(3) $\overline{unf}(H_n(X)=0$, *for* $n\geq 2$.

Proof. Let $unf : X\to\overline{X}$ be unfolding such that $unf(X)=unf(\bigcup_{m=1}^{\infty}C_m)=\bigcup_{m=1}^{\infty}unf(C_m)$ and $unf(C_m)$ is the circles of radius m and centre $(m,0)$ as in Figure **??**. Then we get the induced unfolding $\overline{unf} : H_n(X)\to H_n(\overline{X})$ such that $\overline{unf}(H_n(X))=H_n(unf(X))$.

Thus for $n = 0$, $\overline{unf}(H_0(X)) = H_0(unf(X)) \approx Z$.

Now, for $n = 1$ we want to show that, $H_1(unf(X))$ is a free group on a countable set of generators. It follows from $unf(X) = \bigcup_{m=1}^{\infty} unf(C_m)$ that $unf(X)$ is the union of the circles of radius m and center $(m, 0)$, since $unf(X)$ and $\bigvee_{\infty} S^1$ are homotopy equivalent, we have $H_1(unf(X)) \approx H_1(\bigvee_{\infty} S^1)$ and so $H_1(unf(X)) \approx H_1(Z) \oplus H_1(Z) \oplus \cdots \infty)$. Therefore, $\overline{unf}_m(H_1(X))$ is a free abelian group on a countable set of generators. Moreover, for $n \geq 2$. Clearly we can get $\overline{unf}(H_n(X) = 0$.

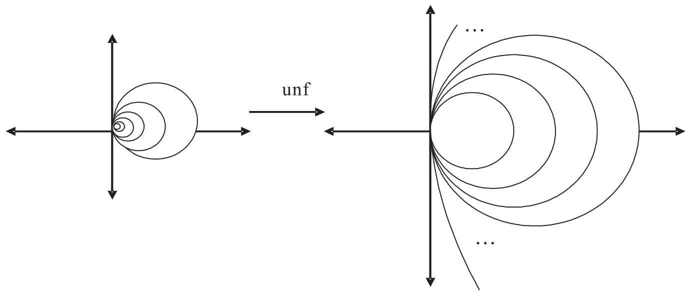

Figure 4:

Theorem 3. *Let X be the subspace of R^2 that is the union of the circles C_m of radius m and center $(m, 0)$ for $m = 1, 2, \ldots$. Then there is a folding $F : X \to \overline{X}$ without singularity, which induces a folding $\overline{F} : H_n(X) \to H_n(\overline{X})$ such that*

(1) $\overline{F}(H_0(X)) \approx Z$.

(2) $\overline{F}(H_1(X))$ *is uncountable.*

(3) $\overline{F}(H_n(X)) = 0$, *for* $n \geq 2$.

Proof. Let $F : X \to \overline{X}$ be a folding without singularity such that $F(X) = F(\bigcup_{m=1}^{\infty} C_m) = \bigcup_{m=1}^{\infty} F(C_m)$ and $F(C_m)$ is the circle of radius $\frac{1}{m}$ and centre $(\frac{1}{m}, 0)$ as in Figure 5.

Then the induced folding $\overline{F} : H_n(X) \to H_n(\overline{X})$ satisfies $\overline{F}(H_n(X)) = H_n(F(\overline{X}))$. Thus for $n = 0$,$\overline{F}(H_0(X)) = H_0(F(\overline{X})) \approx Z$. Now, for $n = 1$ we want to shows that $H_1(F(X))$ is uncountable. Consider the retraction $r_m : F(X) \to F(C_m)$ which collapsing all $F(C_i)$ except $F(C_m)$ to origin. Each r_m induces a surjection. $\overline{r}_m : H_1(F(X)) \to H_1(F(C_m)) \approx Z$, where the origin is a base point. Then the product of the $\overline{r}_m$ is a homomorphism $r : H_1(F(X)) \to \prod_{\infty} Z$ to the direct product of infinite number of copies of Z and clearly, we can prove that r is onto, it follows from $\prod_{\infty} Z$ uncountable that $H_1(F(X))$ is uncountable. Since $\overline{F}(H_1(X)) = H_1(F(X))$ it follows that $\overline{F}(H_1(X))$ is uncountable. Moreover, for $n \geq 2$.Clearly we can get $\overline{F}(H_n(X)) = 0$, for $n \geq 2$.

