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PARTIAL SUMS FOR CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we define and study new classes of meromorphic functions in the punctured disk by using their partial sums.

1. Introduction

Let Σ_α denote the class of functions of the form

$$f(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{\infty} a_n z^{n+\alpha}, \quad \Big(0 \leq \alpha < 1, \Big), \tag{1}$$

which are analytic in the punctured unit disk $U := \{z \in \mathbb{C}, 0 < |z| < 1\}$.

A function $f \in \Sigma_\alpha$ belongs to the class $\mathcal{S}_\alpha(A, B)$, the class of meromorphically α-valent starlike functions if and only if $f \neq 0$, and for $-1 \leq A < B \leq 1$,

$$-\{\frac{zf'(z)}{f(z)}\} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in U).$$

A function $f \in \Sigma_\alpha$ belongs to the class $\mathcal{C}_\alpha(A, B)$, the class of meromorphically α-valent convex functions if and only if $f' \neq 0$, and

$$-\{1 + \frac{zf''(z)}{f'(z)}\} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in U).$$

The class $\Sigma_0 \equiv \Sigma$, was studied by many authors (see [1, 2, 3, 4, 5]). Note that the authors defined and studied the class Σ_α for normalized analytic functions in an open disk (see [6, 7]).

In the present paper, we are motivated with the work done by Silverman [8], and will investigate in similar manner the ratio of a function of the form (1) to its sequence of partial sums

$$f_k(z) = \frac{1}{z^{1+\alpha}} + \sum_{n=1}^{k} a_n z^{n+\alpha}, \quad \Big(0 \leq \alpha < 1, \Big), \tag{2}$$

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when the coefficients are sufficiently small. More precisely, we will determine sharp lower bounds for

$$\Re\Big\{\frac{f(z)}{f_k(z)}\Big\}, \Re\Big\{\frac{f_k(z)}{f(z)}\Big\}, \Re\Big\{\frac{f'(z)}{f_k'(z)}\Big\}, \text{ and } \Re\Big\{\frac{f_k'(z)}{f'(z)}\Big\}.$$

2. Preliminary results

First we prove sufficient conditions for $f \in \Sigma_\alpha$ to be in the classes $\mathcal{S}_\alpha(A,B)$ and $\mathcal{C}_\alpha(A,B)$.

Theorem 2.1. *Let $f \in \Sigma_\alpha$. If*

$$\sum_{n=1}^{\infty} \Big[(n+\alpha)(1+B) + (A+1)\Big]|a_n| \le (B-A) - \alpha(1-B), \quad (z \in U) \tag{3}$$

holds and $B(1+\alpha) > A+\alpha$, then $f \in \mathcal{S}_\alpha(A,B)$.

Proof. Assume that $f \in \Sigma_\alpha$ and satisfies (3). It is sufficient to show that

$$\Big|\frac{1+\frac{zf'(z)}{f(z)}}{B\Big(\frac{zf'(z)}{f(z)}\Big)+A}\Big| < 1,$$

that is

$$\Big|\frac{f(z)+zf'(z)}{Af(z)+Bzf'(z)}\Big| < 1.$$

Consider

$$\begin{aligned} \Big|\frac{f(z)+zf'(z)}{Af(z)+Bzf'(z)}\Big| &= \Big|\frac{-\frac{\alpha}{z^{\alpha+1}} + \sum_{n=1}^{\infty} a_n(n+\alpha+1)z^{n+\alpha}}{\frac{A}{z^{\alpha+1}} + \sum_{n=1}^{\infty} Aa_n z^{n+\alpha} - \frac{B(1+\alpha)}{z^{\alpha+1}} + \sum_{n=1}^{\infty}(n+\alpha)Ba_n z^{n+\alpha}}\Big| \\ &\le \frac{\alpha + \sum_{n=1}^{\infty}(n+\alpha+1)|a_n|}{[B(1+\alpha)-A] - \sum_{n=1}^{\infty}[(n+\alpha)B+A]|a_n|}. \end{aligned} \tag{4}$$

Hence (4) is bounded by 1, if

$$\begin{aligned} &\alpha + \sum_{n=1}^{\infty}(n+\alpha+1)|a_n| \le [B(1+\alpha)-A] - \sum_{n=1}^{\infty}[(n+\alpha)B+A]|a_n| \\ &\Rightarrow \sum_{n=1}^{\infty}(n+\alpha+1)|a_n| + \sum_{n=1}^{\infty}[(n+\alpha)B+A]|a_n| \le [B(1+\alpha)-A] - \alpha \\ &\Rightarrow \sum_{n=1}^{\infty}[(n+\alpha)(1+B)+(1+A)]|a_n| \le (B-A) - \alpha(1-B), \end{aligned}$$

where $B(1+\alpha) > A+\alpha$. This completes the proof.

