TAMKANG JOURNAL OF MATHEMATICS
Volume 41, Number 1, 97-107, Spring 2010

IMPLICIT TYPE VOLTERRA INTEGRODIFFERENTIAL EQUATION

B. G. PACHPATTE

Abstract. In this paper we study the existence, uniqueness and other properties of solutions of a certain nonlinear implicit type Volterra integrodifferential equation. The Banach fixed point theorem and a certain integral inequality with explicit estimate are used to establish the results.

1. Introduction

Let R^n denotes the real n-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$. We denote by $R_+ = [0, \infty)$ the given subset of R, the set of real numbers and $E = R^n \times R^n$ and $C(A, B)$ the class of continuous functions from the set A to the set B. In 1996 A. Constantin [3] studied the global existence of solution of the integrodifferential equation

$$x'(t) = F\Big(t, x(t), \int_0^t K(t, s, x(s))\, ds\Big), \quad x(0) = 0, \tag{1.1}$$

by using the topological transversality argument and a certain integral inequality with explicit estimate on the unknown function (see also [7, 10]). In **(author?)** [5, p.185] C. Corduneanu dealt with the Volterra functional-differential equation of the form

$$x'(t) + (Lx')(t) + f(x(t)) = g(t), \tag{1.2}$$

in which the operator L is defined by

$$(Lx)(t) = \int_0^t k(t, s)\, x(s)\, ds, \tag{1.3}$$

for $t \in R_+$. Many higher order differential equations of considerable interest can be reduced to the equation of the form (1.2). An important example of this is the Liénard equation with memory in the restoring force which we write as

$$x'' + f(x)\, x' + \int_0^t C(t, s)\, g(x(s))\, ds = e(t). \tag{1.4}$$

Received May 20, 2008; revised October 5, 2009.
2000 *Mathematics Subject Classification*. 34G20, 34A60.
Key words and phrases. Implicit type, Volterra integrodifferential equation, integral inequality, Banach fixed point theorem, Bielecki type norm, existence and uniqueness, boundedness and continuous dependence.

Integrating both sides of equation (1.4) from 0 to $t \in R_+$, one may arrive at a slight variant of equation of the form (1.2). The vast literature exists dealing with the special and even more general versions of equations (1.1), (1.2) and (1.4) by using different techniques (see [1, 5, 6, 9, 10]). Owing to the importance of equations of the forms (1.1), (1.2) and (1.4) arising in many physical problems, the simple, unified and concise treatment of these equations is desired. The main objective of the present paper is to study the existence, uniqueness and other properties of solutions of the following general implicit type Volterra integrodifferential equation

$$y'(t) = f\left(t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right), \tag{1.5}$$

with the given initial condition

$$y(0) = y_0, \tag{1.6}$$

for $t \in R_+$, where $f \in C(R_+ \times R^n \times R^n \times R^n, R^n)$ and for $0 \le s \le t < \infty$, $g \in C\left(R_+^2 \times R^n \times R^n, R^n\right)$. The main tools employed in the analysis are based on the applications of the well known Banach fixed point theorem (see **(author?)** [5, p. 37]) coupled with Bielecki type norm [2] and a suitable integral inequality with explicit estimate (see **(author?)** [8, Theorems 1.4.1 and 3.5.5]).

2. Existence and uniqueness

For a function $y(t)$ and its derivative $y'(t)$ in $C(R_+, R^n)$ we denote by $|y(t)|_E = |y(t)| + |y'(t)|$. Let S be the space of those functions $(\phi(t), \phi'(t)) \in E$ which are continuous for $t \in R_+$ and fulfil the condition

$$|\phi(t)|_E = O(\exp(\lambda t)), \tag{2.1}$$

for $t \in R_+$, where $\lambda > 0$ is a constant. In the space S we define the norm

$$|\phi|_S = \sup_{t \in R_+} \left[|\phi(t)|_E \exp(-\lambda t)\right]. \tag{2.2}$$

It is easy to see that S with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $N \ge 0$ such that $|\phi(t)|_E \le N \exp(\lambda t)$ for $t \in R_+$. Using this fact in (2.2) we observe that

$$|\phi|_S \le N. \tag{2.3}$$

We need the following integral inequality similar to those of given in **(author?)** [8, Theorems 1.4.1 and 3.3.5]. We shall give it in the following lemma for completeness.

