PARTIAL SUMS FOR CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we define and study new classes of meromorphic functions in the punctured disk by using their partial sums.

1. Introduction

Let \( \Sigma_\alpha \) denote the class of functions of the form
\[
f(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{\infty} a_n z^{n+\alpha}, \quad (0 \leq \alpha < 1),
\]
which are analytic in the punctured unit disk \( U := \{ z \in \mathbb{C}, 0 < |z| < 1 \} \).

A function \( f \in \Sigma_\alpha \) belongs to the class \( S_\alpha(A, B) \), the class of meromorphically \( \alpha \)-valent starlike functions if and only if \( f \neq 0 \), and for \( -1 \leq A < B \leq 1 \),
\[
- \left\{ \frac{zf'(z)}{f(z)} \right\} \prec 1 + Az \frac{1}{1 + Bz}, \quad (z \in U).
\]

A function \( f \in \Sigma_\alpha \) belongs to the class \( C_\alpha(A, B) \), the class of meromorphically \( \alpha \)-valent convex functions if and only if \( f' \neq 0 \), and
\[
- \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \prec 1 + Az \frac{1}{1 + Bz}, \quad (z \in U).
\]

The class \( \Sigma_0 \equiv \Sigma \), was studied by many authors (see [1, 2, 3, 4, 5]). Note that the authors defined and studied the class \( \Sigma_\alpha \) for normalized analytic functions in an open disk (see [6, 7]).

In the present paper, we are motivated with the work done by Silverman [8], and will investigate in similar manner the ratio of a function of the form (1) to its sequence of partial sums
\[
f_k(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{k} a_n z^{n+\alpha}, \quad (0 \leq \alpha < 1),
\]
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when the coefficients are sufficiently small. More precisely, we will determine sharp lower bounds for
\[ \Re \left\{ \frac{f(z)}{f_k(z)} \right\}, \Re \left\{ \frac{f_k(z)}{f(z)} \right\}, \Re \left\{ \frac{f'(z)}{f(z)} \right\}, \text{ and } \Re \left\{ \frac{f'(z)}{f'(z)} \right\}. \]

2. Preliminary results

First we prove sufficient conditions for \( f \in \Sigma_\alpha \) to be in the classes \( S_\alpha(A, B) \) and \( C_\alpha(A, B) \).

**Theorem 2.1.** Let \( f \in \Sigma_\alpha \). If
\[ \sum_{n=1}^{\infty} (n+\alpha)(1+\alpha) + (A+1) |a_n| \leq (B-A) - \alpha(1-B), \quad (z \in U) \] (3)
holds and \( B(1+\alpha) > A + \alpha \), then \( f \in S_\alpha(A, B) \).

**Proof.** Assume that \( f \in \Sigma_\alpha \) and satisfies (3). It is sufficient to show that
\[ \left| \frac{1 + zf'(z)}{f(z) + Af(z) + Bzf'(z)} \right| < 1, \]
that is
\[ \left| \frac{f(z) + zf'(z)}{Af(z) + Bzf'(z)} \right| < 1. \]

Consider
\[ \left| \frac{f(z) + zf'(z)}{Af(z) + Bzf'(z)} \right| = \left| \frac{-\frac{d}{dz} + \sum_{n=1}^{\infty} a_n(n+\alpha + 1)z^{n+\alpha}}{A + \sum_{n=1}^{\infty} Aa_nz^{n+\alpha} - \frac{B(1+\alpha)}{\alpha} + \sum_{n=1}^{\infty} (n+\alpha)Ba_nz^{n+\alpha}} \right| \] (4)
Hence (4) is bounded by 1, if
\[ \alpha + \sum_{n=1}^{\infty} (n+\alpha + 1)|a_n| \leq [B(1+\alpha) - A] - \sum_{n=1}^{\infty} [(n+\alpha)B + A]|a_n| \]
\[ \Rightarrow \sum_{n=1}^{\infty} (n+\alpha + 1)|a_n| + \sum_{n=1}^{\infty} [(n+\alpha)B + A]|a_n| \leq [B(1+\alpha) - A] - \alpha \]
\[ \Rightarrow \sum_{n=1}^{\infty} [(n+\alpha)(1+B) + (1+A)]|a_n| \leq (B - A) - \alpha(1 - B), \]
where \( B(1+\alpha) > A + \alpha \). This completes the proof.