In a similar manner, we can prove the following result

Theorem 2.2. *Let $f \in \Sigma_\alpha$. If*

$$\sum_{n=1}^{\infty}(n+\alpha)\Big[(n+\alpha)(1+B)+(A+1)\Big]|a_n| \leq (1+\alpha)\Big[(B-A)-\alpha(1-B)\Big], \quad (z \in U) \quad (5)$$

holds and $B(1+\alpha) > A+\alpha$, then $f \in \mathcal{C}_\alpha(A,B)$.

Note that when $\alpha = 0$, Theorem 2.1 and Theorem 2.2 reduce to Theorem 2.2 and Theorem 2.1 in [5] respectively. Further, we note that these sufficient conditions are also necessary for functions of the form (1) when $\alpha = 0, A = 2\mu - 1, B = 1$ with positive or negative coefficients ([1, 2, 3]).

3. Main results

We consider in this section partial sums of functions in the classes $\mathcal{S}_\alpha(A,B)$ and $\mathcal{C}_\alpha(A,B)$ and obtain the sharp lower bounds for the ratio of real part of $f(z)$ to $f_k(z)$ and $f'(z)$ to $f'_k(z)$. In the sequel, we will make use of the generalized result such that $\Re\{(1+w_\alpha(z))/(1-w_\alpha(z))\} > 0, \ (z \in U)$ if and only if $w_\alpha(z) = \sum_{n=1}^{\infty} c_n z^{n+\alpha}$ satisfies the inequality $|w_\alpha(z)| < |z|$.

Theorem 3.1. *Let f be given by (1) and satisfies (3) then*

$$\Re\Big\{\frac{f(z)}{f_k(z)}\Big\} \geq \frac{2(1+k+\alpha+A)}{2(k+\alpha)+(2+A+B)}, \quad (z \in U). \quad (6)$$

The result is sharp for every k with extremal function

$$f(z) = \frac{1}{z^{1+\alpha}} + \frac{(B-A)-\alpha(1-B)}{2(k+\alpha)+(2+A+B)} z^{k+1+\alpha}, \quad k \geq 0. \quad (7)$$

Proof. Assume that $f \in \Sigma_\alpha$ and satisfies (3). Consider

$$\begin{aligned}\frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}&\Big[\frac{f(z)}{f_k(z)} - \frac{2(1+k+\alpha+A)}{2(k+\alpha)+(2+A+B)}\Big]\\ &= \frac{1+\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}{1+\sum_{n=1}^{k} a_n z^{n+2\alpha+1}}\\ &:= \frac{1+w_\alpha(z)}{1-w_\alpha(z)}\end{aligned}$$

where

$$w_\alpha(z) = \frac{\frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}{2+2\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}$$

and

$$|w_\alpha(z)| \le \frac{\frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{k}|a_n| - \frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty}|a_n|}.$$

Now $|w_\alpha(z)| \le 1$ if and only if

$$2\Big[\frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\Big]\sum_{n=k+1}^{\infty}|a_n| \le 2-2\sum_{n=1}^{k}|a_n|$$

which is equivalent to

$$\sum_{n=1}^{k}|a_n| + \Big[\frac{2(k+\alpha)+(2+A+B)}{(B-A)-\alpha(1-B)}\Big]\sum_{n=k+1}^{\infty}|a_n| \le 1. \tag{8}$$

It is suffices to show that the left hand side of (8) is bounded above by

$$\sum_{n=1}^{\infty}\Big[\frac{2(n+\alpha)+A+B}{(B-A)-\alpha(1-B)}\Big]|a_n|$$

which is equivalent

$$\sum_{n=1}^{k}\Big[\frac{2(n+A+\alpha)}{(B-A)-\alpha(1-B)}\Big]|a_n| + \sum_{n=k+1}^{\infty}\Big[\frac{2(n-k-1+\alpha)}{(B-A)-\alpha(1-B)}\Big]|a_n| \ge 0.$$