Lemma. *Let $u(t), f(t) \in C(R_+, R_+)$ and for $0 \le s \le t < \infty$, $e(t,s)$, $\frac{\partial}{\partial t} e(t,s), k(t,s) \in C\left(R_+^2, R_+\right)$ and $c \ge 0$ is a constant. If*

$$u(t) \le c + \int_0^t \left[f(s) u(s) + e(t,s) u(s) + \int_0^s k(s,\sigma) u(\sigma)\, d\sigma\right] ds, \tag{2.4}$$

for $t \in R_+$, *then*

$$u(t) \leq c \exp\left(\int_0^t [f(s) + A(s)]\, ds\right), \tag{2.5}$$

for $t \in R_+$, *where*

$$A(t) = e(t,t) + \int_0^t \left\{k(t,\sigma) + \frac{\partial}{\partial t} e(t,\sigma)\right\} d\sigma. \tag{2.6}$$

Proof. Define a function $w(t)$ by the right hand side of (2.4). Then $w(t) \geq 0, w(0) = c, u(t) \leq w(t)$, $w(t)$ is nondecreasing in t and

$$\begin{aligned} w'(t) &= f(t)\, u(t) + e(t,t)\, u(t) + \int_0^t \frac{\partial}{\partial t} e(t,s)\, u(s)\, ds + \int_0^t k(t,\sigma)\, u(\sigma)\, d\sigma \\ &\leq f(t)\, w(t) + e(t,t)\, w(t) + \int_0^t \frac{\partial}{\partial t} e(t,s)\, w(s)\, ds + \int_0^t k(t,\sigma)\, w(\sigma)\, d\sigma \\ &\leq [f(t) + A(t)]\, w(t). \end{aligned} \tag{2.7}$$

The inequality (2.7) implies the estimate

$$w(t) \leq c \exp\left(\int_0^t [f(s) + A(s)]\, ds\right). \tag{2.8}$$

Using (2.8) in $u(t) \leq w(t)$ we get the required inequality in (2.5).

Now we are in a position to formulate the main result of this section.

Theorem 1. *Assume that*

(i) *the functions* f, g *in equation* (1.5) *satisfy the conditions*

$$|f(t,y,z,u) - f(t,\bar{y},\bar{z},\bar{u})| \leq k\,[|y-\bar{y}| + |z-\bar{z}|] + |u-\bar{u}|, \tag{2.9}$$

$$|g(t,s,y,z) - g(t,s,\bar{y},\bar{z})| \leq h(t,s)\,[|y-\bar{y}| + |z-\bar{z}|], \tag{2.10}$$

where $k \geq 0$ *is a constant and* $h \in C\left(R_+^2, R_+\right)$,

(ii) *for* λ *as in* (2.1)

(a) *there exists a nonnegative constant* α *such that* $\alpha < 1$ *and*

$$H(t) + \int_0^t H(s)\, ds \leq \alpha \exp(\lambda t), \tag{2.11}$$

for $t \in R_+$, *where*

$$H(t) = k \exp(\lambda t) + \int_0^t h(t,\sigma) \exp(\lambda\sigma) d\sigma, \tag{2.12}$$

(b) *there exists a nonnegative constant* β *such that*

$$|y_0| + \left| f\left(t, 0, 0, \int_0^t g(t, \sigma, 0, 0)\, d\sigma\right)\right| + \int_0^t \left| f\left(s, 0, 0, \int_0^s g(s, \sigma, 0, 0)\, d\sigma\right)\right| ds \leq \beta \exp(\lambda t). \tag{2.13}$$

Then the initial value problem (IVP for short) (1.5)–(1.6) *has a unique solution on* R_+.

Proof. Let $y(t) \in S$ and define the operator T by

$$(Ty)(t) = y_0 + \int_0^t f\left(s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right) ds. \tag{2.14}$$

Differentiating both sides of (2.14) with respect to t we get

$$(Ty)'(t) = f\left(t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right). \tag{2.15}$$

