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COMMUTATIVE GROUP ALGEBRAS AND PRÜFER GROUPS

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Abstract. Suppose G is a multiplicatively written abelian p-group, where p is a prime, and F is a field of arbitrary characteristic. The main results in this paper are that none of the Sylow p-group of all normalized units $S(FG)$ in the group ring FG and its quotient group $S(FG)/G$ cannot be Prüfer groups. This contrasts a classical conjecture for which $S(FG)/G$ is a direct factor of a direct sum of generalized Prüfer groups whenever F is a perfect field of characteristic p.

I. Introduction

Throughout the present paper, let G be an arbitrary multiplicative abelian group, let R be a commutative unitary ring of prime characteristic p and let K be a field of characteristic different from p. As usual, RG and KG are the group algebras over R and K respectively, $S(RG)$ is the p-torsion component of the group of all normalized units $V(RG)$ in RG (note that $V(RG) = S(RG)$ when G is a p-group), $S(KG)$ is the group of all normalized p-elements in KG, and G_p is the maximal p-primary subgroup of G. Moreover, $N(R)$ denotes the Baer radical (often called nil-radical) of R. Given a subgroup H of G and a subring L of R containing the same identity, $I(LG; H)$ denotes the relative augmentation ideal of LG with respect to H. Terminology and notations follow [12], [15] and [16].

For instance, following [12] and [13], we shall say that the abelian p-group A is a *Prüfer group* if A^{p^ω} is cyclic of order p and $A/A^{p^\omega} = \oplus_{n<\omega} Z(p^n)$, where $Z(p^n)$ is a cyclic group of order p^n. Certainly, A is an infinite countable group, and the non-zero Ulm-Kaplansky functions of A/A^{p^ω} are equal to 1. In fact, we know that $BG^{p^\omega}/G^{p^\omega}$ is a p-basic subgroup of G/G^{p^ω} whenever B is a p-basic subgroup of G. So, for the n-Ulm-Kaplansky functions f_n, where $n \geq 0$ is an integer, it follows that $f_n(G/G^{p^\omega}) = f_n(BG^{p^\omega}/G^{p^\omega})$ and $f_n(G) = f_n(B)$ (see, for instance, [12, v. I, section 34, p. 170 and p. 173, Exercise 2]). But $BG^{p^\omega}/G^{p^\omega} \cong B$, therefore $f_n(G/G^{p^\omega}) = f_n(G)$ whenever $n \geq 0$. Consequently, if A is a Prüfer abelian p-group, then $f_\alpha(A) = 1$ for each $\alpha \leq \omega$ and $f_\alpha(A) = 0$ for each $\alpha > \omega$.

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From [19, Theorem 6], an abelian p-group A such that A^{p^ω} is countable and A/A^{p^ω} is a direct sum of cyclic groups can be decomposed as $C \times D$, where C is countable and D is a direct sum of cyclic groups. In the case of Prüfer groups we observe that $D = 1$.

In the theory of commutative group algebras there exists a long-standing conjecture (see, e.g., [1]) stating that $S(RG)/G$ is a direct factor of a direct sum of generalized Prüfer groups, defined as in [12, v. II, section 81, pp.103-104], whenever G is a p-group and R is a perfect ring with characteristic p.

Under this point of view, we are motivated to give a criterion in order to check whether $S(RG)$ and $S(KG)$ are Prüfer groups. The same purpose we pursue for $S(RG)/G$ and $S(KG)/G$, provided G is a p-group. The main results of this article show that none of $S(RG)/G$ and $S(KG)/G$ cannot be Prüfer's groups, against ignoring "a direct sum" in the above conjecture. For some positive solutions of this conjecture the reader can see [1], [9], [14], [17] and [18].

II. Main results

The present paper extends [2, Section D]. The theorems are distributed into two sections.

Commutative modular group algebras and Prüfer groups

The next two lemmas are straightforward and, therefore, their proofs are omitted.

Lemma 1. *If g is an element of finite order of G, then*

$$|\langle g\rangle| = o(g).$$

where $\langle g\rangle$ is a finite cyclic group and $o(g)$ is the order of g.

Lemma 2. (a) *Assume G is p-primary, and $k \in \mathbb{N}$. Then $S^{p^k}(RG) = 1$ if and only if $G^{p^k} = 1$.*

(b) *Assume R has no nilpotent elements and $k \in \mathbb{N}$. Then $S^{p^k}(RG) = 1$ if and only if $G_p^{p^k} = 1$.*

Lemma 3.([3, p. 8]) *We have $S(RG) = G_p$ if and only if one of the following conditions is true:*

(i) $G = 1$;

(ii) $G \neq G_p = 1$ *and* $N(R) = 0$;

(iii) $|G| = 2$ *and* $|R| = 2$.

