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ON THE ORDER AND THE LOWER ORDER OF DIFFERENTIAL POLYNOMIALS

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Abstract. Suppose that f is a meromorphc function with order $\sigma(f)$ and lower order $\mu(f)$. Suppose that $P[f]$ is a differential polynomial of f. In this paper, it is shown that the order and the lower order of $P[f]$ are equal to the order and the lower order of f under certain conditions on the degree of the differential polynomial $P[f]$, *i.e.*, $\sigma(P)=\sigma(f)$ and $\mu(P)=\mu(f)$. This result improves previous results.

1. Introduction

Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the usual notations, such as $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $T(r,f)$ and etc., of Nevanlinna theory, see *e.g.*, [4, 9]. We denote by $S(r,f)$ any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to+\infty$, possibly outside a set of finite linear measure E. Throughout this paper we denote by a_j any small meromorphic function satisfying $T(r,a_j)=S(r,f)$ with $j=1,2,\ldots,n$. As usual, the order $\sigma(f)$ and the lower order $\mu(f)$ of f are defined by

$$\sigma(f):=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}\quad\text{and}\quad \mu(f):=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r},$$

respectively.

There has been quite a number of researches on the value distribution of differential polynomials since the 1940s. For example, H. Milloux [5] (see also [4, Theorem 3.3]) showed in 1940 that if f is a transcendental meromorphic function having only a finite number of zeros and poles, then the function ψ defined by

$$\psi:=\sum_{j=0}^{n}a_jf^{(j)}$$

assumes every finite complex value except possibly zero infinitely many times or else ψ is

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identically constant, where a_j are small functions of f with $j = 0, 1, \dots, n$. Another example is given by W. K. Hayman. In fact, he [3] (see also [4, Theorem 3.5]) showed in 1959 that if f is a transcendental meromorphic function, then either f assumes every finite value infinitely many times or every derivative of f assumes every finite value except possibly zero infinitely many times.

We note that their results are true no matter what the growth of the meromorphic function f is. However, if we restrict its growth, then people can obtain other kind of results of the function f and its differential polynomial $P[f]$. Specifically, mathematicians have paid attention to the study of the relations between the order and the lower order of the transcendental meromoprhic function f and its differential polynomials $P[f]$. In particular, L. R. Sons [8, Theorem 3] showed in 1969 that if f is a transcendental meromorphic function of finite order σ and lower order μ, then $\sigma(f) = \sigma(P)$ and $\mu(f) = \mu(P)$, where $P[f]$ is a monomial given by

$$P[f] = (f)^{n_0}(f')^{n_1}\cdots(f^{(k)})^{n_k},$$

$n_0 \geq 1$, $n_k \geq 1$, $n_i \geq 0$ for $1 \leq i \leq k-1$. Later in 1985, A. P. Singh [6] generalized L. R. Sons' result to functions of infinite order and also to a class of homogeneous differential polynomial $P[f]$. In fact, he showed that

Theorem A. *Let f be a transcendental meromorphic function and let $P[f]$ be a non-zero homogeneous differential polynomial of degree n in the form*

$$\begin{aligned} P[f] = a_0(f)^{\nu_0}(f')^{\nu_1}\cdots(f^{(i)})^{\nu_i} + b_0(f)^{\mu_0}(f')^{\mu_1}\cdots(f^{(j)})^{\mu_j} \\ + \cdots + c_0(f)^{\delta_0}(f')^{\delta_1}\cdots(f^{(k)})^{\delta_k} \end{aligned}$$

satisfying that each of the exponents of f, i.e., $\nu_0, \mu_0, \dots, \delta_0$, is an integer greater than or equal to 1 and $a_0, b_0, \dots, c_0$ are small functions of f. Then we have $\sigma(P) = \sigma(f)$.

In this direction, A. P. Singh and R. S. Dhar [7, Theorem 1] enriched the Theorem A to

Theorem B. *Let f be a transcendental meromorphic function and $P[f]$ be a non-constant differential polynomial of degree $\overline{d}(P)$ in the form*

$$P[f] := a + Q[f] = a + \sum_{j=1}^{n} a_j(f)^{l_{0j}}(f')^{l_{1j}}\cdots(f^{(k)})^{l_{kj}}$$

where $T(r, a) = S(r, f)$, each of the exponents of f is an integer greater than or equal to 1 and $\sum_{j=1}^{n} d(M_j) > (n-1)\overline{d}(Q)$. Then we must have $\sigma(P) = \sigma(f)$.

