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GENERALIZED f-NONEXPANSIVE R-SUBWEAKLY COMMUTING MULTIVALUED MAPS

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Abstract. We prove coincidence point theorems for the generalized f-nonexpansive R-subweakly commuting multivalued maps. Our results generalize and extend well known results for noncommuting maps.

1. Introduction and Preliminaries

In 1941, Kakutani [7] generalized the Brouwer fixed point theorem to multivalued mappings. Subsequently, Schauder fixed point theorem was extended to multivalued version by Bohenblast and Karlin [2]. On the other hand, Nadler [10], in 1969 extended the well known Banach's contraction mapping principle to multivalued contractions. Since then, this discipline has been further developed by Daffer and Kaneko [3], Mizoguchi and Takahashi [9], Beg and Azam [1], Itoh and Takahashi [5]and so on. Introducing the notion of multivalued R-subweakly commuting mappings, Shahzad [14] has established the validity of Latif and Tweddle's[8] result for this new class of mappings, thereby improving the results of Dotson [4], Jungck and Sessa [6]and Latif and Tweddle [8].

In this paper, we prove the coincidence point theorem for generalized f-nonexpansive R-subweakly commuting multivalued mapping and also obtain common fixed point. Our results extend well known results of Shahzad [12-15], Latif and Tweddle [8] etc.

Let $X = (X, d)$ be a metric space and S, a nonempty subset of X. We denote by $CB(S)$, the family of nonempty closed bounded subsets of S and by $K(S)$, the family of nonempty compact subsets of S. Let H be the Hausdorff metric on $CB(S)$ induced by the metric d and $T : S \to CB(S)$ a multivalued map.

We need the following basic definitions to prove our main results.

Definition 1.1. A multivalued map $T : S \to CB(S)$ is said to be a contraction if there exists $0 \leq \lambda < 1$ such that $H(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in S$. If $\lambda = 1$, then T is called nonexpansive.

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Definition 1.2. Let $f : S \to S$ be a continuous map. A multivalued map $T : S \to CB(S)$ is called an f - contraction if there exists $0 \leq \lambda < 1$ such that $H(Tx, Ty) \leq \lambda d(fx, fy)$ for all $x, y \in S$. If $\lambda = 1$, then T is called an f-nonexpansive map.

Definition 1.3. Let S be a nonempty subset of a Banach space X. Then the set S is called $p-starshaped$ with $p \in S$ if $\lambda x + (1-\lambda)p \in S$ for all $x \in S$ and all real λ with $0 \leq \lambda \leq 1$.

A point $x \in S$ is a fixed point of $T : S \to CB(S)$ if $x \in Tx$. Let the set of fixed points of T be denoted by $F(T)$ and the set of coincidence points of f and T is represented by $C(f, T)$.

Definition 1.4.

1. The pair $\{f, T\}$ is called commuting if $Tfx = fTx$ for all $x \in S$.
2. The pair $\{f, T\}$ is called R-weakly commuting if for all $x \in S$, $fTx \in CB(S)$ and there exists $R > 0$ such that $H(fTx, Tfx) \leq Rd(fx, Tx)$.
3. Suppose S is p-starshaped, then the pair $\{f, T\}$ is called R-subweakly commuting if for all $x \in S$, $fTx \in CB(S)$ and there exists $R > 0$ such that $H(fTx, Tfx) \leq Rd(fx, A_\lambda x)$ for every $0 \leq \lambda \leq 1$ where $A_\lambda x = \lambda Tx + (1-\lambda)p$ and $d(fx, A_\lambda x) = inf\{\|fx - y_\lambda\| : y_\lambda \in A_\lambda x\}$.

Obviously, Commuting maps are R-subweakly commuting, but the converse is not true in general. However, R-subweakly commuting maps commutes at their coincidence points. Moreover, R-subweakly commuting maps are R-weakly commuting and the converse is not true in general.

Definition 1.5. A multivalued map $T : S \to CB(S)$ is said to be demiclosed at $y_0 \in X$ if whenever $\{x_n\} \subset S$ and $\{y_n\} \subset X$ with $y_n \in Tx_n$ are sequences such that $\{x_n\}$ converges weakly to x_0 and $\{y_n\}$converges strongly to y_0 in X, then $y_0 \in Tx_0$.

We shall make use of the following useful lemma.