Proposition 1. *$V(RG)$ is a cyclic p-group if and only if one of the following equalities holds:*

(1) $G = 1$;

(2) $p = |G| = |R| = 2$.

Proof. Obviously, if $V(RG)$ is cyclic, then G is cyclic. From Lemmas 1 and 2(a), we deduce $|V(RG)| = |G|$ and so $V(RG) = G$. Lemma 3 concludes the necessity.

Conversely, the result follows again by Lemma 3.

We shall now generalize Proposition 1.

Proposition 2. *$S(RG)$ is cyclic if and only if one of the following is valid:*

(1) $G = 1$;

(2) $G \neq G_p = 1$ and $N(R) = 0$;

(3) $|G| = 2$ and $|R| = 2$;

(4) $G_p = 1$, $|G| = 2$ and $|N(R)| = p \geq 3$.

Proof. First, assume that $S(RG)$ is cyclic. We distinguish two cases, namely:

Case 1: $N(R^{p^i}) = N^{p^i}(R) \neq 0$ for all $i \in \mathbb{N}$.

(a) Since there is $g \in G$ with $g^n \neq 1$ for every $n \in \mathbb{N}$, G is either torsion-free or mixed. Let $0 \neq r \in N(R)$. For $m \in \mathbb{N}$ we construct the infinite number of different elements $1+r(1-g^{p^m}) \in S(R\langle g\rangle)$ where $S(R\langle g\rangle)$ is cyclic, hence finite, as a subgroup of $S(RG)$. Note that $\langle g\rangle$ is an infinite cyclic group in this situation. Therefore, $S(R\langle g\rangle) = 1$, i.e., $g = 1$ which is a contradiction.

(b) By what we have just shown in the previous point, G must be torsion.

   (b.1) $N(R^{p^{i'}}) = N(R^{p^{i'+1}})$ for some $i' \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Thus $1+I(N(R^{p^{i'}})G_q; G_q) = 1$ being both a divisible and cyclic group since G_q is p-divisible whenever $q \neq p$ is a prime. Therefore, $G_q = 1$, i.e., G is p-primary.

   (b.2) $N(R^{p^i}) \neq N(R^{p^{i+1}})$ for each $i \in \mathbb{N}_0$. Then the sequence $N(R)$, $N(R^p)$, $N(R^{p^2})$, …, $N(R^{p^i})$, … has an infinite number of different members. Choose the elements $x_i = 1+r_i^{p^i}(1-g_q)$ where $r_i \in N(R)$; $r_i^{p^i} \in N(R^{p^i})$ with $r_i^{p^i} \neq r_{i+1}^{p^{i+1}}$ and $g_q \in G_q$. Thus $x_i \neq x_{i+1}$ and $S(RG_q)$ is infinite cyclic, i.e., $S(RG_q) = 1$, whence $G_q = 1$. Finally, we conclude that G is p-torsion.

In that aspect Proposition 1 substantiates our claim.

Case 2: $N(R^{p^j}) = 0$ for some $j \in \mathbb{N}_0$.

If P is a commutative ring with unity of characteristic p such that $N(P) = 0$, as in Proposition 1, $S(PG)$ cyclic yields $S(PG) = G_p$, because we have $\exp(S(PG)) = \exp(G_p)$ by Lemma 2(b).

Consequently, $S(RG)$ cyclic gives $S(R^{p^j}G)$ is cyclic, hence in view of the conclusions above, $S(R^{p^j}G) = G_p$. So, Lemma 3 leads us to $G = 1$, or $G \neq G_p = 1$, or $G = G_p \neq 1$, $|R^{p^j}| = 2$, $|G| = 2$. We will consider the following two subcases.

(2.1) Assume $G_p \neq 1$. Hence R is a field which follows from Proposition 1 since G is a p-group.
We also note that R^{p^j} is a field and $N(R) = 0$, whence R is a field. If $0 \neq r \in R$, then $0 \neq r^{p^j} \in R^{p^j}$. So, there exists some $0 \neq \alpha \in R^{p^j}$ such that $r^{p^j} \cdot \alpha = r(r^{p^j-1} \cdot \alpha) = 1$. Therefore, r is invertible in R and this allows us to deduce that R is a field.