This paper is organized as follows: In §2, we give a very brief review on the differential polynomials of a meromorphic function f. After that, the main results of this paper are stated.

In §3, the necessary lemmas are stated and proofs of our main results are given in §4. In §5, two remarks about our main results and previous results are given.

2. Definitions and the main results

This paper concerns the value distribution of the differential polynomials of a meromorphic function f, so we shall give a brief review to the definitions of the study here. For a positive integer j, by a *monomial in* f we mean an expression of the type

$$M_j[f] = a_j(f)^{n_{0j}}(f')^{n_{1j}}\cdots(f^{(k)})^{n_{kj}}, \tag{2.1}$$

where $n_{0j}, n_{1j}, \ldots, n_{kj}$ are non-negative integers. We define $d(M_j) = \sum_{i=0}^{k} n_{ij}$ as the *degree* of $M_j[f]$ and $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$ as the *weight* of $M_j[f]$. Next a *differential polynomial in* f is a finite sum of such monomials (2.1), *i.e.*,

$$P[f] = \sum_{j=1}^{n} a_j M_j[f]. \tag{2.2}$$

We define

$$\overline{d}(P) = \max_{1\le j\le n}\{d(M_j)\}, \quad \underline{d}(P) = \min_{1\le j\le n}\{d(M_j)\} \quad \text{and} \quad \Gamma_P = \max_{1\le j\le n}\{\Gamma_{M_j}\}$$

as the *degree*, the *lower degree* and the *weight* of $P[f]$, respectively. If, in particular, $\overline{d}(P) = \underline{d}(P)$, then $P[f]$ is called *homogeneous* and *non-homogeneous* otherwise.

We have two main results which are stated as follows:

Theorem 2.1. *Suppose that f is a transcendental meromorphic function and $P[f]$ is a non-constant differential polynomial* (2.2) *with $T(r, a_j) = S(r, f)$, where $j = 1, 2, \ldots, n$, satisfying that each n_j of the exponents of f is a positive integer. If the constant*

$$d := \sum_{j=1}^{n} d(M_j) - (n-1)[\overline{d}(P) - \underline{d}(P)] > 0, \tag{2.3}$$

then we must have $\sigma(P) = \sigma(f)$ and $\mu(P) = \mu(f)$.

Theorem 2.2. *Suppose that f is a meromorphic function satisfying the condition*

$$\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r, f). \tag{2.4}$$

If $P[f]$ is a non-constant differential polynomial, each term in $P[f]$ contains f and $d > 0$, then we have

$$\sigma(P) = \sigma(f), \mu(P) = \mu(f) \quad \textit{and} \quad \overline{N}(r, P) + \overline{N}\left(r, \frac{1}{P}\right) = S(r, P).$$

3. Lemmas

For the proofs of our main results, we need the following two lemmas due to W. Doeringer.

Lemma 3.1. [1, Lemma 1] *Let f be a non-constant meromorphic function. If $P[f]$ is a differential polynomial in f with arbitrary meromorphic coefficients a_j, $1 \le j \le n$, then we have*

$$m(r,P) \le \overline{d}(P)m(r,f) + \sum_{j=1}^{n} m(r,a_j) + S(r,f)$$

and

$$N(r,P) \le \Gamma_P N(r,f) + \sum_{j=1}^{n} N(r,a_j) + O(1).$$

Lemma 3.2. [1, Lemma 3] *Let $T_1(r)$ and $T_2(r)$ be real valued, non-negative and non-decreasing functions defined for $r > r_0 > 0$ and satisfying $T_1(r) = O(T_2(r))$ as $r \to +\infty$. Then we have*

$$\limsup_{r\to+\infty} \frac{\log^+ T_1(r)}{\log r} \le \limsup_{r\to+\infty} \frac{\log^+ T_2(r)}{\log r}$$

and

$$\liminf_{r\to+\infty} \frac{\log^+ T_1(r)}{\log r} \le \liminf_{r\to+\infty} \frac{\log^+ T_2(r)}{\log r}.$$

In particular, this implies that for meromorphic functions f_1 and f_2 with $T(r,f_1) = O(T(r,f_2))$, $r \to +\infty$, possibly outside a set of finite linear measure, the inequalities $\mu(f_1) \le \mu(f_2)$ and $\sigma(f_1) \le \sigma(f_2)$ hold.