First we shall show that Ty maps S into itself. Evidently Ty is continuous on R_+ and $Ty \in R^n$. We verify that (2.1) is fulfilled. From (2.14), (2.15), using the hypotheses and (2.3), we have

$$\begin{aligned}
&|(Ty)(t)| + |(Ty)'(t)| \\
&\leq |y_0| + \int_0^t \left| f\left(s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right) \right. \\
&\quad \left. - f\left(s, 0, 0, \int_0^s g(s, \sigma, 0, 0)\, d\sigma\right)\right| ds \\
&\quad + \int_0^t \left| f\left(s, 0, 0, \int_0^s g(s, \sigma, 0, 0)\, d\sigma\right)\right| ds \\
&\quad + \left| f\left(t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right) \right. \\
&\quad \left. - f\left(t, 0, 0, \int_0^t g(t, \sigma, 0, 0)\, d\sigma\right)\right| \\
&\quad + \left| f\left(t, 0, 0, \int_0^t g(t, \sigma, 0, 0)\, d\sigma\right)\right| \\
&\leq \beta \exp(\lambda t) + \int_0^t \left\{ k\left[|y(s)| + |y'(s)|\right] + \int_0^s h(s, \sigma)\left[|y(\sigma)| + |y'(\sigma)|\right] d\sigma \right\} ds \\
&\quad + k\left[|y(t)| + |y'(t)|\right] + \int_0^t h(t, \sigma)\left[|y(\sigma)| + |y'(\sigma)|\right] d\sigma \\
&\leq \beta \exp(\lambda t) + |y|_S \left\{ \int_0^t H(s)\, ds + H(t) \right\} \\
&\leq [\beta + N\alpha] \exp(\lambda t).
\end{aligned} \tag{2.16}$$

From (2.16), we observe that

$$|Ty|_S \leq [\beta + N\alpha],$$

and hence it follows that $Ty \in S$. This proves that T maps S into itself.

Next, we verify that the operator T is a contraction map. Let $y(t), z(t) \in S$. From (2.14), (2.15) and using the hypotheses, we have

$$\begin{aligned}
&|(Ty)(t) - (Tz)(t)| + \left|(Ty)'(t) - (Tz)'(t)\right| \\
&\leq \int_0^t \left| f\left(s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right) \right. \\
&\quad \left. - f\left(s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right) \right| ds \\
&\quad + \left| f\left(t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right) \right. \\
&\quad \left. - f\left(t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right) \right| \\
&\leq \int_0^t \left\{ k[|y(s) - z(s)| + |y'(s) - z'(s)|] \right. \\
&\quad \left. + \int_0^s h(s, \sigma)[|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|]\, d\sigma \right\} ds \\
&\quad + k[|y(t) - z(t)| + |y'(t) - z'(t)|] \\
&\quad + \int_0^t h(t, \sigma)[|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|]\, d\sigma \\
&\leq |y - z|_S \left\{ \int_0^t H(s)\, ds + H(t) \right\} \\
&\leq |y - z|_S\, \alpha \exp(\lambda t). \qquad (2.17)
\end{aligned}$$

From (2.17), we observe that

$$|Ty - Tz|_S \leq \alpha\, |y - z|_S.$$

Since $\alpha < 1$, it follows from Banach fixed point theorem (see **(author?)** [5, p.37]) that T has a unique fixed point in S. The fixed point of T is however a solution of IVP (1.5)–(1.6). The proof is complete.

Remark 1. The norm $|\cdot|_S$ defined in (2.2) was first used by Bielecki [2] for proving global existence and uniqueness of solutions of ordinary differential equations. For a detailed discussion related to this topic, see [4] and the references cited therein.

The following theorem deals with the uniqueness of solutions of IVP (1.5)–(1.6) in the whole space R^n, without existence part.

Theorem 2. *Assume that the functions f, g in equation (1.5) satisfy the conditions*

$$|f(t,y,z,u) - f(t,\bar{y},\bar{z},\bar{u})| \le d[|y-\bar{y}| + |z-\bar{z}|] + |u-\bar{u}|, \tag{2.18}$$

$$|g(t,s,y,z) - g(t,s,\bar{y},\bar{z})| \le r(t,s)[|y-\bar{y}| + |z-\bar{z}|], \tag{2.19}$$

where d is a nonnegative constant such that $d < 1$ and for $0 \le s \le t < \infty$, $r(t,s)$, $\frac{\partial}{\partial t} r(t,s) \in C(R_+^2, R_+)$. Then the IVP (1.5)–(1.6) has at most one solution on R_+.

