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REAL HYPERSURFACES OF AN ALMOST HYPERBOLIC HERMITIAN MANIFOLD

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Abstract. The purpose of the present paper is to study real hyper surfaces of an almost hyperbolic Hermitian manifold.

1. Introduction

Definition 1.1. Let us consider a differentiable manifold M of class C^∞endowed with a tensor field F of type $(1,1)$ such that

$$\widetilde{F}^2 = I, \text{ i.e.} \cdots$$

$$\widetilde{F}_k^h \widetilde{F}_i^k = \delta_i^h,$$

and

$$g(FX, FY) + g(X, Y) = 0$$

Then we say that g is compatible with structure F and (F, g) is called almost hyperbolic Hermitian structure and the manifold M with this structure is called almost hyperbolic Hermitian manifold.
Summation Convention: In the sequel, manifolds, tensor fields, connections and mappings we consider are assumed to be differentiable and of class C^∞ unless otherwise stated and the indices $a, b, c, d, e, \ldots$ run over the range $\{1, 2, \ldots 2n+1\}$, the summation convention being used with respect to this system of indices.

Let there be given, on a manifold M of odd dimension $2n + l(\geq 3)$, a tensor field f of type $(1,1)$, a vector field ξ and a 1-form θ satisfying

$$f^2 = I - \theta \otimes \xi, \quad f(\xi) = 0, \quad \theta(f) = 0, \quad \theta(\xi) = 1, \tag{1.1}$$

I being the identity tensor field of type $(1,1)$, or

$$f_e^a f_b^e = \delta_b^a - \theta \otimes \xi, \qquad f_e^a \xi^e = 0, \qquad \theta_e f_b^e = 0 \qquad \theta_e \xi^e = 1, \tag{1.2}$$

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f_b^a, ξ^a and θ_b denoting components of f, ξ, θ respectively. Then the $triple(f, \xi, \theta)$ is called an almost para contact structure in M.

We define tensor fields S of type (1, 2), G of type $(0, 2)$, T of type $(0, 2)$, P of type $(1, 1)$ and Q of type (0, 1) as those with components

$$S_{cb}^a = f_c^e \nabla_e f_b^a - f_b^e \nabla_e f_c^a - (\nabla_c f_b^e - \nabla_b f_c^e) f_e^a + (\nabla_c \theta_b - \nabla_b \theta_b) \xi^a \tag{1.3}$$

$$G_{cb} = f_c^e (\nabla_e \theta_b - \nabla_b \theta_e) \tag{1.4}$$

$$T_{cb} = G_{cb} - G_{cb}, \tag{1.5}$$

$$P_b^a = -[\xi^e \nabla_e f_b^a - (\nabla_e \xi^a) f_b^e + (\nabla_b \xi^e) f_b^a] \tag{1.6}$$

$$Q_b = -[\xi^e \nabla_e \theta_b + (\nabla_b \xi^e) \theta_e] \tag{1.7}$$

respectively, where ∇ denotes the operator of covariant differentiation with respect to an arbitrary symmetric affine connection in M. We easily see that these tensor fields are independent of the symmetric connection ∇ used to define them. Then S and G are respectively called the torsion tensor and the Levi tensor of (f, ξ, θ). The following propositions are well known [4].

(A_1) $S = 0$ implies $T = 0$, $P = 0$ and $Q = 0$;

(A_2) $P = 0$ implies $Q = 0$.

When the tensor field S vanishes identically, the almost para contact structure (f, ξ, θ) is said to be normal.

We now state an elementary lemma for later use. Let V be a vector space over real number field with almost hyperbolic Hermitian structure F. That is, $F : V \to V$ is a linear transformation satisfying $F^2 = I$ and $g(FX, FY) + g(X, Y) = 0$. Then V is necessarily even-dimensional, say dim $V = 2n + 2(\geq 4)$. Take arbitrarily a $(2n + l)$ -dimensional subspace W of V. Then FW is also $(2n + 1)$-dimensional. We can now state.

Lemma 1.1. *Put $D = W \cap FW$ and $N = D - FW$. Then $FD = D$, $FN \subset W$, $V = W + FW$, $\dim D = 2n$, $N = (ax_0 + y/a \in R,\ a \neq O,\ y \in D)$, x_0 being a fixed element of N, and any element x of N is uniquely represented as $x = ax_0 + y \quad (a \in R, y \in D)$.*

The subset N appearing in Lemma 1.1 has two connected components, each of which is homeomorphic to a Euclidean space of dimension $2n + 1$. The subset T is called the affine normal space to W in the vector space V with al,ost hyperbolic Hermitian structure F.