To show that the function f given by (7) gives the sharp result, we observe that for

$$z = re^{\frac{\pi i}{k+2+2\alpha}}$$

$$\begin{aligned}\frac{f(z)}{f_k(z)} &= 1+\frac{(B-A)-\alpha(1-B)}{2(k+\alpha)+(2+A+B)}z^{k+2+2\alpha}\\ &\to 1-\frac{(B-A)-\alpha(1-B)}{2(k+\alpha)+(2+A+B)}\\ &= \frac{2(k+\alpha)+(2+A+B)-(B-A)+\alpha(1-B)}{2(k+\alpha)+(2+A+B)}\\ &= \frac{2(1+k+A+\alpha)}{2(k+\alpha)+(2+A+B)}\end{aligned}$$

when $r \to 1^-$. Therefore we complete the proof of Theorem 3.1.

Next result can be found in [5].

Corollary 3.1. *Let f be given by* (1) *and satisfies* (3) *then*

$$\Re\Big\{\frac{f(z)}{f_k(z)}\Big\} \ge \frac{2(1+k+A)}{2k+2+A+B}, \quad (z \in U). \tag{9}$$

The result is sharp for every k with extremal function

$$f(z) = \frac{1}{z} + \frac{(B-A)}{2k+2+A+B} z^{k+1}, \quad k \geq 0. \tag{10}$$

Proof. Assume that $\alpha = 0$.

Moreover, the following result can be found in [8].

Corollary 3.2. *Let the assumptions of Theorem 3.1 hold. Then for f of the form (1) satisfies condition*

$$\sum_{n=1}^{\infty} (n+\mu)|a_n| \leq 1-\mu, \quad (z \in U),$$
$$\Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{k+2\mu}{k+1+\mu}, \quad (z \in U). \tag{11}$$

The result is sharp for every k with extremal function

$$f(z) = \frac{1}{z} + \frac{1-\mu}{k+1+\mu} z^{k+1}, \quad k \geq 0. \tag{12}$$

Proof. Assume that $\alpha = 0$, $A = 2\mu - 1$, $B = 1$.

Theorem 3.2. *Let $f \in \Sigma_\alpha$ and*

$$\sum_{n=1}^{\infty} (n+\alpha)\Big[(n+\alpha)(1+B) + (A+1)\Big]|a_n| \leq (1+\alpha)\Big[(B-A) - \alpha(1-B)\Big], \quad (z \in U)$$

holds, then

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{(k+2)(2k+A+B) + \alpha\Big[(2k+2+\alpha)(1+B) + 2\alpha(1-B) - B + 2A + 1)\Big]}{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B) + (A+1)]}. \tag{13}$$

The result is sharp for every k with extremal function

$$f(z) = \frac{1}{z^{1+\alpha}} + \left[\frac{(1+\alpha)\Big[(B-A) - \alpha(1-B)\Big]}{(k+1)(2k+2+A+B) + \alpha[(2k+2+\alpha)(1+B) + (A+1)]}\right] z^{k+1+\alpha}, \quad k \geq 0. \tag{14}$$

Proof. Let $f \in \Sigma_\alpha$. Then we obtain

$$\frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)\Big[(B-A)-\alpha(1-B)\Big]}\times$$

$$\Big[\frac{f(z)}{f_k(z)} - \frac{(k+2)(2k+A+B)+\alpha\Big[(2k+2+\alpha)(1+B)+2\alpha(1-B)-B+2A+1)\Big]}{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}$$

$$= \frac{1+\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}{1+\sum_{n=1}^{k} a_n z^{n+2\alpha+1}}$$

$$:= \frac{1+w_\alpha(z)}{1-w_\alpha(z)},$$

where

$$w_\alpha(z) = \frac{\frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}{2+2\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}$$

and

$$|w_\alpha(z)| \le \frac{\frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\sum_{n=k+1}^{\infty} |a_n|}{2-2\sum_{n=1}^{k} |a_n| - \frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\sum_{n=k+1}^{\infty} |a_n|}.$$

Since $|w_\alpha(z)| \le 1$ if and only if

$$2\Big[\frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\Big]\sum_{n=k+1}^{\infty}|a_n| \le 2-2\sum_{n=1}^{k}|a_n|,$$

this means

$$\sum_{n=1}^{k}|a_n| + \Big[\frac{(k+1)(2k+2+A+B)+\alpha[(2k+2+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\Big]\sum_{n=k+1}^{\infty}|a_n| \le 1. \tag{15}$$