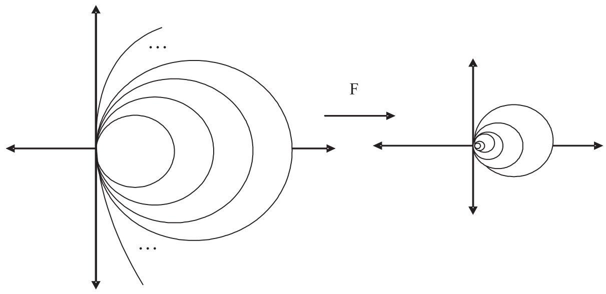

Figure 5:

Theorem 4. *Let X be the subspace of R^2 that is the union of the circles C_m of radius $\frac{1}{m}$ and centre $(\frac{1}{m}, 0)$ for $m = 1, 2, \ldots$. And $F : X \to X$ is a folding such that $F(C_m) \neq C_m$ $\forall m$. Then there are unfoldings $unf : F(X) \subset X \to X$ such that $H_n(\lim_{m\to\infty}(unf_m(F(X)))$ is uncountable.*

Proof. Let $F : X \to X$ be a folding such that $F(C_m) \neq C_m$ $\forall m$, i.e. folding by cut. Then, we can define a sequence of unfoldings $unf_1 : F(X) \to X_1, F(X) \subseteq X_1 \subseteq X, unf_2 : X_1 \to X_2, X_1 \subseteq X_2 \subseteq X, \ldots, unf_m : X_{m-1} \to X_m, X_{m-1} \subseteq X_m \subseteq X$, and so $\lim_{m\to\infty}(unf_m(F(X)) = X$, as in Figure 6, thus $H_n(\lim_{m\to\infty}(unf_m(F(X)))$ is uncountable.

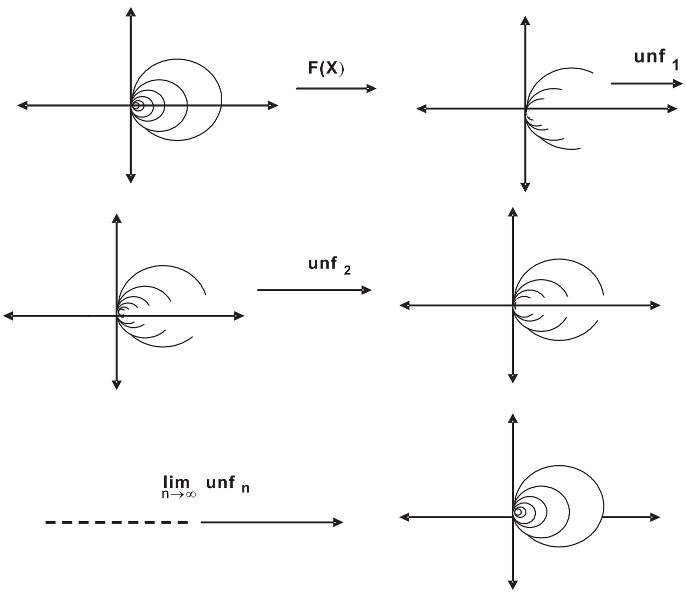

Figure 6:

Theorem 5. *Let $X \subset R^3$ be the union of the circles C_m of radius $\frac{1}{m}$ and centred at $(\frac{1}{m}, 0)$ for $m = 1, 2, \ldots$. Then there are foldings $F_m : X \to X$ and retractions $r_m : X \to C_m$ such that $H_n(F_m(X)) = H_n(r_m(X))$ and $H_n(F_m(X))$ is either 0 or isomorphic to Z.*

Proof. Let $F_m : X \to X$ be a folding such that $F_m(C_m) = C_m \ \forall m = 1, 2, \ldots$ then $F_m(X) = C_m$ as in Figure **??** and so $H_n(F_m(X)) = H_n(C_m)$. Also, consider the retractions $r_m : X \to C_m$, which collapsing all C_i except C_m to the origin and so $r_m(X) = C_m$, thus $H_n(r_m(X)) = H_n(C_m)$. Therefore $H_n(F_m(X)) = H_n(r_m(X))$. Now, if $n = 0, 1$ we have $H_n(F_m(X)) \approx Z$.

Also, if $n \geq 2$. we get $H_n(F_m(X)) \approx 0$.

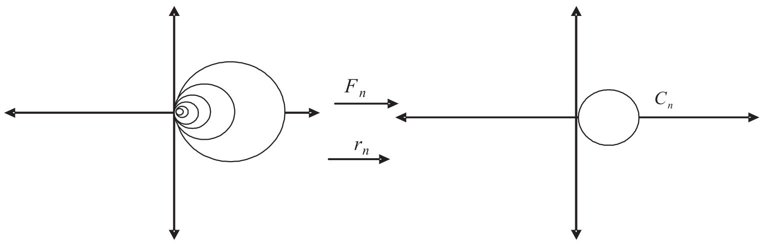

Figure 7:

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