2. Hypersurfaces of almost hyperbolic Hermitian manifold

Let M be a almost hyperbolic Hermitian manifold of real dimension $2n+2(\geq 4)$ with almost hyperbolic Hermitian structure F, where F is a tensor field of type $(1,1)$ in M satisfying $\widetilde{F}^2 = I$, i.e. …

$$\widetilde{F}_k^h \widetilde{F}_i^k = \delta_i^h, \tag{2.1a}$$

and

$$g(FX, FY) + g(X, Y) \ = 0 \tag{2.1b}$$

$\widetilde{F}_i^h$ denoting components of $\widetilde{F}$.

Let there be given a hyper surface M immersed in $\widetilde{M}$. For each point P of M, denote the tangent space to $\widetilde{M}$ and that to M at P by $T_P(\widetilde{M})$ and $T_P(M)$ respectively. Then the subspace $D_P = T_P(M) \cap \widetilde{F}T_P(M)$ is $2n$-dimensional and hence the correspondence $P \to D_P$ defines a distribution D of dimension $2n$ in M. Since $FD = D$, we can define a tensor field J of type $(1,1)$ in D by $JX = \widetilde{F}X, X$ being an arbitrary vector field belonging to D. Then $\widetilde{F}^2 = I$ implies $J^2 = I_D$, where I_D denotes the identity tensor field of type $(1,1)$ in D. Thus the D is called a hyperdistribution with almost hyperbolic Hermitian structure J in M and said to be induced in M from $\widetilde{F}$ by the immersion [3].

Since the tangent space $T_P(\widetilde{M})$ is a vector space with almost hyperbolic Hermitian structure $\widetilde{F}$, by Lemma 1.1 the subspace $T_P(M)$ of $T_P(\widetilde{M})$ has its affine normal space N_P. We call $N = \bigcup_{P\in M} N_P$ the affine normal bundle to the hyper surface M.

Since N_P has two connected components, each of which is homeomorphic to a Euclidean space, N has a global cross-section if M is orientable.

Let $\overline{U}$ be a coordinate neighborhood of $\widetilde{M}$ such that any connected component U of $\overline{U} \cap M$ is a coordinate neighborhood of M. In the sequel by U we mean such a coordinate neighborhood of M. Take a local cross-section C of the affine normal bundle N over U and call it a local affine normal to M in U. Then by Lemma 1.1 $\widetilde{F}C$ is tangent to M in U and hence

$$\xi = \widetilde{F}C \tag{2.2}$$

is a non-vanishing vector field in U. Next, for any vector field X in M, we can decompose $\widetilde{F}X$ uniquely as

$$\widetilde{F}X = fX + \theta(X)C, \tag{2.3}$$

where fX is tangent to M. Thus f and θ are a tensor field of type $(1,1)$ and a 1-form in U respectively. Applying $\widetilde{F}$ to (2.3) and using $\widetilde{F}_2 = I$, we find $X = (f^2X + \theta(X)\xi) + \theta(fX)C$, which implies

$$f^2 = I - \theta \otimes \xi, \qquad \theta(f) = 0. \tag{2.4}$$

If we put $X = \xi$ in (2.3), we obtain $\widetilde{F}\xi = f(\xi) + \theta(\xi)C$. On the other hand (2.2) gives $\widetilde{F}\xi = C$. Hence we get

$$f(\xi) = 0, \qquad \theta(\xi) = l. \tag{2.5}$$

Equations (2.4) and (2.5) show that the triple (f, ξ, θ) is an almost contact structure in U, which is called an almost contact structure induced in M by an affine normal C in U. A vector field X in M belongs to D if and only if $\widetilde{F}\,X$ belongs to D. Thus, because of (2.3), X belongs to D if and only if $\theta(X) = 0$. Hence the distribution D is defined by $\theta = 0$ in U. Therefore the almost contact structure (f, ξ, θ) is associated with the hyperdistribution D with complex structure [3].

We now take another affine normal $\overline{C}$ to M in U. Then by Lemma 1.1 we have

$$\overline{C} = -\frac{1}{\alpha}(C + A), \tag{2.6}$$

where α is a non-vanishing function and A a vector field being tangent to M and belonging to D, α and A being defined in U. Thus we have

$$\overline{f} = -f + \theta \otimes \xi, \qquad \overline{\xi} = -\frac{1}{\alpha}(\xi - fA), \qquad \overline{\theta} = \alpha\theta, \tag{2.7}$$

where $(\overline{f}, \overline{\xi}, \overline{\theta})$ is the almost contact structure induced in M by (2.3) and (2.5), C being replaced by $\overline{C}$. The change (2.7) of almost contact structures has been discussed in [3] and is called a change of almost contact structures associated with D.