(2.2) Assume $G_p = 1$ and $N(R) \neq 0$. Each element of $S(RG)$ will be of the form $1 + r_1 g_1 + \ldots + r_k g_k$, where $0 \neq r_i \in N(R)$ with $\sum_{i=1}^{k} r_i = 0$ and $g_i \in G$; $1 \leq i \leq k$. Clearly, $|N(R)| \leq |S(RG)|$ and $|G| \leq |S(RG)|$ since $1 + r(1-g) = 1 + r'(1-g')$ if and only if $r = r'$ and $g = g'$, where $r, r' \in N(R) \setminus \{0\}$ and $g, g' \in G \setminus \{1\}$. Since $tr \neq 0$ whenever $1 \leq t \leq p-1$, we obtain that $|N(R)| \geq p$. On the other hand, if $|G| \geq 3$, then as above $|N(R)| < |S(RG)|$ whenever $|S(RG)| < \aleph_0$.

Now consider $S(RG) = \{1, v, v^2, \ldots, v^{p^m-1} | v^{p^m} = 1\}$, where $1 \leq m \leq j$; thus $|S(RG)| = p^m$. Moreover, $|R| \geq 4$, $|N(R)| \leq p^m$ and $|G| < p^m$; we observe that $|G| = p^m$ means $G = G_p = 1$.

Suppose $j = 1$ (hence $m = 1$). But $|S(RG)| = p$ and $p \geq 3$, since in the remaining case when $p = 2$ we deduce $|G| < 2$, i.e., $G = 1$. Because of the above given inequalities, we obtain $|N(R)| = p$ so $N(R) = \{0, r, 2r, \ldots, (p-1)r | r^2 = 0\}$ and $N^p(R) = N(R^p) = 0$. Utilizing the above ideas, it follows that $|G| = 2$.

Now, let $m \geq 2$, so $j \geq 2$. Since $S(R^{p^{j-1}}G) \subseteq S(RG)$ is cyclic and $S^p(R^{p^{j-1}}G) = 1$, we conclude that $|S(R^{p^{j-1}}G)| = p$. Hereafter, the above step can be successfully employed to deduce that $|G| = 2$; of course $N(R^{p^{j-1}}) \neq 0$, otherwise $1 \leq m \leq j-1$ and the first point will be started for $j = 2$. That is why, $|N(R)| = p^m$ and $|N(R^{p^{j-1}})| = p$.

Evidently, every element of $S(RG)$ is of the form $1 + r(1-g)$, where $r \in N(R)$. If $r \in N(R)$, it follows that $\{0, r, 2r, \ldots, (p-1)r\} \subseteq N(R)$. Likewise, if $r \in N(R)$ with $r^s = 0$, $s \geq 2$ and $r^{s-1} \neq 0$, it holds that the set $\{r, r^2, \ldots, r^{s-1}\}$ of $s-1$ elements is a set of different nilpotent elements. All of these arguments lead us to the evaluation that $|N(R)| \geq 1 + (s-1)(p-1)$ for $s \geq 2$. Since the sum of nilpotent elements is a nilpotent, there are $s-2$ nilpotent elements by considering the different sums $r + r^2, \ldots, r + r^{s-1}$. Finally, we compute that $|N(R)| \geq (s-1)(p-1) + 1 + s - 2 = (s-1)p$, $s \geq 2$; of course there is an exact estimation of the cardinality of the finite ideal $N(R)$, but however this inequality works.

So, if $S(RG)$ is cyclic of $\exp(S(RG)) = p^m$ for $m \geq 2$, it is routine to verify that there exists a nilpotent element $\beta \in R$ such that $\beta^{p^m} = 0$, $\beta^{p^m-1} \neq 0$ for $m \geq 2$ and $S(RG) = \langle 1 + \beta(1-g) \rangle$.

By what we have just shown for $s = p^m$ with $m \geq 2$, it follows that $|N(R)| \geq (p^m - 1)p > p^m$ whenever $p \geq 2$, $m \geq 2$; notice that $p^m - 1 > p^{m-1}$ holds even for $p \geq 3$, $m \geq 1$. This contradicts our equality $|N(R)| = p^m$.

Finally, we find that $m < 2$ and $j < 2$ when $S(RG)$ is cyclic. The result follows.