Proof. Let $y_1(t)$ and $y_2(t)$ be two solutions of IVP (1.5)–(1.6) and $u(t) = |y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|$. Then by hypotheses, we have

$$\begin{aligned}
u(t) \le & \int_0^t \Big| f\Big(s, y_1(s), y_1'(s), \int_0^s g(s,\sigma,y_1(\sigma),y_1'(\sigma))\,d\sigma\Big) \\
& - f\Big(s, y_2(s), y_2'(s), \int_0^s g(s,\sigma,y_2(\sigma),y_2'(\sigma))\,d\sigma\Big)\Big| ds \\
& + \Big| f\Big(t, y_1(t), y_1'(t), \int_0^t g(t,\sigma,y_1(\sigma),y_1'(\sigma))\,d\sigma\Big) \\
& - f\Big(t, y_2(t), y_2'(t), \int_0^t g(t,\sigma,y_2(\sigma),y_2'(\sigma))\,d\sigma\Big)\Big| \\
\le & \int_0^t \Big\{ d[|y_1(s) - y_2(s)| + |y_1'(s) - y_2'(s)|] \\
& + \int_0^s r(s,\sigma)[|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|]d\sigma \Big\} ds \\
& + d[|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|] \\
& + \int_0^t r(t,\sigma)[|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|]\,d\sigma.
\end{aligned} \tag{2.20}$$

From (2.20), we observe that

$$u(t) \le \frac{1}{1-d} \int_0^t \Big\{ du(s) + r(t,s)u(s) + \int_0^s r(s,\sigma)u(\sigma)\,d\sigma \Big\} ds. \tag{2.21}$$

Now a suitable application of Lemma (with $e(t,s) = k(t,s) = r(t,s)$ and $c = 0$) to (2.21) yields

$$|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)| \le 0,$$

which implies $y_1(t) = y_2(t)$ for $t \in R_+$. Thus there is at most one solution to the IVP (1.5)–(1.6) on R_+.

3. Boundedness and continuous dependence

In this section we shall study the boundedness of solutions of IVP (1.5)–(1.6) and the continuous dependence of solutions of equation (1.5) on the given initial data and the functions involved therein.

The following theorem contains the estimate on the solution of IVP (1.5)–(1.6).

Theorem 3. *Assume that the functions f, g in equation (1.5) satisfy the conditions*

$$|f(t,y,z,u)| \leq \gamma\,[|y| + |z|] + |u|, \tag{3.1}$$

$$|g(t,s,y,z)| \leq q(t,s)\,[|y| + |z|], \tag{3.2}$$

where γ is a nonnegative constant such that $\gamma < 1$ and for $0 \leq s \leq t < \infty$, $q(t,s), \frac{\partial}{\partial t} q(t,s) \in C\left(R_+^2, R_+\right)$. If $y(t)$, $t \in R_+$ is any solution of IVP (1.5)−(1.6), then

$$|y(t)| + |y'(t)| \leq \frac{|y_0|}{1-\gamma} \exp\left(\int_0^t \left[\frac{\gamma}{1-\gamma} + \bar{A}(s)\right] ds\right), \tag{3.3}$$

for $t \in R_+$, where

$$\bar{A}(t) = \frac{1}{1-\gamma}\left[q(t,t) + \int_0^t \left\{q(t,\sigma) + \frac{\partial}{\partial t} q(t,\sigma)\right\} d\sigma\right]. \tag{3.4}$$

Proof. Using the fact that $y(t)$ is a solution of IVP (1.5)−(1.6), the hypotheses and following closely the proof of Theorem 2, we have

$$\begin{aligned} |y(t)| + |y'(t)| \leq |y_0| &+ \int_0^t \left\{\gamma\,[|y(s)| + |y'(s)|] + \int_0^s q(s,\sigma)\,[|y(\sigma)| + |y'(\sigma)|] d\sigma\right\} ds \\ &+ \gamma\,[|y(t)| + |y'(t)|] + \int_0^t q(t,\sigma)\,[|y(\sigma)| + |y'(\sigma)|] d\sigma. \end{aligned} \tag{3.5}$$

From (3.5), we observe that

$$\begin{aligned} |y(t)| + |y'(t)| \leq \frac{|y_0|}{1-\gamma} &+ \frac{1}{1-\gamma}\int_0^t \Big\{\gamma\,[|y(s)| + |y'(s)|] + q(t,s)\,[|y(s)| + |y'(s)|] \\ &+ \int_0^s q(s,\sigma)\,[|y(\sigma)| + |y'(\sigma)|]\, d\sigma\Big\} ds. \end{aligned} \tag{3.6}$$

Now a suitable application of Lemma to (3.6) yields (3.3).

Remark 2. We note that the estimate obtained in (3.3) yields not only the bound on the solution $y(t)$ of IVP (1.5)−(1.6) but also the bound on $y'(t)$ for $t \in R_+$. If the estimate on the right hand side in (3.3) is bounded, then the solution $y(t)$ of IVP (1.5)−(1.6) and also $y'(t)$ are bounded on R_+.