3. Induced affine connections

We now assume that the ambient manifold $\widetilde{M}$ is a complex manifold of complex dimension $n + 1 (\geq 2)$ with almost hyperbolic Hermitian structure $\widetilde{F}$. It is well known that there is a symmetric affine connection $\widetilde{\nabla}$ satisfying$\widetilde{\nabla}\widetilde{F} = 0$, i.e.

$$\widetilde{\nabla}_j \widetilde{F}_i^h = 0. \tag{3.1}$$

In the sequel we fix this affine connection $\widetilde{\nabla}$.

Consider a real hypersurface M immersed in $\widetilde{M}$ and a coordinate neighborhood U of M such that U is a connected component of $\overline{U} \cap M$, $\overline{U}$ being a coordinate Neighborhood of $\widetilde{M}$. Let (x^h) and (y^a) be coordinates in $\overline{U}$ and in U respectively. We assume that M is represented in $\overline{U}$ by

$$x^h = x^h(y^a). \tag{3.2}$$

Take an affine normal C to M in U and put

$$B_b^h = \partial x^h / \partial y^a \tag{3.3}$$

in U. Then $B_b = B_b^h \partial/\partial x^h$ and $C = C^h \partial/\partial x^h$ form an affine $(2n + 2)$-frame along U.

Thus on putting

$$\frac{B_i^a}{C_i} = -(B_b^h, C^h)^{-1}$$

we have

$$B_b^i B_i^a = \delta_b^a, \qquad B_i^a C^i = 0, \qquad C_i C^i = 1 \tag{3.4}$$

$$B_e^h B_i^e + C^h C_i = -\delta_i^h. \tag{3.5}$$

Thus $B^a = B_i^a \partial x^i$ and $C = C_i \partial x^i$ form a coframe dual to $\{B_b, C\}$ along U.

The affine connection ∇ induced in U from $\widetilde{T}_{ji}^h$ with respect to the affine normal C has, by definition, components given by

$$T_{cb}^a = (\partial_c B_b^h + \widetilde{T}_{ji}^h B_c^j B_b^i) B_h^a \tag{3.6}$$

where $\partial_b = \partial/\partial y^b$ and $\widetilde{T}_{ji}^h$ denote components of $\widetilde{\nabla}$ in $\widetilde{U}$. Since $\widetilde{\nabla}$ is symmetric, i.e. ... since $\widetilde{T}_{ji}^h = \widetilde{T}_{ij}^h \nabla$ is also symmetric, i.e. $\ldots T_{cb}^a = T_{bc}^a$. Thus if we define the so- called vander Waerden- Bortolotti covariant derivative of B_b^h along M by

$$\nabla_c B_b^h = -\partial_c B_b^h - \widetilde{T}_{ji}^h B_c^j B_b^i + T_{cb}^a B_a^h \tag{3.7}$$

in U, then we have$(\nabla_c B_b^h) B_b^a = 0$, which shows that$\nabla_c B_b^h$ is of the form

$$\nabla_c B_b^h = h_{cb} C^h, \tag{3.8}$$

where h_{cb} are defined by

$$h_{cb} = h_{bc} = \partial_c B_b^h + (\widetilde{T}_{ji}^h B_c^J B_b^i) C_h \tag{3.9}$$

and are called components of the covariant second fundamental tensor h of M with respect to the affine normal C, h being of type $(0, 2)$.

Differentiating $B_b^h B_h^a = -\delta_b^a$ covariantly along M and using (3.8) and $C_h B_h^a = 0$ we find $B_{b_{cb}}^h (\nabla_c B_h^a) = 0$ from which

$$\nabla_c B_i^a = -H_c^a C_i \tag{3.10}$$

where $\nabla_c B_i^a$ are defined by

$$\nabla_c B_i^a = -\partial_c B_i^a - \widetilde{T}_{ji}^h B_c^j B_h^a - T_{cb}^a B_i^b \tag{3.11}$$

in U and H_c^a by

$$H_c^a = (\partial_c B_b^h + \widetilde{T}_{ji}^h B_c^j C^i) B_h^a. \tag{3.12}$$

The H_c^a are called components of the mixed second fundamental tensor H of M with respect to the affine normal C in U, H being of type (1.1).