Conversely, assume that conditions (1)−(4) are satisfied. For points (1)−(3) we directly apply Lemma 3 to conclude that $S(RG) = G_p$ and since G_p is cyclic, the result follows. If now (4) holds, it is easily verified that every element of $S(RG)$ is of the type $1 + nr(1-g) = (1 + r(1-g))^n$, where $0 \leq n \leq p-1$ and $r \in N(R)$ with $r^2 = 0$. So, $S(RG)$ is a cyclic group of order p.

Remark 1. *$1 \neq S(RG)$ cyclic yields that $N(R)$ is finite. On the other hand, $1 \neq S(RG)$ cyclic implies that R is perfect provided $G_p \neq 1$. From [7, Example 9], $N(R) = 0$. Actually, Cases 1 and 2 in Proposition 2 may be reduced to $N(R) = 0$ and $G_p \neq 1$; or $N(R) \neq 0$ and $G_p = 1$, or $N(R) \neq 0$ and $G_p \neq 1$.*

The reader can see also [4, Theorem, pp.262-263] where we have established a criterion for $S(RG)$ to be a direct sum of cyclic groups of the same order p^t for $t \geq 1$.

The next constructions illustrate Proposition 2.

Example 1. Consider the following rings and their nil-radicals.

(1) $R = \{0, 1, -1\}$, char $(R) = 3$, $N(R) = \{0\}$;

(2) $R = \{0, 1, r, 1 - r | r^2 = 0\}$, char $(R) = 2$, $N(R) = \{0, r\}$;

(3) $R = \{0, 1, -1, r, 2r, r+1, r-1, -r-1, 1-r | r^2 = 0\}$, char $(R) = 3$, $N(R) = \{0, r, 2r\}$;

(4) $N(R) = \{0, r, r^2, r + r^2 | r^3 = 0\}$, char $(R) = 2$;

(5) $N(R) = \{0, r, 2r, r^2, 2r^2, r + r^2, -r - r^2, r - r^2, r^2 - r | r^3 = 0\}$, char $(R) = 3$.

Suppose now that $|S(RG)| = 4$ ($p = 2, m = 2$), that $|G| = 3$ with $G = \langle g \rangle$ and that R is as in 2). Therefore, $S(RG) = \{1, 1 + r(1 - g), 1 + r(1 - g^2), 1 + rg(1 - g)\}$. Because $S^2(RG) = 1, S(RG)$ is not cyclic but is a direct sum of the two cyclic groups $\{1, 1 + r(1 - g)\}$ and $\{1, 1 + r(1 - g^2)\}$ each of which is with order 2; we observe that $(1 + r(1 - g))(1 + r(1 - g^2)) = 1 + rg(1 - g)$.

Let now $p = 3$ and $m = 2$, as well as $|G| = 2$ and $N(R)$ be as in 5). Consequently, $S(RG) = \{1 + \alpha(1 - g) | \alpha \in N(R)\}$ is of power 9. But $S(RG)$ is not cyclic of order 9 since $S^p(RG) = 1$, i.e., $(1 + \alpha(1 - g))^3 = 1 + \alpha^3(1 - g^3) = 1$; note that $N^p(R) = 0$.

Proposition 3. *Suppose that G is a p-group. Then $V(RG)/G$ is cyclic if and only if one of the following conditions is true:*

(1) $G = 1$;

(2) $p = |R| = |G| = 2$;

(3) $p = |G| = 2$ *and* $|R| = 4$;

(4) $p = |R| = 2$, $G^2 \neq 1$ *and* $|G| = 4$;

(5) $p = |R| = |G| = 3$.

Proof. Let $V(RG)/G$ be a nontrivial cyclic group of order p^m for $m \geq 1$. So, $\overline{V} = V(RG)/G = \{\overline{1}, \overline{v}, \overline{v}^2, \ldots, \overline{v}^{p^m - 1} | \overline{v}^{p^m} = 1\}$ for some element $\overline{v} = (r_1 g_1 + \ldots + r_n g_n)G$; $0 \neq r_i \in R$ with $\sum_{i=1}^{n} r_i = 1$, $g_i \in G$; $1 \leq i \leq n$. If G is decomposable, then by [1, Theorem (Direct Factor)], $V(RG)/G$ must be decomposable (see also [9]). But this is impossible and thus G is indecomposable.