4. Proofs of the main results

4.1. Proof of Theorem 2.1.

By the Lemma 3.1 and the fact that $\overline{d}(P) \le \Gamma_P$, we have

$$T(r,P) \le \Gamma_P T(r,f) + S(r,f). \tag{4.1}$$

Since $P[f]$ is non-constant, we have $P[f] \not\equiv 0$ and then

$$\frac{1}{f^{\overline{d}(P)-\underline{d}(P)}} = \frac{1}{P[f]} \cdot \frac{P[f]}{f^{\overline{d}(P)-\underline{d}(P)}}$$

which gives

$$T\left(r, \frac{1}{f^{\overline{d}(P)-\underline{d}(P)}}\right) \le T\left(r, \frac{1}{P}\right) + T\left(r, \sum_{j=1}^{n} \frac{a_j M_j[f]}{f^{d(M_j)}} \cdot \frac{f^{d(M_j)}}{f^{\overline{d}(P)-\underline{d}(P)}}\right). \tag{4.2}$$

Thus the inequality (4.2) together with the First Fundamental Theorem imply that

$$\begin{aligned}[\overline{d}(P)-\underline{d}(P)]T(r,f) &\le T\Big(r,\frac{1}{f^{\overline{d}(P)-\underline{d}(P)}}\Big)+O(1)\\ &\le T(r,P)+\sum_{j=1}^{n}T\Big(r,\frac{a_jM_j[f]}{f^{d(M_j)}}\Big)\\ &\quad+\sum_{j=1}^{n}T\Big(r,\frac{1}{f^{\overline{d}(P)-\underline{d}(P)-d(M_j)}}\Big)+O(1)\end{aligned}\tag{4.3}$$

which deduces that

$$dT(r,f)\le T(r,P)+\sum_{j=1}^{n}T\Big(r,\frac{M_j[f]}{f^{d(M_j)}}\Big)+S(r,f).\tag{4.4}$$

Now for all positive integers p, we have

$$N\Big(r,\frac{f^{(p)}}{f}\Big)\le p\Big[\overline{N}(r,f)+\overline{N}\Big(r,\frac{1}{f}\Big)\Big]\le p\Big[\overline{N}(r,P)+\overline{N}\Big(r,\frac{1}{P}\Big)\Big]\tag{4.5}$$

because each of n_{0j} is an integer ≥ 1, where $j=1,2,\dots,n$. Furthermore, we have

$$\begin{aligned}m\Big(r,\frac{M_j[f]}{f^{d(M_j)}}\Big) &= m\Big(r,\frac{f^{n_{0j}}(f')^{n_{1j}}\cdots(f^{(k)})^{n_{kj}}}{f^{\sum_{j=0}^{k}n_{ij}}}\Big)\\ &= m\Big(r,\frac{f^{n_{0j}}(f')^{n_{1j}}\cdots(f^{(k)})^{n_{kj}}}{f^{n_{0j}}\cdot f^{n_{1j}}\cdots\cdot f^{n_{kj}}}\Big)\\ &= S(r,f)\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}N\Big(r,\frac{M_j[f]}{f^{d(M_j)}}\Big) &= n_{1j}N\Big(r,\frac{f'}{f}\Big)+n_{2j}N\Big(r,\frac{f''}{f}\Big)+\cdots+n_{kj}N\Big(r,\frac{f^{(k)}}{f}\Big)\\ &\le (n_{1j}+2n_{2j}+\cdots+kn_{kj})\Big[\overline{N}(r,P)+\overline{N}\Big(r,\frac{1}{P}\Big)\Big].\end{aligned}\tag{4.7}$$

Thus we have

$$T\Big(r,\frac{M_j[f]}{f^{d(M_j)}}\Big)\le 2\big[\Gamma_{M_j}-d(M_j)\big]T(r,P)+S(r,P)+S(r,f).\tag{4.8}$$