The next theorem deals with the continuous dependence of solutions of equation (1.5) on given initial values.

Theorem 4. *Assume that the functions f, g in equation (1.5) satisfy the conditions (2.18), (2.19). Let $y_1(t)$ and $y_2(t)$ be the solutions of equation (1.5) with the given initial conditions*

$$y_1(0) = c_1, \tag{3.7}$$

and

$$y_2(0) = c_2, \tag{3.8}$$

respectively, where c_1, c_2 *are constants. Then*

$$|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)| \le \frac{|c_1 - c_2|}{1-d} \exp\left(\int_0^t \left[\frac{d}{1-d} + B(s)\right] ds\right), \tag{3.9}$$

for $t \in R_+$, *where*

$$B(t) = \frac{1}{1-d}\left[r(t,t) + \int_0^t \left\{r(t,\sigma) + \frac{\partial}{\partial t} r(t,\sigma)\right\} d\sigma\right]. \tag{3.10}$$

Proof. Let $u(t) = |y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|$ for $t \in R_+$. Following the proof of Theorem 3 and using the hypotheses, we have

$$\begin{aligned} u(t) \le |c_1 - c_2| + \int_0^t & \Big\{ d\left[|y_1(s) - y_2(s)| + |y_1'(s) - y_2'(s)|\right] \\ & + \int_0^s r(s,\sigma)\left[|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|\right] d\sigma \Big\} ds \\ & + d\left[|y_1(t) - y_2(t)| + |y_1'(t) - y_2'(t)|\right] \\ & + \int_0^t r(t,\sigma)\left[|y_1(\sigma) - y_2(\sigma)| + |y_1'(\sigma) - y_2'(\sigma)|\right] d\sigma. \end{aligned} \tag{3.11}$$

From (3.11), we observe that

$$u(t) \le \frac{|c_1 - c_2|}{1-d} + \frac{1}{1-d}\int_0^t \left\{ du(s) + r(t,s)\,u(s) + \int_0^s r(s,\sigma)\,u(\sigma)\,d\sigma \right\} ds. \tag{3.12}$$

Now an application of Lemma to (3.12) yields the bound in (3.9), which shows the dependency of solutions of equation (1.5) on given initial values.

Next, we consider the IVP (1.5)–(1.6) and the corresponding IVP

$$z'(t) = F\left(t, z(t), z'(t), \int_0^t G(t,\sigma,z(\sigma),z'(\sigma))\,d\sigma\right), \qquad z(0) = z_0, \tag{3.13}$$

for $t \in R_+$, where $F \in C(R_+ \times R^n \times R^n \times R^n, R^n)$ and for $0 \le s \le t < \infty$, $G \in C\left(R_+^2 \times R^n \times R^n, R^n\right)$.

The following theorem deals with the continuous dependence of solutions of IVP (1.5)–(1.6) on the functions involved therein.

Theorem 5. *Assume that the functions* f, g *in equation* (1.5) *satisfy the conditions*

(2.18), (2.19). Let $z(t)$ be a solution of equation (3.13) and suppose that

$$
\begin{aligned}
&|y_0 - z_0| + \int_0^t \left| f\left(s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right. \\
&\quad \left. - F\left(s, z(s), z'(s), \int_0^s G(s, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right| ds \\
&\quad + \left| f\left(t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right. \\
&\quad \left. - F\left(t, z(t), z'(t), \int_0^t G(t, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right| \le \varepsilon,
\end{aligned} \tag{3.14}
$$

where y_0, f, g and z_0, F, G are as in IVP (1.5)–(1.6) and IVP (3.13), $\varepsilon > 0$ is an arbitrary small constant. Then the solution $y(t)$ of IVP (1.5)–(1.6) depends continuously on the functions involved therein.