Since $(V(RG)/G)^{p^m} = \overline{1}$, we have $V^{p^m}(RG)G = G$, i.e., $V(R^{p^m} G^{p^m}) = G^{p^m}$. Using Lemma 3, we derive $G^{p^m} = 1$, or $G^{p^m} \neq 1$, $p = |R^{p^m}| = |G^{p^m}| = 2$. Both R and G are finite. Moreover either $G^{p^k} \neq 1$ and $|R^{p^k}| > 2$, or $|G^{p^k}| > 2$, for $0 \leq k < m$.

We shall distinguish five cases:

Case 1: $p = 2$, $m = 1$. Thus $V(RG)/G = \{\overline{1}, \overline{v} | \overline{v}^2 = 1\}$. Assume that $|R| > 2$. Hence, $\overline{1} \neq [1 + r(1 - g)]G = \overline{v}$ for $0, 1 \neq r \in R$ and $1 \neq g \in G$ with $[1 + r(1 - g)]^2 \in G$, i.e., $1 + r^2(1 - g^2) \in G$. This is equivalent to $r^2 = 0$ or $r^2 = 1$ or $g^2 = 1$. Besides, we consider $[1 + r'(1 - h)]G$ for some $r' \in R \setminus \{0, 1, r\}$ and $h \in G \setminus \{1, g\}$ such that $[1 + r'(1 - h)]G = [1+r(1-g)]G$, i.e., $r' = 1+r$ and $h = g^{-1}$. Consequently, $[1+r(1-g)]G \neq [1+r(1-g^{-1})]G$ when $g \neq g^{-1}$. That is why, $|G| = 2$ and $|R| = 4$.

Let us now $|R| = 2$, whence $|G| > 2$ and more precisely $|G| \geq 4$. This will be studied in the next case.

Case 2: $p = 2$, $m \geq 2$. Starting with $m = 2$, $V(LG)/G \subseteq V(RG)/G$ is cyclic of order 2 or 4 where $L = \{0, 1\} \leq R$. Since $G^4 = 1$ and $G^2 \neq 1$, or $G^4 \neq 1$ with $|R^4| = |G^4| = 2$, there exists $1 \neq g \in G$ such that $g^2 \neq 1$, $g^3 \neq 1$ whence $\{1, g, g^2, g^3 | g^4 = 1\} \leq G$.

First, $|V(LG)/G| = 2$, and $|G| = 4$, i.e., $G = \{1, g, g^2, g^3 | g^4 = 1\}$, so $G^2 = \{1, g^2\}$. Since $V(RG) = G \cup \{1 + g + g^2, 1 + g + g^3, 1 + g^2 + g^3, g + g^2 + g^3\}$, it is immediate that $\overline{V} = \{\overline{1}, [1 + g + g^2]G\}$.

If now $|G| = 8$, i.e., $G = \{1, g, \ldots, g^7 | g^8 = 1\}$, we observe that $\overline{V}$ contains five different elements, that are $\overline{1}$, $(1 + g + g^2)G$, $(1 + g + g^3)G$, $(1 + g + g^2 + g^3 + g^4)G$ and $(1 + g + g^2 + g^3 + g^4 + g^5 + g^6)G$. This contradicts the power of $\overline{V}$ which is precisely 4.

If $|G| = 4$ and $|R| = 4$, then $G = \{1, g, g^2, g^3 | g^4 = 1\}$ and $R = \{0, 1, r, 1 + r\}$. Furthermore, we see that $\overline{V} = \{\overline{1}, (1+g+g^2)G, (1+r-rg)G, (1+r-rg^2)G, (1+r-rg^3)G\}$ consists of five different elements because $1 + r \neq -r$, but this is not true. Thus $|R| < 4$ or $|G| < 4$, i.e., $|R| \leq 2$ or $|G| \leq 2$.

Next, for $m \geq 3$ we observe that $V(L^{p^{m-2}} G^{p^{m-2}})G/G \cong V(L^{p^{m-2}} G^{p^{m-2}})/G^{p^{m-2}}$ is a cyclic group of order p^2 as a subgroup of $V(LG)/G$. This is exactly the previous step. Thus, when $m \geq 2$, $V(RG)/G$ is not cyclic.

Case 3: $p = 3$, $m = 1$. Since $p = 3$ there is $r \in R$ with $r \neq 0, 1$, hence $\{0, 1, -1\} \subseteq R$. Moreover, $G^3 = 1$ and $G^2 \neq 1$. Let us now $|R| = 3$, i.e., $R = \{0, 1, -1\}$ and $|G| = 3$, that is, $G = \{1, g, g^2 | g^3 = 1\}$. Therefore, $V(RG) = \{1, 1 + g(1 - g), 1 - g(1 - g), -g - g^2, -1 - g, g, g^2\}$ and $\overline{V} = \{\overline{1}, [1 + g(1 - g)]G, [-1 - g]G\}$. Moreover, we calculate that $[1+g(1-g)]G = [1+g-g^2]G = [1-g+g^2]G = [1+2g+g^2]G = [-1-g]^2G = [(-1-g)G]^2$.