Lemma 1.6.([10]) *Let $A, B \in CB(S)$ and $\alpha > 1$. Then for each $x \in A$, there exists an element $y \in B$ such that $d(x, y) \leq \alpha H(A, B)$.*

2. Main Results

Theorem 2.1. *Let X be a complete metric space. Suppose f is a continuous self mapping of X and $T : X \to CB(X)$ a continuous multivalued mapping such that $T(X) \subset f(X)$. If the pair $\{f, T\}$ is R - weakly commuting and there exists $0 \leq k < 1$ such that*

$$H(Tx, Ty) \leq k\ max\{d(fx, fy), dist(fx, Tx), dist(fy, Ty), \frac{1}{2}[dist(fx, Ty) + dist(fy, Tx)]\}$$

for all $x, y \in X$, *then* $C(f,T) \neq \phi$.

Proof. Let $x_0 \in X$ be arbitrary. Now choose a real number α such that $1 < \alpha < \frac{1}{k}$. Since $T(X) \subset f(X)$, there exists $x_1 \in X$ such that $fx_1 \in Tx_0$. By Lemma 1.6, there exists $u_1 \in Tx_1$ and $\alpha > 1$ such that

$$d(u_1, fx_1) \leq \alpha H(Tx_1, Tx_0)$$

Then there exists $x_2 \in X$ such that $u_1 = fx_2$. Therefore $fx_2 \in Tx_1$.

$$d(fx_2, fx_1) \leq \alpha H(Tx_1, Tx_0)$$

Continuing in this fashion, we get

$$\begin{aligned} d(fx_n, fx_{n-1}) &\leq \alpha H(Tx_{n-1}, Tx_{n-2}). \\ &\leq \alpha k \ \max\{d(fx_{n-1}, fx_{n-2}), dist(fx_{n-1}, Tx_{n-1}), \\ &\quad dist(fx_{n-2}, Tx_{n-2}), \frac{1}{2}[dist(fx_{n-1}, Tx_{n-2}) + dist(fx_{n-2}, Tx_{n-1})]\} \\ &\leq \alpha k \ \max\{d(fx_{n-1}, fx_{n-2}), d(fx_{n-1}, fx_n), \\ &\quad d(fx_{n-2}, fx_{n-1}), \frac{1}{2}[d(fx_{n-2}, fx_n)]\} \\ &\leq \alpha k d(fx_{n-1}, fx_{n-2}) \end{aligned}$$

This shows that $\{fx_n\}$ is a Cauchy sequence in X. As X is complete, there exists $z \in X$ such that have

$$lim_{n\to\infty} fx_n = z$$

Now, we shall show that z is the coincidence point of f and T.
As $fx_n \in Tx_{n-1}$ and T is continuous, it follows that $H(Tfx_n, Tz) \to 0$ as $n \to \infty$.
Now by Lemma 1.6 and as the pair $\{f, T\}$ is R-weakly commuting

$$\begin{aligned} d(ffx_n, Tz) &\leq H(fTx_{n-1}, Tz) \\ &\leq H(fTx_{n-1}, Tfx_{n-1}) + H(Tfx_{n-1}, Tz) \\ &\leq Rd(fx_{n-1}, Tx_{n-1}) + H(Tfx_{n-1}, Tz) \end{aligned}$$

On letting $n \to \infty$, we have $d(fz, Tz) \to 0$. Therefore $fz \in Tz$, that is $z \in C\{f, T\}$.
Hence proved that $C\{f, T\} \neq \phi$.

Remark 2.2. The above theorem generalizes corollary 6 of Shahzad and Kamran [13].

Remark 2.3. For single valued version of the above theorem, one may refer to Shahzad [15].

Theorem 2.4. *Let S be a nonempty closed and bounded subset of a Banach Space X, $f : S \to S$ be a continuous affine mapping with respect to p, and $T : S \to CB(S)$ be a* *continuous multivalued mapping such that* $T(S) \subset f(S)$*. Suppose* S *is p-starshaped with* $p \in F(f)$ *and the pair* $\{f, T\}$ *is R-subweakly commuting satisfying*

$$H(Tx,Ty) \leq max\{d(fx,fy), dist(fx, A_\lambda x), dist(fy, A_\lambda y), \frac{1}{2}[dist(fx, A_\lambda y) + dist(fy, A_\lambda x)]\} \tag{2.1}$$

for all $x, y \in S$*, where* $A_\lambda x = \lambda Tx + (1-\lambda)p$ *for* $\lambda \in [0,1]$*, and* $dist(fx, A_\lambda x) = inf\{\|fx - y_\lambda\| : y_\lambda \in A_\lambda x\}$*. Further, if* $(f-T)S$ *is closed, then* $C(f,T) \neq \phi$*. If in addition,* $y \in C(f,T)$ *implies the existence of* $lim_{n\to\infty} f^n y$*, then* $F(f) \cap F(T) \neq \phi$*.*

Proof. Choose a sequence $\{\lambda_n\} \subset (0,1)$ such that $\lambda_n \to 1$ as $n \to \infty$.