Proof. Let $u(t) = |y(t) - z(t)| + |y'(t) - z'(t)|$ for $t \in R_+$. From the hypotheses, we have

$$
\begin{aligned}
u(t) &\le |y_0 - z_0| + \int_0^t \left| f\left(s, y(s), y'(s), \int_0^s g(s, \sigma, y(\sigma), y'(\sigma)) d\sigma\right)\right. \\
&\quad \left. - f\left(s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right| ds \\
&\quad + \int_0^t \left| f\left(s, z(s), z'(s), \int_0^s g(s, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right. \\
&\quad \left. - F\left(s, z(s), z'(s), \int_0^s G(s, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right| ds \\
&\quad + \left| f\left(t, y(t), y'(t), \int_0^t g(t, \sigma, y(\sigma), y'(\sigma))\, d\sigma\right)\right. \\
&\quad \left. - f\left(t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right| \\
&\quad + \left| f\left(t, z(t), z'(t), \int_0^t g(t, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right. \\
&\quad \left. - F\left(t, z(t), z'(t), \int_0^t G(t, \sigma, z(\sigma), z'(\sigma))\, d\sigma\right)\right| \\
&\le \varepsilon + \left\{ \int_0^t d\,[|y(s) - z(s)| + |y'(s) - z'(s)|] \right. \\
&\quad \left. + \int_0^s r(s, \sigma)\,[|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|] d\sigma \right\} ds \\
&\quad + d\,[|y(t) - z(t)| + |y'(t) - z'(t)|] \\
&\quad + \int_0^t r(t, \sigma)\,[|y(\sigma) - z(\sigma)| + |y'(\sigma) - z'(\sigma)|] d\sigma.
\end{aligned} \tag{3.15}
$$

From (3.15), we observe that

$$u(t) \leq \frac{\varepsilon}{1-d} + \frac{1}{1-d}\int_0^t \left\{ du(s) + r(t,s)\,u(s) + \int_0^s r(s,\sigma)u(\sigma)\,d\sigma \right\} ds. \tag{3.16}$$

Now a suitable application of Lemma to (3.16) yields

$$u(t) \leq \frac{\varepsilon}{1-d} \exp\left(\int_0^t \left[\frac{d}{1-d} + B(s) \right] ds \right), \tag{3.17}$$

for $t \in R_+$, where $B(t)$ is given by (3.10). From (3.17) it follows that the solutions of IVP (1.5)−(1.6) depends continuously on the functions involved therein.

Remark 3. We note that our approach to the study of IVP (1.5)−(1.6) is different from those in [3,5,7,10] and we believe that the results given here are of independent interest. We also note that the idea employed here can be extended to the study of higher order integrodifferential equation of the form

$$y^{(n)}(t) = f\left(t, y(t), \ldots, y^{(n-1)}(t), y^{(n)}(t), Qy(t)\right), \tag{3.18}$$

with the given initial conditions

$$y^{(k)}(0) = c_k, \qquad k = 0, 1, \ldots, n-1, \tag{3.19}$$

where

$$Qy(t) = \int_0^t g\left(t, \sigma, y(\sigma), \ldots, y^{(n-1)}(\sigma), y^{(n)}(\sigma)\right) d\sigma. \tag{3.20}$$

Naturally, these considerations will make the analysis more complicated. However, the detailed discussion of such results is left to another place.

Acknowledgement

The author wishes to express his gratitude to the referee for several helful suggestions.

References

[1] R. Bellman and K. L. Cooke, Differential–Difference Equations, Academic Press, New York, 1963.

[2] A. Bielecki, *Une remarque sur la méthod de Banach–Cacciopoli–Tikhonov dans la théorie des équations difféentilles ordinaries*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Phys. Astr. **4**(1956), 261–164.

[3] A. Constantin, *Topological Transversality: Application to an integrodifferential equation*, J. Math. Anal. Appl. **197**(1996), 855–863.

[4] C. Corduneanu, *Bielecki's method in the theory of integral equations*, Ann. Univ. Mariae Curie–Sklodowskia, Section A **38**(1984), 23–40.

[5] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, 1991.

[6] R. K. Miller, Nonlinear Volterra Integral Equations, W.A. Benjamin, Menlo Park CA, 1971.

[7] B. G. Pachpatte, *Applications of the Leray–Schauder alternative to some Volterra integral and integrodifferential equations*, Indian J. Pure and Appl. Math. **26**(1995), 1161–1168.

[8] B. G. Pachpatte, Integral and Finite Difference Inequalities and Applications, North–Holland Mathematics Studies 205, Elsevier Science B.V. 2006.

[9] B. G. Pachpatte, *On higher order Volterra–Fredholm integrodifferential equation*, Fasciculi Mathematici Nr. **37**(2007), 35–48.

[10] A. Sikorska, *Existence theory for nonlinear Volterra integral and differential equations*, J. Inequal. and Appl. **6**(2001), 325–338.

57 Shri Niketan Colony, Near Abhinay Talkies, Aurangabad 431 001 (Maharashtra) India.

E-mail: bgpachpatte@gmail.com