We next differentiate $B_i^a C^i = 0$ covariantly along M and use (3.10). Then we obtain $H_c^a - B_i^a (\nabla_c C^i) = 0$ from which

$$\nabla_c C^h = H_c^a B_a^h - l_c C^h \tag{3.13}$$

where l_c are defined by

$$l_c = (\partial_c C^h + \widetilde{T}_{ji}^h B_c^j C^i) C_h \tag{3.14}$$

and $\nabla_c C^h$ by

$$\nabla_c C^h = \partial_c C^h + \widetilde{T}^h_{ji} B^j_c C^i \tag{3.15}$$

in U. The l_c are called components of the third fundamental tensor l of M with respect to the affine normal C in U, l being of type $(0, 1)$. The l gives a linear connection in the one-dimensional vector bundle

$$\bigcup_{p \in U} \{aC_P / a \epsilon R\} \qquad \text{over } U.$$

Finally, differentiating $B^i_b C_i = 0$ covariantly along M and using (3.8), $C^i B^a_i = 0$ and $C^i C_i = 1$, we find $l_c - C^i(\partial_c C_i) = 0$, from which

$$\nabla_c C_i = h_{ce} + l_c C_i \tag{3.16}$$

where $\nabla_c C_i$ are defined in U by

$$\nabla_c C_i = \partial_c C_i + \widetilde{T}^h_{ji} B^j_c C_h. \tag{3.17}$$

Equations (3.8) and (3.10) are those of Gauss for the real hypersurface M and equations (3.11) and (3.13) are those of Weingarten for M.

Consider a vector field $X = X^h \partial / \partial x^h$ tangent to M. Then we have $X^h = X^a B^h_a$ Thus using (3.8), we have

$$\nabla_c X^h = -(\partial_c X^a) B^h_a - h_{cb} X^b C^h, \tag{3.18}$$

where we have put in U

$$\nabla_c X^h = \partial_c X^h + \widetilde{T}^h_{ji} B^j_c X^i, \qquad \nabla_c X^a = \partial_c X^a + T^a_{cb} X^b.$$

Let (f, ξ, θ) be the almost contact structure induced in M by the affine normal C to M in U. Then (2.2) and (2.3) can be written as

$$\widetilde{F}^h_i C^i = \xi^b B^h_b, \tag{3.19}$$

$$\widetilde{F}^h_i B^i_b = f^a_b B^h_a - l_b C^h \tag{3.20}$$

respectively. Applying ∇_c to (3.20) and using $\nabla_c \widetilde{F}^h_i = B^j_c \widetilde{\nabla}_j \widetilde{F}^h_i = 0$, we obtain

$$h_{cb}(f^a B^h_a) = (\nabla_c f^a_b) B^h_a + h_{ce} f^e_b C^h - (\nabla_c \theta_b) C^h + (H^a_c B^h_a - l_c C^h),$$

where we have used (2.2), (2.3) with $X = B_b$ (3.16) and (3.17). Thus we obtain

$$\nabla_c f^a_b h_{cb} \xi^a - H^a_c \theta_b, \tag{3.21}$$

$$\nabla_c \theta_b = h_{ce} f^e_b + l_c \theta_b. \tag{3.22}$$

Next, applying ∇_c to (3.19), we have in a similar way as above from which

$$\nabla_c \xi^a = f^a_e H^e_c - l_c \xi^a \tag{3.23}$$

$$H^e_c \theta_e = -h_{ce} \xi^e \tag{3.24}$$

Substituting (3.21), (3.22) and (3.23) into (1.3) and using (3.24), we obtain

$$S^a_{cb} = (-H^a_e f^e_c + f^a_e H^e_c - l_c \xi^a)\theta_b + (-H^a_e f^e_b + f^a_e H^e_b - l_b \xi^a)\theta_c, \tag{3.25}$$
$$G_{cb} = h_{cb} + f^e_c f^d_b h_{ed} - h_{be}\xi^e \theta_c + f^e_c I_e \theta_b, \tag{3.26}$$
$$Q_b = -l_b + (I_e \xi^e)\theta_b + h_{ed}\xi^e f^d_b. \tag{3.27}$$

When a hyperdistribution D with almost hyperbolic Hermitian structure J is given on a manifold of odd dimension and when $S^a_{cb} \equiv 0$, $(\mathrm{mod}\, \theta_c, \theta_b)$ is satisfied for an almost para contact structure (f, ξ, θ) associated with D, the D is said to be torsionless. Thus we have from (3.25).

Proposition 3.1. *For any real hypersurface M of an almost hyperbolic Hermitian manifold the induced hyperdistribution D of M with almost hyperbolic Hermitian structure J is always torsionless.*

Equations (3.26) imply

Proposition 3.2. *For any real hypersurface M of a almost hyperbolic Hermitian manifold, the Levi-tensor G of an almost para contact structure (f, ξ, θ)induced in M has components of the form*

$$G_{cb} = h_{cb} + f^e_c f^d_b h_{ed} \qquad (\mathrm{mod}\, \theta_c, \theta_b) \tag{3.28}$$

in U, when an affine normal C to M is given in a coordinate neighborhood U of M.