In a similar manner, we can prove the following result.
Theorem 2.2. Let $f \in \Sigma_{\alpha}$. If

$$
\sum_{n=1}^{\infty} (n+\alpha)(n+\alpha)(1+B)+(A+1)|a_n| \leq (1+\alpha)(B-A)-\alpha(1-B), \quad (z \in U) \quad (5)
$$

holds and $B(1+\alpha) > A + \alpha$, then $f \in C_{\alpha}(A, B)$.

Note that when $\alpha = 0$, Theorem 2.1 and Theorem 2.2 reduce to Theorem 2.2 and Theorem 2.1 in [5] respectively. Further, we note that these sufficient conditions are also necessary for functions of the form (1) when $\alpha = 0, A = 2\mu - 1, B = 1$ with positive or negative coefficients ([1, 2, 3]).

3. Main results

We consider in this section partial sums of functions in the classes $S_{\alpha}(A, B)$ and $C_{\alpha}(A, B)$ and obtain the sharp lower bounds for the ratio of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$. In the sequel, we will make use of the generalized result such that $\Re \{((1+w_{\alpha}(z))/(1-w_{\alpha}(z))) > 0, \quad (z \in U)$ if and only if $w_{\alpha}(z) = \sum_{n=1}^{\infty} c_n z^{n+\alpha}$ satisfies the inequality $|w_{\alpha}(z)| < |z|$.

Theorem 3.1. Let $f$ be given by (1) and satisfies (3) then

$$
\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{2(1+k+\alpha+A)}{2(k+\alpha)+(2+A+B)}, \quad (z \in U).
$$

The result is sharp for every $k$ with extremal function

$$
f(z) = \frac{1}{z^{k+\alpha}} + \frac{(B-A)-\alpha(1-B)}{2(k+\alpha)+(2+A+B)} z^{k+1+\alpha}, \quad k \geq 0.
$$

Proof. Assume that $f \in \Sigma_{\alpha}$ and satisfies (3). Consider

$$
\frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \left\{ \frac{f(z)}{f_k(z)} = \frac{2(1+k+\alpha+A)}{2(k+\alpha)+(2+A+B)} \right\}
$$

$$
= 1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
$$

$$
\frac{1}{1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1}}
$$

$$
:= \frac{1 + w_{\alpha}(z)}{1 - w_{\alpha}(z)}
$$

where

$$
w_{\alpha}(z) = \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
$$

$$
2 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
$$
and

$$|w_\alpha(z)| \leq \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} |a_n|.$$ 

Now $|w_\alpha(z)| \leq 1$ if and only if

$$2 \left[ \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \right] \sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=1}^{k} |a_n|$$

which is equivalent to

$$\sum_{n=1}^{k} |a_n| + \left[ \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)} \right] \sum_{n=k+1}^{\infty} |a_n| \leq 1. \tag{8}$$

It is sufficient to show that the left hand side of (8) is bounded above by

$$\sum_{n=1}^{\infty} \left[ \frac{2(n+\alpha)+A+B}{(B-A)-\alpha(1-B)} \right] |a_n|$$

which is equivalent

$$\sum_{n=1}^{k} \left[ \frac{2(n+\alpha)+A+B}{(B-A)-\alpha(1-B)} \right] |a_n| + \sum_{n=k+1}^{\infty} \left[ \frac{2(n-k-1+\alpha)}{(B-A)-\alpha(1-B)} \right] |a_n| \geq 0.$$ 

To show that the function $f$ given by (7) gives the sharp result, we observe that for $z = re^{\pi i k + 2+2\alpha}$

$$\frac{f(z)}{f_k(z)} = 1 + \frac{(B-A)-\alpha(1-B)}{2(k+\alpha)+(2+A+B)} r^{k+2+2\alpha}$$

$$\rightarrow 1 - \frac{(B-A)-\alpha(1-B)}{2(k+\alpha)+(2+A+B)}$$

$$= \frac{2(k+\alpha)+(2+A+B)-(B-A)+\alpha(1-B)}{2(k+\alpha)+(2+A+B)}$$

$$= \frac{2(1+k+A+\alpha)}{2(k+\alpha)+(2+A+B)}$$

when $r \rightarrow 1^-$. Therefore we complete the proof of Theorem 3.1.