Hence inequalities (4.4) and (4.8) imply that

$$dT(r,f)\le T(r,P)+\sum_{j=1}^{n}2\big[\Gamma_{M_j}-d(M_j)\big]T(r,P)+S(r,P)+S(r,f).$$

By (4.1), we must have $S(r,P)=S(r,f)$ so that

$$dT(r,f)\le d_1T(r,P)+S(r,f),\tag{4.9}$$

where $d_1 = 2\sum_{j=1}^{n}\left[\Gamma_{M_j} - d(M_j)\right] + 1 > 0$. Since $d > 0$ by hypothesis, it follows that

$$T(r,f) \le d_2 T(r,P) + S(r,f), \tag{4.10}$$

where $d_2 = \frac{d_1}{d}$. Hence inequalities (4.1), (4.10) and the Lemma 3.2 show that

$$\sigma(P) = \sigma(f) \quad \text{and} \quad \mu(P) = \mu(f),$$

as required.

4.2. Proof of Theorem 2.2

By the condition (2.4), we have from the estimate (4.5) that

$$N\left(r, \frac{f^{(p)}}{f}\right) = S(r,f)$$

for integers p. Thus the inequality (4.7) gives

$$N\left(r, \frac{M_j[f]}{f^{d(M_j)}}\right) = S(r,f)$$

for each $j = 1, 2, \ldots, n$ and combining (4.6), we obtain that

$$T\left(r, \frac{M_j[f]}{f^{d(M_j)}}\right) = S(r,f) \tag{4.11}$$

for each $j = 1, 2, \ldots, n$. Therefore it follows from (4.4) that

$$d\,T(r,f) \le T(r,P) + S(r,f), \tag{4.12}$$

where d is defined by (2.3). By the inequalities (4.1), (4.12) and the Lemma 3.2, we have

$$\sigma(f) = \sigma(P) \quad \text{and} \quad \mu(f) = \mu(P),$$

as required.

Furthermore, since $\overline{N}(P) \le \overline{N}(r,f) + S(r,f)$, the condition (2.4) implies that

$$\overline{N}(P) = S(r,f) = S(r,P) \tag{4.13}$$

and for any j,

$$\begin{aligned}\overline{N}\left(r, \frac{1}{P}\right) &\le \overline{N}\left(r, \frac{f^{d(M_j)}}{P}\right) + \overline{N}\left(r, \frac{1}{f^{d(M_j)}}\right)\\ &\le T\left(r, \frac{f^{d(M_j)}}{P}\right) + \overline{N}\left(r, \frac{1}{f^{d(M_j)}}\right)\end{aligned}$$

$$\le T\left(r, \frac{P}{f^{d(M_j)}}\right) + \overline{N}\left(r, \frac{1}{f}\right) + O(1)$$
$$\le \sum_{j=1}^{n} T\left(r, \frac{M_j[f]}{f^{d(M_j)}}\right) + \overline{N}\left(r, \frac{1}{f}\right) + O(1),$$

so that

$$\overline{N}\left(r, \frac{1}{P}\right) = S(r, f) = S(r, P) \tag{4.14}$$

by the conditions (2.4) and (4.11). Hence it follows from (4.13) and (4.14) that

$$\overline{N}(r, P) + \overline{N}\left(r, \frac{1}{P}\right) = S(r, P).$$

This completes the proof of the theorem.

5. Further remarks

In this section, several comparisons between our main results and previous results are given as follows:

Remark 1. In our Theorem 2.1, if we take $n = 1$, then we get a result which is similar to the one of L. R. Sons stated in the introduction. If $\overline{d}(P) = \underline{d}(P)$, *i.e.*, $P[f]$ is homogeneous, then we have the Theorem A. In addition, if $\underline{d}(P) = 0$, then we get the Theorem B.

Remark 2. In 1986, H. S. Gopalakrishna and S. S. Bhoosnurmath [2] obtained the results $\sigma(P) = \sigma(f)$ and $\overline{N}(r, P) + \overline{N}\left(r, \frac{1}{P}\right) = S(r, P)$ by assuming that the condition (2.4) holds and $P[f]$ is a *homogeneous* differential polynomial in f which does not reduce to a constant. Thus it is easy to see that if the differential polynomial is homogeneous in the Theorem 2.2, then we have a result which is similar to the result obtained by them.

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