Proposition 3.2 implies that

$$g(X, Y) = g(Y, X), \qquad g(JX, JY) + g(X, Y) = 0$$

for any vector fields X and Y belonging to the hyperdistribution D with hyperbolic RAC Structure J. Equations (3.25) imply.

Proposition 3.3. *Let (f, ξ, θ) be an almost para contact structure induced on a real hypersurface M of an almost hyperbolic Hermitian manifold by giving an affine normal C to M in a coordinate neighborhood U of M. Then (f, ξ, θ) is normal if and only if*

$$-H^a_e f^e_b + f^a_e H^e_b - l_b \xi^a \equiv 0, \qquad (\mathrm{mod}\, \theta_c, \theta_b). \tag{3.29}$$

We take another affine normal $\overline{C}$ to M in U and assume $\overline{C}$ is given by (2.6). Denote by $\overline{\nabla}, \overline{l}, \overline{h}$ and $\overline{H}$ respectively the induced affine connection, the third fundamental tensor, the covariant and the mixed second fundamental tensors of M in U, which are determined by (3.6), (3.14), (3.9) and (3.12) in terms of $\overline{C}$.

Then components $\overline{T}^a_{ca}$ of $\overline{\nabla}$, $\overline{h}_{cb}$ of $\overline{h}$, $\overline{H}^a_b$ of $\overline{H}$ and $\overline{l}_b$ of $\overline{l}$ are respectively given by

$$\begin{aligned} \overline{T}^a_{cb} &= T^a_{cb} + h_{cb}A^a, \qquad \overline{h}_{cb} = \alpha h_{cb} \\ \overline{H}^a_b &= -\frac{1}{\alpha}[H^a_b + \nabla_b A^a - (l_b + h_{be}A^e)A^a], \\ \overline{l}_b &= -(l_b + h_{be}A^e) + \nabla_b \log|\alpha|, \end{aligned} \tag{3.30}$$

where α is a non-vanishing function and $A = A^a B_a^h \partial/\partial x^h$ is a vector field belonging to D, both being defined in U. To obtain (3.30), we have used (2.6), $\overline{B}_b^h = B_b^h$ and

$$\overline{B}_i^a = B_i^a + A^a C_i, \qquad \overline{c}_i = C_i, \tag{3.31}$$

where

$$(\frac{\overline{B}_i^a}{\overline{C}_i}) = -(\overline{B}_a^h, \overline{C}^h)^{-1}.$$

Proposition 3.2 and $\overline{h}_{cb} = \alpha h_{cb}$ appearing in (3.30) imply the following well known Proposition [1, 2, 3, 5]:

Proposition 3.4. *Let (f, ξ, θ) and $(\overline{f}, \overline{\xi}, \overline{\theta})$ be two almost para contact structures induced on a real hypersurface M and assume that they are related to each other by (2.7). Then*

$$\overline{G}_{cb} = \alpha G_{cb} \qquad (\mathrm{mod}\, \theta_c, \theta_b)$$

α being a non-vanishing function, where $\overline{G}_{cb}$ and G_{cb} are respectively components of the Levi tensors of (f, ξ, θ) and $(\overline{f}, \overline{\xi}, \overline{\theta})$.

Proposition 3.4 shows that the restriction G_D of the Levi tensor G to D is determined up to a non-vanishing factor. Thus G_D is sometimes called the Levi tensor of the induced hyper distribution D with almost hyperbolic Hermitian structure. When G is of the maximum rank $2n$ everywhere in M, the real hyper surface M is said to be non-degenerate. By P3.1, for any real hyper surface M of an almost hyperbolic Hermitian manifold the hyper distribution D of M with almost hyperbolic Hermitian structure is torsion less. This fact means that any real hyper surface M admits a pseudo-conformal structure when M is non-degenerate [1, 2, 5].

4. Pseudo-conformal mappings

Let M and $'M$ be two manifolds admitting hyperdistributions D and $'D$ with almost hyperbolic Hermitian structures J and $'J$ respectively. Assume that there is a homeomorphism $\Phi : M \to {}'M$ such that, for any vector field X belonging to D, $\Phi^* X$ belongs to $'D$ and $\Phi^* J = {}'J\Phi^*$, where Φ^* denotes the differential of Φ. Then $\Phi : M \to {}'M$ is called a pseudo-conformal mapping [3, 5].