Next result can be found in [5].

**Corollary 3.1.** Let $f$ be given by (1) and satisfies (3) then

$$\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{2(1+k+A)}{2k+2+A+B}, \quad (z \in U). \tag{9}$$
The result is sharp for every $k$ with extremal function

$$f(z) = \frac{1}{z} + \frac{(B - A)}{2k + 2 + A + B}z^{k+1}, \; k \geq 0.$$  \hspace{1cm} (10)

**Proof.** Assume that $\alpha = 0$.

Moreover, the following result can be found in [8].

**Corollary 3.2.** Let the assumptions of Theorem 3.1 hold. Then for $f$ of the form (1) satisfies condition

$$\sum_{n=1}^{\infty} (n + \mu)|a_n| \leq 1 - \mu, \; (z \in U),$$

$$\Re\left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{k + 2\mu}{k + 1 + \mu}, \; (z \in U).$$  \hspace{1cm} (11)

The result is sharp for every $k$ with extremal function

$$f(z) = \frac{1}{z} + \frac{1 - \mu}{k + 1 + \mu}z^{k+1}, \; k \geq 0.$$  \hspace{1cm} (12)

**Proof.** Assume that $\alpha = 0$, $A = 2\mu - 1$, $B = 1$.

**Theorem 3.2.** Let $f \in \Sigma_\alpha$ and

$$\sum_{n=1}^{\infty} (n + \alpha)[(n + \alpha)(1 + B) + (A + 1)]|a_n| \leq (1 + \alpha)[(B - A) - \alpha(1 - B)], \; (z \in U)$$

holds, then

$$\Re\left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k+2)(2k+A+B)+\alpha[(2k+2+\alpha)(1+B)+2\alpha(1-B)-B+2A+1]}{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}.$$  \hspace{1cm} (13)

The result is sharp for every $k$ with extremal function

$$f(z) = \frac{1}{z^{1+\alpha}} + \left[ \frac{(1 + \alpha)[(B - A) - \alpha(1 - B)]}{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]} \right]z^{k+\alpha}, \; k \geq 0.$$  \hspace{1cm} (14)
Proof. Let $f \in \Sigma_\alpha$. Then we obtain

\[
\begin{aligned}
\frac{(k + 1)(2k + 2 + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)]}{(1 + \alpha)(B - A) - \alpha(1 - B)}
\times
\left[ f(z) - \frac{(k + 2)(2k + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + 2\alpha(1 - B) - B + 2A + 1]}{(k + 1)(2k + 2 + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)]} \right]
\end{aligned}
\]

\[
\frac{1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + (k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1}}
\]

\[
:= \frac{1 + w_\alpha(z)}{1 - w_\alpha(z)},
\]

where

\[
w_\alpha(z) = \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)(B-A) - \alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]

\[
2 + 2 \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)(B-A) - \alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]

and

\[
|w_\alpha(z)| \leq \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{2 - 2 \sum_{n=1}^{k} |a_n| - \frac{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)(B-A) - \alpha(1-B)} \sum_{n=k+1}^{\infty} |a_n|} \sum_{n=k+1}^{\infty} |a_n|.
\]

Since $|w_\alpha(z)| \leq 1$ if and only if

\[
2 \left[ \frac{(k + 1)(2k + 2 + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)]}{(1 + \alpha)(B - A) - \alpha(1 - B)} \right] \sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=1}^{k} |a_n|,
\]

this means

\[
\sum_{n=1}^{k} |a_n| + \left[ \frac{(k + 1)(2k + 2 + A + B) + \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)]}{(1 + \alpha)(B - A) - \alpha(1 - B)} \right] \sum_{n=k+1}^{\infty} |a_n| \leq 1.
\]