If we suppose that $|R| > 3$ or $|G| > 3$, i.e., $|R| \geq 9$ or $|G| \geq 9$, it is not difficult to obtain in the same manner that $|\overline{V}| > 3$, which is a contradiction.

Case 4: $p = 3$, $m \geq 2$. Start with $m = 2$. Certainly, $G^9 = 1$ and so $|G| = 9$ since otherwise if $|G| = 3$ it follows that $G^3 = 1$, a contradiction. Thus $G = \{1, g, g^2, \ldots, g^8 | g^9 = 1\}$ and as above $R \supseteq \{0, 1, -1\}$. Consider the elements $(1 + g - g^k)G$ for $2 \leq k \leq 8$. It is only a technical matter to check that $(1+g-g^k)G \neq (1+g-g^j)G$ whenever $g^k \neq g^j$ and $2 \leq j \leq 8$, because $1 \neq -1$. Moreover, two different elements are also $(1 + g + g^2 + g^3)G$ and $(1 + g + g^2 - g^3 - g^4)G$. A crucial approach here is that the canonical forms of these elements are with different lengths. Consequently, $\overline{V} \supseteq \{\overline{1}, (1 + g - g^k)G$ for $2 \leq k \leq 8$, $(1+g+g^2+g^3)G$, $(1+g+g^2-g^3-g^4)G\}$ contains ten elements. This gives a contradiction and finishes the step $m = 2$.

When $m \geq 3$ we have $|G| \geq 27$ and, therefore, we can copy the idea from Case 2.

Case 5: $p \geq 5$. Begin with $m = 1$. Since the characteristic of R is p and $R \supseteq \{0, 1, 2, \ldots, p-1\}$, it holds that $|R| \geq p$. Moreover, $G^p = 1$ and $G^s \neq 1$ for $1 \leq s \leq p-1$. It is a routine technical exercise to verify that $(1+(1-g^k))G = (2-g^k)G \neq (2-g^j)G = (1+(1-g^j))G$ when $g^k \neq g^j$ and $1 \leq k \neq j \leq p-1$ for some $g \in G$. In this way $(1+g(1-g))G \neq (2-g^k)G$ for all $1 \leq k \leq p-1$. Finally, $\overline{V}$ contains the set of $p+1$ different elements $\{\overline{1}, (2-g)G, (2-g^2)G, \ldots, (2-g^{p-1})G, (1+g(1-g))G\}$ while $|\overline{V}| = p$. This contradiction shows that this case cannot happen. After this, because $V(R^{p^{m-1}}G^{p^{m-1}})/G^{p^{m-1}} \cong V(R^{p^{m-1}}G^{p^{m-1}})G/G \subseteq V(RG)/G$ is cyclic of order p whenever $m \geq 2$, we conclude that the case is contradictory. This completes the necessity.

As for the sufficiency, we observe that for the first four situations we have $|V(RG)/G| = 1$, hence $V(RG)/G = \overline{1}$, or $|V(RG)/G| = 2$. The fifth dependence was considered in Case 3 above.

Example 2. There are four special commutative unitary rings of power 4 and with characteristic 2 which illustrate the criteria in Propositions 2 and 3. Specifically, they are the following:

(1) $R = \{0, 1, r, 1+r | r^2 = 0\}$, $N(R) = \{0, r\}$;

(2) $R = \{0, 1, r, 1+r | r^2 = 1\}$, $N(R) = \{0, 1+r\}$;

(3) $R = \{0, 1, r, 1+r | r^2 = r\}$, $N(R) = \{0\}$ and R has two zero divisors $\{r, 1+r\}$ which are idempotents, so R is perfect;

(4) $R = \{0, 1, r, 1+r | r^2 = 1+r\}$, $N(R) = \{0\}$ and R has three units $\{1, r, 1+r\}$, i.e., R is a perfect field.