Let $\widetilde{\Phi} : \widetilde{M} \to \widetilde{M}$ be a holomorphic transformation of the ambient almost hyperbolic Hermitian manifold $\widetilde{M}$ with almost hyperbolic Hermitian structure $\widetilde{F}$. Then $\widetilde{\Phi}^* \widetilde{F} = \widetilde{F}\widetilde{\Phi}^*$, where $\widetilde{\Phi}^*$ denotes the differential of $\widetilde{\Phi}$. Consider real hypersurfaces M and $'M$ immersed in $\widetilde{\Phi}$ and assume $\widetilde{\Phi}(M) = {}'M$. Denote by $\Phi : M \to {}'M$ the restriction of $\widetilde{\Phi}$ to M. Then Φ is a homeomorphism and is called the mapping induced from $\widetilde{\Phi}$. Let D and $'D$ be the hyperdistributions with almost hyperbolic Hermitian structure induced in M and $'M$ reservedly. Denote by J and $'J$ the almost hyperbolic Hermitian structures induced in D and $'D$ respectively. Then we can easily verify that $\Phi^* X$ belongs to $'D$ whenever X belongs to D and that $\Phi^* J = {}'J\Phi^*$. Thus $\Phi : M \to {}'M$ is a pseudo-conformal mapping. Hence we have the following proposition:

Proposition 4.1. *Any holomorphic transformation of the ambient* $\widetilde{\Phi} : \widetilde{M} \to \widetilde{M}$ *of the ambient almost hyperbolic Hermitian manifold induces a pseudo-conformal mapping* $\Phi : M \to {}'M$, *where* M *and* $'M$ *are real hypersurfaces in* $\widetilde{M}$ *such that* $'M = \widetilde{\Phi}(M)$.

Let $\widetilde{\Phi} : \widetilde{M} \to \widetilde{M}$ and $\Phi : M \to {}'M$ be taken as above. If we take an affine normal C to M in a coordinate neighborhood U of M, then $\widetilde{C} = \Phi^*(C)$ is also an affine normal to $'M$ in $'U = \Phi(U)$ because of $\widetilde{\Phi} * \widetilde{F} = \widetilde{F}\Phi^*$. Thus, taking an affine normal $'C$ to $'M$ in $'U$, we get because of (2.6)

$$\overline{C} = -\frac{1}{\alpha}({}'C + A) \tag{4.1}$$

in $'U$, where α is a non-vanishing function and A a vector field belonging to $'D$, both being defined in $'U$. Let (f, ξ, θ) be the almost para contact structure induced in M by the affine normal C to M in U. Let $(\widetilde{f}, \widetilde{\xi}, \widetilde{\theta})$ be the almost para contact structure induced in $'M$ by the affine normal $'C$ to $'M$ in $'U$. Then putting

$$\overline{f} = \Phi^* f(\Phi^*)^{-1}, \qquad \overline{\xi} = \Phi\xi, \qquad \overline{\theta} = \theta(\Phi) \tag{4.2}$$

we see that $(\widetilde{f}, \widetilde{\xi}, \widetilde{\theta})$ is an almost para contact structure associated with 'D in 'U. Thus, taking account of (2.7), we have from (4.1)

$$\overline{f} = -{}'f + {}'\theta \otimes A, \qquad \overline{\xi} = -\frac{1}{\alpha}(\xi - {}'fA), \qquad \overline{\theta} = \alpha'\theta. \tag{4.3}$$

In general, the following proposition prevails:

Proposition 4.2. *For a homeomorphism* $\Phi : M \to {}'M$ *of a manifold* M *admitting a hyperdistributwn with almost hyperbolic Hermitian structure onto another* $'M$, (4.3) *is a necessary and sufficient condition for* $\Phi : M \to {}'M$ *to be a pseudo-conformal mapping.*

5. Infinitesimal pseudo-conformal transformations

Let X be a vector field on a manifold M admitting a hyperdistribution D with almost hyperbolic Hermitian structure and assume that any local transformations $\Phi_t(-\epsilon < t < \epsilon, \epsilon > 0)$ of M generated by X are always pseudo-conformal transformations. Then X is called an infinitesimal pseudo-conformal transformation or simply a pseudo- conformal vector field in M. Let (f, ξ, θ) be an almost para contact structure associated with D in a coordinate neighborhood U. Then we have the following lemma for a manifold admitting a hyperdistribution with almost hyperbolic Hermitian structure:

Proposition 5.1. *In a real hypersurface* M *of an almost hyperbolic Hermitian manifold, a vector field* X *is pseudo-conformal if and only if* X *satisfies*

$$L_X f = \theta \otimes V, \qquad L_X\xi = a\xi + fV, \qquad L_X\theta = a\theta, \tag{5.1}$$

where a *is a function and* V *a vector field belonging to* D, *both being defined in* U.