Thus by the assumption of the theorem, the left hand side of (15) is bounded above by

\[
\sum_{n=1}^{\infty} |a_n|.
\]
if
\[
\frac{1}{(1 + \alpha)((B - A) - \alpha(1 - B))}\left\{ \sum_{n=1}^{k} \left[n(2n + A + B) + \alpha[(2n + \alpha)(1 + B) + (A + 1)]\right]
\right.
\left. - (1 + \alpha)((B - A) - \alpha(1 - B))\right\}[a_n] + \sum_{n=k+1}^{\infty} \left[n(2n + A + B) + \alpha[(2n + \alpha)(1 + B) + (A + 1)]\right]|a_n|
\]
\[
+ (A + 1) - (k + 1)(2k + 2 + A + B) - \alpha[(2k + 2 + \alpha)(1 + B) + (A + 1)]|a_n| \geq 0.
\]

Which completes the proof of Theorem 3.2.

The following result can be found in [5].

**Corollary 3.3.** Let \( f \) be given by (1) and satisfies (5) then
\[
\Re\left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k + 2)(2k + A + B)}{(k + 1)(2k + 2 + A + B)} \quad (z \in U).
\] (16)
The result is sharp for every \( k \) with extremal function
\[
f(z) = \frac{1}{z} + \frac{(B - A)}{(k + 1)(2k + 2 + A + B)}z^{k+1}, \quad k \geq 0.
\] (17)

**Proof.** Assume that \( \alpha = 0 \).

Further, the next result can be found in [8].

**Corollary 3.4.** Let the assumptions of Theorem 3.2 hold. Then for \( f(z) \) of the form (1) satisfies condition
\[
\sum_{n=1}^{\infty} n(n + \mu)|a_n| \leq 1 - \mu, \quad (z \in U),
\]
\[
\Re\left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{(k + 2)(k + \mu)}{(k + 1)(k + 1 + \mu)} \quad (z \in U).
\] (18)
The result is sharp for every \( k \) with extremal function
\[
f(z) = \frac{1}{z} + \frac{1 - \mu}{(k + 1)(k + 1 + \mu)}z^{k+1}, \quad k \geq 0.
\] (19)

**Proof.** Assume that \( \alpha = 0, A = 2\mu - 1, B = 1 \).

We next determine the bounds for \( \Re\left\{ \frac{f_k(z)}{f(z)} \right\} \) of functions in the classes \( S_\alpha(A, B) \) and \( C_\alpha(A, B) \).
Theorem 3.3. Let \( f \in \Sigma_\alpha \) such that
\[
\sum_{n=1}^{\infty} \left[ (n + \alpha)(1 + B) + (A + 1) \right] |a_n| \leq (B - A) - \alpha(1 - B), \quad (z \in U)
\]
holds. Then
\[
\Re \left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{2(k + 1) + A + B + \alpha(1 + B)}{k + 2 + 2\alpha B}, \quad (z \in U).
\]
(20)
Equations hold for the functions given by (7).

Proof. Let \( f \in \Sigma_\alpha \), then we have
\[
\frac{k + 2 + 2\alpha B}{(B - A) - \alpha(1 - B)} \left( \frac{f_k(z)}{f(z)} \right) - \frac{2(k + 1) + A + B + \alpha(1 + B)}{k + 2 + 2\alpha B}
\]
\[
= 1 + \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]
\[
1 + \sum_{n=1}^{\infty} a_n z^{n+2\alpha+1}
\]
\[
:= 1 + w_\alpha(z)
\]
\[
1 - w_\alpha(z).
\]
where
\[
w_\alpha(z) = \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]
\[
2 + 2 \sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}
\]
\[
with
\]
\[
|w_\alpha(z)| \leq \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} |a_n|
\]
\[
2 \sum_{n=1}^{k} |a_n| - \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} |a_n|
\]
\[
Note that |w_\alpha(z)| \leq 1 if and only if
\]
\[
2 \left[ \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)} \sum_{n=k+1}^{\infty} |a_n| \right] \sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2 \sum_{n=1}^{k} |a_n|
\]
which implies
\[
\sum_{n=k+1}^{\infty} |a_n| \leq 1.
\]
(21)
From the assumption of the theorem, we can observe that the left hand side of (21) is bounded above by
\[
\sum_{n=1}^{\infty} \left[ \frac{2n + A + B + \alpha(1 + B)}{(B - A) - \alpha(1 - B)} \right] |a_n|.
\]
Hence the proof.