Then for each n, define $T_n : S \to CB(S)$ as $T_n x = (1-\lambda_n)p + \lambda_n Tx$ for each $x \in S$.

Then for each n , $T_n(S) \subset f(S)$, since f is affine with respect to p and $T(S) \subset f(S)$. Also, for all $x, y \in S$

$$\begin{aligned} H(T_n x, T_n y) &= \lambda_n H(Tx,Ty) \\ &\leq \lambda_n \max\{\|fx - fy\|, dist(fx, A_{\lambda_n} x), dist(fy, A_{\lambda_n} y), \\ &\quad \frac{1}{2}[dist(fx, A_{\lambda_n} y) + dist(fy, A_{\lambda_n} x)]\} \\ &= \lambda_n \max\{\|fx - fy\|, dist(fx, T_n x)), dist(fy, T_n y), \\ &\quad \frac{1}{2}[dist(fx, T_n y) + dist(fy, T_n x)]\} \end{aligned}$$

Therefore, each T_n is a generalized f - contraction.

Further, it follows from the R-subweak commutativity of the pair $\{f, T\}$ that $fTx \in CB(S)$. Moreover, as f is affine with respect to p, we have $fT_n x \in CB(S)$ and

$$\begin{aligned} H(T_n fx, fT_n x) &= \lambda_n H(Tfx, fTx) \\ &\leq R\lambda_n \text{ dist}(fx, T_n x) \end{aligned}$$

for all $x \in S$. Thus $\{f, T_n\}$ is $R\lambda_n$ - weakly commuting for each n.

By theorem 2.1, $C(f, T_n) \neq \phi$.

Therefore $fx_n \in T_n x_n$ for some $x_n \in S$. That is $fx_n \in \lambda_n Tx_n + (1-\lambda_n)p$.

Hence there exists $y_n \in Tx_n$ such that $fx_n = \lambda_n y_n + (1-\lambda_n)p$.

Hence, $fx_n - y_n = (1-\lambda_n)(p - y_n)$. Since $T(S)$ is bounded, $fx_n - y_n \to 0$ as $n \to \infty$.

The closedness of $(f-T)S$, further implies that $0 \in (f-T)S$.

Hence $C(f,T) \neq \phi$.

As a consequence of the R-subweak commutativity property, the pair $\{f, T\}$ commutes on $C(f,T)$ and it follows that $f^n y = f^{n-1} fy \in f^{n-1} Ty = Tf^{n-1} y$ for some $y \in C(f,T)$.

Let $lim_{n\to\infty} f^n y = x_0$. Then taking $n \to \infty$, we get $x_0 \in F(T)$.

Also $x_0 \in F(f)$. Thus $F(T) \cap F(f) \neq \phi$.

Theorem 2.5. *Let* S *be a nonempty weakly compact p-starshaped subset of a Banach space* X*. Suppose* $f : S \to S$ *is a continuous affine mapping with respect to* p *where* $p \in$

$F(f)$, and $T : S \to K(S)$ is a continuous multivalued mapping such that $T(S) \subset f(S)$. If the pair $\{f, T\}$ is R-subweakly commuting satisfying (2.1) and $(f - T)S$ is demiclosed at 0, then $C(f, T) \neq \phi$. If in addition, $y \in C(f, T)$ implies the existence of $lim_{n\to\infty} f^n y$, then $F(f) \cap F(T) \neq \phi$.

Proof. As in the proof of Theorem 2.4, $fx_n - y_n \to 0$ as $n \to \infty$. By the weak compactness of S, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y \in S$ weakly. But f being affine and continuous, is weakly continuous and the weak topology is Hausdorff, we have $fy = y$. As S is bounded, $fx_m - y_m \in fx_m - Tx_m \to 0$ as $m \to \infty$. Now, since $(f - T)$ is demiclosed at 0, we have $0 \in (f - T)y$. Hence $C(f, T) \neq \phi$. Again as in the proof of Theorem 2.4, $F(f) \cap F(T) \neq \phi$.

Remark 2.6. Theorem 2.4 generalizes Theorem 2.1 of Shahzad [14]. For single valued version, Theorem 2.2 of Shahzad may be referred [15].

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