Thus by the assumption of the theorem, the left hand side of (15) is bounded above by

$$\sum_{n=1}^{\infty}\Big[\frac{n(2n+A+B)+\alpha[(2n+\alpha)(1+B)+(A+1)]}{(1+\alpha)[(B-A)-\alpha(1-B)]}\Big]|a_n|$$

if

$$\frac{1}{(1+\alpha)[(B-A)-\alpha(1-B)]}\Big\{\sum_{n=1}^{k}\Big[n(2n+A+B)+\alpha[(2n+\alpha)(1+B)+(A+1)]$$
$$-(1+\alpha)[(B-A)-\alpha(1-B)]\Big]|a_n|+\sum_{n=k+1}^{\infty}\Big[n(2n+A+B)+\alpha[(2n+\alpha)(1+B)]$$
$$+(A+1)-(k+1)(2k+2+A+B)-\alpha[(2k+2+\alpha)(1+B)+(A+1)]\Big]|a_n|\Big\}\geq 0.$$

Which completes the proof of Theorem 3.2.

The following result can be found in [5].

Corollary 3.3. *Let f be given by (1) and satisfies (5) then*

$$\Re\Big\{\frac{f(z)}{f_k(z)}\Big\}\geq\frac{(k+2)(2k+A+B)}{(k+1)(2k+2+A+B)},\ \ (z\in U). \tag{16}$$

The result is sharp for every k with extremal function

$$f(z)=\frac{1}{z}+\frac{(B-A)}{(k+1)(2k+2+A+B)}z^{k+1},\ \ k\geq 0. \tag{17}$$

Proof. Assume that $\alpha=0$.

Further, the next result can be found in [8].

Corollary 3.4. *Let the assumptions of Theorem 3.2 hold. Then for $f(z)$ of the form (1) satisfies condition*

$$\sum_{n=1}^{\infty}n(n+\mu)|a_n|\leq 1-\mu,\ \ (z\in U),$$
$$\Re\Big\{\frac{f(z)}{f_k(z)}\Big\}\geq\frac{(k+2)(k+\mu)}{(k+1)(k+1+\mu)},\ \ (z\in U). \tag{18}$$

The result is sharp for every k with extremal function

$$f(z)=\frac{1}{z}+\frac{1-\mu}{(k+1)(k+1+\mu)}z^{k+1},\ \ k\geq 0. \tag{19}$$

Proof. Assume that $\alpha=0$, $A=2\mu-1$, $B=1$.

We next determine the bounds for $\Re\Big\{\frac{f_k(z)}{f(z)}\Big\}$ of functions in the classes $\mathcal{S}_\alpha(A,B)$ and $\mathcal{C}_\alpha(A,B)$.

Theorem 3.3. *Let $f \in \Sigma_\alpha$ such that*

$$\sum_{n=1}^{\infty}\Big[(n+\alpha)(1+B)+(A+1)\Big]|a_n| \le (B-A)-\alpha(1-B), \ \ (z\in U)$$

holds. Then

$$\Re\Big\{\frac{f_k(z)}{f(z)}\Big\} \ge \frac{2(k+1)+A+B+\alpha(1+B)}{k+2+2\alpha B}, \ \ (z\in U). \tag{20}$$

Equalities hold for the functions given by (7).

Proof. Let $f \in \Sigma_\alpha$, then we have

$$\begin{aligned}\frac{k+2+2\alpha B}{(B-A)-\alpha(1-B)}&\Big\{\frac{f_k(z)}{f(z)} - \frac{2(k+1)+A+B+\alpha(1+B)}{k+2+2\alpha B}\Big\}\\ &= \frac{1+\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}{1+\sum_{n=1}^{k} a_n z^{n+2\alpha+1}}\\ &:= \frac{1+w_\alpha(z)}{1-w_\alpha(z)},\end{aligned}$$

where

$$w_\alpha(z) = \frac{\frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}{2+2\sum_{n=1}^{k} a_n z^{n+2\alpha+1} + \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty} a_n z^{n+2\alpha+1}}$$

with

$$|w_\alpha(z)| \le \frac{\frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=1}^{k}|a_n| - \frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\sum_{n=k+1}^{\infty}|a_n|}.$$