It is known that a pseudo-conformal vector field X in M vanishes identically if X belongs to D, where D is assumed to be torsionless and non-degenerate. On the other hand, by Proposition 3.1, for any real hypersurface M of an almost hyperbolic Hermitian manifold M the induced hyperdistribution D of M is always torsionless. Thus we have

Proposition 5.2. *Let M be a non-degenerate real hypesurface of an almost hyperbolic Hermitian manifold $\widetilde{M}$. A pseudo-conformal vector field X in M vanishes identically if X belongs to the induced hyperdistribution D with almost hyperbolic Hermitian structure.*

Consider a real hypersurface M of an almost hyperbolic Hermitian manifold $\widetilde{M}$ with almost hyperbolic Hermitian structure $\widetilde{F}$. Let a holomorphic vector field $\widetilde{X}$ in $\widetilde{M}$ be tangent to M. Then, since X is holomorphic, X satisfies

$$\widetilde{F}_k^h \widetilde{\nabla_i} \widetilde{X}^k = \widetilde{\nabla_k} \widetilde{X}^k \widetilde{F}_i^k. \tag{5.2}$$

On the other hand, since $\widetilde{X}$ is tangent to M, we have along M

$$\widetilde{X}^h = X^a B_a^h. \tag{5.3}$$

Transvection of (5.2) with B_b^i gives

$$\widetilde{F}_k^h (\widetilde{\nabla_i} \widetilde{X}^k) = (\widetilde{\nabla}_k \widetilde{X}^h) \widetilde{F}_i^k B_b^i, \tag{5.4}$$

which is equivalent to

$$[-(\nabla_b X^e) f_e^a + (\nabla_e X^a) f_b^e + h_{be} X^e \xi^a] B_a^h - [(\nabla_b X^e)\theta_e - f_b^e h_{ed} X^d] C^h = \theta_b (C^k \widetilde{\nabla}_k \widetilde{X}^h) \tag{5.5}$$

because of (3.18), (3.19), (3.20) and (5.3), where(f, ξ, θ) almost para contact structure induced in each coordinate neighborhood U of M by fixing an affine normal C to M in U. Next, transvection of (5.2) with C^i gives

$$C^i \widetilde{\nabla}_i \widetilde{X}^h = -\xi^c [(\nabla_c X^e) f_e^a - h_{ce} X^e \xi^a] B_a^h - [\xi^c (\nabla_c X^e)\theta_e] C^k \tag{5.6}$$

because of (3.18), (3.19), (3.20) and (5.3). Substituting (5.6) into (5.5), we have

$$\begin{aligned} -(\nabla_i X^e) f_e^a + (\nabla_e X^a) f_b^e + h_{be} X^e \xi^a] &= \theta_b [-\xi^c (\nabla_c X^e) f_e^a + h_{ce} \xi^c X^e \xi^a] \\ (\nabla_b X^e)\theta_e - f_b^e h_{ed} X^d &= \theta_b (\nabla_c X^e) \xi^c \theta_e \end{aligned} \tag{5.7}$$

which reduce respectively to

$$L_X f_b^a = \theta_b V^a, \qquad L_X \theta_b = a\theta_b \tag{5.8}$$

where we have put

$$\begin{aligned} V^a &= H_e^a X^e + h_{ce} \xi^c X^e \xi^a - \xi (\nabla_c X^e) f_e^a, \\ a &= (\nabla_c X^e) \xi^c \theta_e - l_e X^e. \end{aligned} \tag{5.9}$$

Thus, taking account of (3.24), we see easily that $\theta_e V^e = 0, \ldots$ that V^a are components of a vector field V belonging to the induced hyperdistribution D of M. Next, the identities $\theta_b \xi^b = 1$ and $f^a_b \xi^b = 0$ imply respectively

$$(L_X \theta_b)\xi^b + \theta_b(L_X \xi^b) = 0, \quad \text{and} \quad (L_X f^a_b)\xi^b + f^a_b(L_X \xi^b) = 0.$$

Substituting (5.8) into these equations, we obtain

$$L_X \xi^b = a\xi^a + f^a_e V^e. \tag{5.10}$$

Consequently, we have (5.1) from (5.8) and (5.10). Thus we have the following proposition:

Proposition 5.3. *Let M be a real hypersurface immersed in an almost hyperbolic Hermitian manifold. If a holomorphic vector field $\widetilde{X}$ in $\widetilde{M}$ is tangent to M, then the restriction X of $\widetilde{X}$ to M is a pseudo-conformal vector field in M.*

Let (f, ξ, θ) be an almost para contact structure induced in a coordinate neighborhood U of M and assume that ξ is a pseudo-conformal vector field in U. Then (f, ξ, θ) is said to be regular [5]. If this is the case, (5.1) implies

$$P^a_b = L_\xi f^a_b = 0,$$

because$L_\xi \xi = 0$ and (5.1) gives $V = 0$ and $a = 0$. Therefore (3.25) and (3.28) implies

$$-H^a_e f^e_b + f^a_e H^e_b - l_b \xi^a \qquad (\bmod \theta_b)$$

if and only if (f, ξ, θ) is regular.Thus we have, from Proposition 3.3.

Proposition 5.4. *In a real hypersurface M immersed in an almost hyperbolic Hermitian manifold, an induced almost para contact structure of M is normal if and only if it is regular.*

Proposition 5.3 is however a consequence of Proposition 3.1.

Let (Z^λ) be a system of complex coordinates in a coordinate neighborhood $\widetilde{U}$ of the ambient almost hyperbolic Hermitian manifold M. Then we have

Proposition 5.5. *The condition (5.2) is equivalent to each of the following conditions*

$$\frac{\partial \widetilde{X}^\lambda}{\partial \overline{z}^\mu} = 0 \tag{5.11}$$

$$d\widetilde{X}^\lambda \wedge dz^1 \wedge \cdots \wedge dz^{n+1} = 0 \tag{5.12}$$

It is easily verified that condition (5.8) is equivalent to (5.7) which is equivalent to (5.5) and hence to (5.4). Thus we have

Proposition 5.6. *Condition* (5.8) *is equivalent to* (5.4) *or to*

$$d\widetilde{X}^{\lambda} \wedge d(z^1 oi) \wedge \cdots \wedge d(z^{n+1} oi) = 0 \tag{5.13}$$

where $i : M \to \widetilde{M}$ *is the immersion of* M.

Let there be given a vector field X in M and put $\widehat{X}^h = X^a B_a^h$ where X^a are components of X in U. Then we have

Proposition 5.7. *The condition* (5.8) *for a vector field* $X^a \partial/\partial y^a$ *tangent to* M *is equivalent to the condition*

$$d\widetilde{X}^{\lambda} \wedge d(z^1 oi) \wedge \cdots \wedge d(z^{n+1} oi) = 0 \tag{5.14}$$

where $i : M \to \widetilde{M}$ *is the immersion of* M *and* $\widetilde{X}^h = X^a B_a^h$.

We now assume that M is a real hypersurface analytically immersed in M and that X is an analytic vector field in M. Then, as is well known, the differential equation (5.12) with unknown functions $\widetilde{X}^{\lambda}(z^{\mu}, \overline{z}^{\mu})$ has a local solution $\widetilde{X}^{\lambda}$ satisfying the boundary condition

$$(\widetilde{X}^{\lambda} oi) = \widetilde{X}^{\lambda}$$

along M, when X satisfy condition (5.13) [5]. Therefore, taking account of Proposition 5.3 and Lemmas 5.5, 5.6 and 5.7, we can prove the following proposition:

Proposition 5.8. *Let* M *be a real hypersurface analytically immersed in an almost hyperbolic Hermitian manifold* $\widetilde{M}$. *Then an analytic vector field* X *in* M *is pseudo-conformal if and only if, for any point* P *belonging to* M, *there are a neighborhood* O *of* M *containing* P *and a homomorphic vector field* $\widetilde{X}$ *in* $\widetilde{O}$*such that* X *is the restriction of* $\widetilde{X}$ *to* $\widetilde{O} \cap M$.

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References

[1] E. Cartan, *Sur la gomtrie pseudo-conforme des hypersurfaces de deux variables complexes. I*, Ann. Math. Pura Appl. (4) **11** (1932), 17–90 (or Oeuvres. II, 2, 1231–1304); II, Ann. Scuola Norm. Sup. Pisa, (2) **1** (1932), 333–354 (or Oeuvres. III, 2, 1217-1238).

[2] S. S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. **133** (1974), 219–271.

[3] S. Ishihara, *Distributions with complex structure*, Kodai Math. J., **1**(1978), 164–276.

[4] S. Sasaki, Almost contact manifolds, Part I, Lecture Notes, Tohoku University, Sendai, Japan, 1965.

[5] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of n complex variables*, J. Math. Soc. Japan **14** (1962), 397–429.

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