We are now prepared to proceed by proving the main assertions. In the next two theorems we use results on Ulm-Kaplansky invariants of $V(RG)/G$ provided G is a p-group and R is a perfect ring of prime characteristic p (see details in [10, p. 138], Theorem 6 and p. 141, Remark]). Utilizing the same ideas, it easily follows that these Ulm-Kaplansky invariants are either infinite or zero when G is infinite and R is not necessarily perfect.

Theorem 1. *Suppose G is a p-group or R is a ring with no nilpotent elements. Then $S(RG)$ cannot be a Prüfer group.*

Proof. (1) Assume $G = G_p$ and by contradiction, let $S(RG) = V(RG)$ be a Prüfer group. Thus $V^{p^\omega}(RG) = V(R^{p^\omega}G^{p^\omega})$ is cyclic of order p, hence Proposition 1 and its proof guarantee that $V(R^{p^\omega}G^{p^\omega}) = G^{p^\omega}$. On the other hand, $V(RG)$ is countable and so $V(RG)/G$ is countable. Since G is nice in $V(RG)$ (cf., [10, p.135, Lemma 1]), we deduce that $(V(RG)/G)^{p^\omega} = V^{p^\omega}(RG)G/G = \overline{1}$, whence $V(RG)/G$ is separable. By the second Prüfer's theorem (see [12, v. I, Theorem 17.3]), $V(RG)/G$ is a direct sum of cyclic groups. Furthermore, because of the purity of G in $V(RG)$, a result due to L. Kulikov (e.g., [12,

v. I, Theorem 28.2]) is applicable to obtain $V(RG) \cong G \times (V(RG)/G)$. Therefore, $V(RG)/G^{p^\omega} \cong (G/G^{p^\omega}) \times (V(RG)/G)$. Assume $G = G^{p^\omega}$. Then G is both cyclic and divisible. This gives a contradiction when $G \neq 1$. Thus we conclude that $G \neq G^{p^\omega}$, whence G is infinite because $G^{p^\omega} \neq 1$. Moreover, $V(RG)/G^{p^\omega} = V(RG)/V^{p^\omega}(RG)$ has Ulm-Kaplansky functions equal to 1. On the other hand, conforming with [12, v. I, section 37, p.185, Exercise 8], these invariants for $V(RG)/G^{p^\omega}$ are equal to the sum of the Ulm-Kaplansky invariants of G/G^{p^ω} and $V(RG)/G$ respectively. Moreover, [10, p.138, Theorem 6] applies to show that $V(RG)/G$ has infinite Ulm-Kaplansky invariants when either G or R is infinite and $G \neq G^p$; as early observed $G \neq G^p$ holds. If both G and R are finite, then $S(RG)$ is obviously finite whence it is not Prüfer. Consequently, $V(RG) = G$, so Lemma 3 leads us to $|R| = |G| = 2$, which is the desired contradiction with the infinite cardinality of the Prüfer groups.

(2) Assume $N(R) = 0$. This case can be processed similarly as that in (1).

Theorem 2. *Suppose G is a p-group and R is a commutative unitary ring of prime characteristic p. Then $V(RG)/G$ cannot be a Prüfer group.*

Proof. Assume the contrary. In view of the definition and our assumption $(V(RG)/G)^{p^\omega} = V^{p^\omega}(RG)G/G = V(R^{p^\omega}G^{p^\omega})G/G \cong V(R^{p^\omega}G^{p^\omega})/G^{p^\omega}$ is cyclic of order p, whence $G^{p^\omega} \neq V(R^{p^\omega}G^{p^\omega})$ and thus G is infinite. However, by the above commentaries, $V(RG)/G$ should be with Ulm-Kaplansky functions precisely 1. But, complying with [10, p.138, Theorem 6], when G is infinite we deduce that these invariants computed for $V(RG)/G$ are infinite or 0. So, we obtain the wanted contradiction.

In case G is finite, we yield that $V(RG)/G$ is bounded whence it is not a Prüfer group.

The following illustrates Theorem 1.

Example 3. Consider $V(RG) = \oplus_{n<\omega}\mathrm{Z}(p^n)$. Evidently, $|V(RG)| = \aleph_0 \iff |R| + |G| = \aleph_0$ with $|R| \leq \aleph_0$ and $|G| \leq \aleph_0$. Besides, $V^{p^\omega}(RG) = 1 \iff G^{p^\omega} = 1$ and, for each $n \geq 1$, $V^{p^n}(RG) \neq 1 \iff G^{p^n} \neq 1$. By assumption, for every $k \geq 0$, the k-Ulm-Kaplansky invariants of $V(RG)$ are 1, while owing to [22, Theorem 7] they are equal to 0 or to $\max(|R^{p^k}|, |G^{p^k}|)$ if either $|R^{p^k}| \geq \aleph_0$ or $|G^{p^k}| \geq \aleph_0$. In the case $k = 0$, we obtain a contradiction. That is why, $V(RG) \neq \oplus_{n<\omega}\mathrm{Z}(p^n)$.