Note that $|w_\alpha(z)| \le 1$ if and only if

$$2\Big[\frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\Big]\sum_{n=k+1}^{\infty}|a_n| \le 2-2\sum_{n=1}^{k}|a_n|$$

which implies

$$\sum_{n=1}^{k}|a_n| + \Big[\frac{2(k+1)+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\Big]\sum_{n=k+1}^{\infty}|a_n| \le 1. \tag{21}$$

From the assumption of the theorem, we can observe that the left hand side of (21) is bounded above by

$$\sum_{n=1}^{\infty}\Big[\frac{2n+A+B+\alpha(1+B)}{(B-A)-\alpha(1-B)}\Big]|a_n|.$$

Hence the proof.

The following result can be found in [5].

Corollary 3.5. *Let the assumptions of Corollary 3.1 hold. Then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{2(k+1)+A+B}{k+2}, \quad (z \in U). \tag{22}$$

Proof. Assume that $\alpha = 0$.

Further, the next result can be found in [8].

Corollary 3.6. *Let the assumptions of Corollary 3.2 hold. Then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{k+1+\mu}{k+2}, \quad (z \in U). \tag{23}$$

Proof. Assume that $\alpha = 0$, $A = 2\mu - 1$, $B = 1$.

In the same manner, we can prove the following result

Theorem 3.4. *Let f be given by (1) and satisfies (5) then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k+1)2(2k+2+A+B)+\alpha\nu}{2(k+1)(k+2)-(B-A)+\alpha\omega}, \quad (z \in U). \tag{24}$$

where

$$\nu := \big[(k+1+\alpha)(1+B)+(A+1)+(k+1)(B+1)\big]$$

and

$$\omega := \big[(k+1+\alpha)(1+B)+(A+1)+(k+1)(B+1)+(B-A)-(\alpha+1)(1-B)\big].$$

Equalities hold for the function given by (14).

Corollary 3.7. *Let the assumptions of Corollary 3.3 hold. Then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k+1)2(2k+2+A+B)}{2(k+1)(k+2)-(B-A)}, \quad (z \in U). \tag{25}$$

Proof. Assume that $\alpha = 0$.

Further, the next result can be found in [8].

Corollary 3.8. *Let the assumptions of Corollary 3.4 hold. Then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{(k+1)(k+1+\mu)}{(k+1)(k+2)-k(1-\mu)}, \quad (z \in U). \tag{26}$$

Proof. Assume that $\alpha = 0$, $A = 2\mu - 1$, $B = 1$.

We turn to ratios involving derivatives (see [9]). In the similar manner, we can prove the following results and so the details may be omitted.

Theorem 3.5. *Let f be given by (1) and satisfies (3) with $A = -B$. Then*

$$\Re\Big\{\frac{f'(z)}{f_k'(z)}\Big\} \geq 0, \quad (z \in U), \tag{27}$$

$$\Re\Big\{\frac{f_k'(z)}{f'(z)}\Big\} \geq \frac{1+2\alpha}{2(1+\alpha)}, \quad (z \in U). \tag{28}$$

In both cases, the extremal function is given by (7) with $\alpha = 0$, $A = -B$.

Theorem 3.6. *Let f be given by (1) and satisfies (5). Then*

$$\Re\Big\{\frac{f'(z)}{f_k'(z)}\Big\} \geq \frac{2(k+A+B)+\phi-\alpha[(k+1)(B-A-\alpha(1-B))]}{(2+2k+A+B)+\phi}, \quad (z \in U), \tag{29}$$

$$\Re\Big\{\frac{f_k'(z)}{f'(z)}\Big\} \geq \frac{2k+2+A+B+\phi}{2(k+2)+\phi+\alpha[(k+\alpha)(B-A-(\alpha+1)(1-B))-(k+1)(1-B)]}, \quad (z \in U). \tag{30}$$

where

$$\phi := \alpha\big[(k+1+\alpha)(1+B)+(A+1)+(1+B)(k+1)\big].$$

In both cases, the extremal function is given by (14).

Proof. The proof comes immediately from Theorems 3.1 and 3.3 respectively.

Remark 3.1. We note that $\alpha = 0$ in Theorems 3.5 and 3.6 coincide with the results obtained in [5].

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