The following result can be found in [5].
Corollary 3.5. Let the assumptions of Corollary 3.1 hold. Then
\[
\Re\left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{2(k+1) + A + B}{k+2}, \quad (z \in U).
\] (22)

Proof. Assume that \( \alpha = 0 \).

Further, the next result can be found in [8].

Corollary 3.6. Let the assumptions of Corollary 3.2 hold. Then
\[
\Re\left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{k + 1 + \mu}{k+2}, \quad (z \in U).
\] (23)

Proof. Assume that \( \alpha = 0, A = 2\mu - 1, B = 1 \).

In the same manner, we can prove the following result

Theorem 3.4. Let \( f \) be given by (1) and satisfies (5) then
\[
\Re\left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k+1)2(2k+2 + A + B) + \alpha \nu}{2(k+1)(k+2) - (B - A) + \alpha \omega}, \quad (z \in U).
\] (24)

where \( \nu := [(k+1 + \alpha)(1 + A) + (A+1)+(k+1)(B+1)] \)

and \( \omega := [(k+1 + \alpha)(1 + A) + (A+1)+(k+1)(B+1) + (B - A) - (\alpha + 1)(1- B)]. \)

Equalities hold for the function given by (14).

Corollary 3.7. Let the assumptions of Corollary 3.3 hold. Then
\[
\Re\left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k+1)2(2k+2 + A + B)}{2(k+1)(k+2) - (B - A)}, \quad (z \in U).
\] (25)

Proof. Assume that \( \alpha = 0 \).

Further, the next result can be found in [8].

Corollary 3.8. Let the assumptions of Corollary 3.4 hold. Then
\[
\Re\left\{ \frac{f_k(z)}{f(z)} \right\} \geq \frac{(k+1)(k+1 + \mu)}{(k+1)(k+2) - k(1- \mu)}, \quad (z \in U).
\] (26)

Proof. Assume that \( \alpha = 0, A = 2\mu - 1, B = 1 \).

We turn to ratios involving derivatives (see [9]). In the similar manner, we can prove the following results and so the details may be omitted.
Theorem 3.5. Let $f$ be given by (1) and satisfies (3) with $A = -B$. Then

$$\Re \left\{ \frac{f'(z)}{f_k'(z)} \right\} \geq 0, \quad (z \in U), \quad (27)$$

$$\Re \left\{ \frac{f_k'(z)}{f'(z)} \right\} \geq \frac{1 + 2\alpha}{2(1 + \alpha)}, \quad (z \in U). \quad (28)$$

In both cases, the extremal function is given by (7) with $\alpha = 0, A = -B$.

Theorem 3.6. Let $f$ be given by (1) and satisfies (5). Then

$$\Re \left\{ \frac{f'(z)}{f_k'(z)} \right\} = \frac{2(k + A + B) + \phi - \alpha[(k + 1)(B - A - \alpha(1 - B))]}{(2 + 2k + A + B) + \phi}, \quad (z \in U), \quad (29)$$

$$\Re \left\{ \frac{f_k'(z)}{f'(z)} \right\} = \frac{2k + 2 + A + B + \phi}{2(k + 2) + \phi + \alpha[(k + \alpha)(B - A - (\alpha + 1)(1 - B)) - (k + 1)(1 - B)]}, \quad (z \in U). \quad (30)$$

where

$$\phi := \alpha [(k + 1 + \alpha)(1 + B) + (A + 1) + (1 + B)(k + 1)].$$

In both cases, the extremal function is given by (14).

Proof. The proof comes immediately from Theorems 3.1 and 3.3 respectively.

Remark 3.1. We note that $\alpha = 0$ in Theorems 3.5 and 3.6 coincide with the results obtained in [5].

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