Another idea to show that the equality $V(RG) = \oplus_{n<\omega}\mathrm{Z}(p^n)$ is not true is like this: If yes, G should be a direct sum of cyclic groups, hence, in virtue of [5] or [6], so is $V(RG)/G$. Furthermore, as we have just seen above, $V(RG) = G \times V(RG)/G$. But then the Ulm-Kaplansky functions argument means that $V(RG) = G$, i.e., by Lemma 3, $V(RG)$ is finite which is against our hypothesis.

Commutative semisimple group algebras and Prüfer groups

Theorem 3. *Suppose G is a p-group and K is the first kind field with respect to p. Then $S(KG)$ cannot be a Prüfer group.*

Proof. Let $S(KG)$ be a Prüfer group. By definition, $S^{p^\omega}(KG)$ is cyclic of order p. Exploiting [20, Theorem 19], $S^{p^\omega}(KG)$ is divisible. Thus $S^{p^\omega}(KG) = 1$, a contradiction.

Note 1: In the situation of Theorem 3, $S(KG)/S^{p^\omega}(KG) \cong S(KG)$ has Ulm-Kaplansky invariants equal to 1. Taking into account [21, Theorem 7], $S(KG)$ possesses Ulm-Kaplansky functions equal to 0 or to $|B|$ where B is the basic subgroup of G. Hence, $B = 1$, i.e., G is divisible. Furthermore, in virtue of [8, Theorem 4], we derive that $S(KG)$ is divisible. But it is reduced, i.e., $S(KG) = 1$, a contradiction.

Theorem 4. *Suppose G is a p-group and K is a field of the first kind with respect to p. Then $S(KG)/G$ cannot be a Prüfer group.*

Proof. According to [11, Proposition 1], $(S(KG)/G)^{p^\omega}$ is always divisible (see also [8]). So, it cannot be a cyclic group of order p. That is why $S(KG)/G$ cannot be a Prüfer group, as asserted.

Note 2: As in Note 1, the Ulm-Kaplansky arguments from [11] are also applicable to deduce that $S(KG)/G$ cannot be, in fact, a Prüfer group.

Theorem 5. *Suppose G is a p-group and K is the second kind field with respect to p. Then $S(KG)$ cannot be a Prüfer group.*

Proof. Owing to [20, p.36 and Theorem 21], we find that $S(KG)$ is a direct sum of co-cyclic groups, hence it is not a Prüfer group.

Theorem 6. *Suppose G is a p-group and K is a field of the second kind with respect to p. Then $S(KG)/G$ cannot be a Prüfer group.*

Proof. If G is finite, $S(KG)$ is finite or divisible whence so is $S(KG)/G$. Therefore, it is not a Prüfer group.

When G is infinite, we employ [20, p.45, Theorem 21] to infer that $S(KG)/G$ need not be a Prüfer group. In fact, if $p \neq 2$, then $S(KG)$ is divisible, whereas if $p = 2$ and $G^p \neq 1$, then $S^p(KG)$ is divisible. Thus in both cases $(S(KG)/G)^p = S^p(KG)G/G$ is divisible, and consequently $(S(KG)/G)^{p^\omega} = (S(KG)/G)^p$ is not cyclic. When $G^p = 1$, we observe that $S(KG)$ is bounded by p, whence the same is $S(KG)/G$.

So, in any event, $S(KG)/G$ is not a Prüfer group, as expected.

Example 4. As in the modular case, one can illustrate in Theorem 3 that the equality $S(KG) = \oplus_{n<\omega} \mathrm{Z}(p^n)$ is not valid by applying [20] and [21]; see the proof of Theorem 3 as well.

Global case

Combining both the modular and semi-simple cases, we establish the following.

Global Theorem 7. *Let G be a p-group and let F be a field of arbitrary characteristic. Then $S(FG)$ and $S(FG)/G$ cannot be Prüfer groups.*

Proof. Each field has characteristic p or characteristic different from p. These fields with characteristic $\neq p$ are either of the first kind with respect to p or of the second kind with respect to p, respectively. Henceforth, the foregoing theorems work.

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