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LOGARITHMICALLY COMPLETE MONOTONICITY PROPERTIES AND CHARACTERIZATIONS OF THE GAMMA FUNCTION

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Abstract. In this paper, the logarithmically complete monotonic properties of the functions $\prod_{i=1}^{n}\frac{\Gamma(x-a_i)}{\Gamma(x-b_i)}$,$\Gamma(x)^{\alpha}\Gamma\Big(x-\sum_{i=1}^{n}a_i\Big)/\prod_{i=1}^{n}\Gamma(x-a_i)$, and $x^r(e/x)^x\Gamma(x)$ are obtained. Some characterizations of the gamma function are deduced.

1. Introduction

The classical gamma function is usually defined for Re$z>0$ as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\mathrm{d}t.$$

It is one of the most important functions in analysis and its applications. The history and development of this function are described in detail [11].

The psi or digamma function, the logarithmic derivative of the gamma function, and the polygamma functions can be defined [20, p.16] as

$$\psi(x)=\frac{\Gamma^{'}(x)}{\Gamma(x)}=-\gamma+\int_0^\infty\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\mathrm{d}t \tag{1}$$

or

$$\psi(x)=-\gamma+\sum_{n=0}^{\infty}\Big(\frac{1}{1+n}-\frac{1}{x+n}\Big), \tag{2}$$

$$\psi^{(k)}(x)=(-1)^{k+1}\int_0^\infty\frac{t^k}{1-e^{-t}}e^{-xt}\mathrm{d}t \tag{3}$$

or

$$\psi^{(k)}(x)=(-1)^{k+1}k!\sum_{i=0}^{\infty}\frac{1}{(x+i)^{k+1}} \tag{4}$$

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for $x > 0$ and $k \in \mathbb{N}$, where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \cdots). \tag{5}$$

If the inequality (5) is strict, then f is said to be strictly completely monotonic on I.

"Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory [8], probability theory [9, 15, 19], physics [12], numerical and asymptotic analysis [17, 28], and combinatorics [6]. A detailed collection of the most important properties of completely monotonic functions can be found in [27, Chapter IV], and in an abstract in [7]." [3, p.446]

A positive function f is said be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \tag{6}$$

for $x \in I$ and $n \in \mathbb{N} := 1, 2, \ldots$. If inequality (6) is strict, then f is said to be strictly logarithmically completely monotonic. The terminology "(strictly) logarithmically completely monotonic function" was introduced in [24]. It is also shown in this paper that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

In the past many articles [1, 13, 18, 25] were published providing some different properties for the ratio $\Gamma(x+1)/\Gamma(x+s)$, where $x > 0$ and $s \in (0, 1)$. In 1986, J. Bustoz and M.E.H. Ismail [10] established the function

$$p(x; a, b) = \frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)} \qquad (a, b > 0),$$

which can be represented in terms of Gauss' hypergeometric series

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$, namely,

$$\frac{\Gamma(x)\Gamma(x+a+b)}{\Gamma(x+a)\Gamma(x+b)} = {}_2F_1(-a, -b, x; 1), \qquad (x > -a-b).$$

They showed that the function $p(x; a, b)$ is completely monotonic on $(0, \infty)$. This generalized a proposition of K. B. Stolarsky [26], who obtained that p is decreasing in x.

In 1997, H. Alzer [2] proved that the function

$$\phi(x) = \prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$$

is completely monotonic on $(0,\infty)$ with $0 \le a_1 \le \cdots \le a_n$, $0 \le b_1 \le \cdots \le b_n$, and $\sum_{i=1}^k a_i \le \sum_{i=1}^k b_i$ for $k = 1, \ldots, n$. This extended Bustoz's result.

Let us extend a_i, b_i to all real numbers. For a_i and b_i $(i = 1, \ldots, n)$ are positive real numbers, define

$$\Phi(x) = \prod_{i=1}^{n} \frac{\Gamma(x - a_i)}{\Gamma(x - b_i)}.$$

Then we obtain that $\Phi(x)$ is logarithmically completely monotonic in the following theorems.

Theorem 1. *Let a_i and b_i be positive real numbers, $m = \max\{a_i, b_i\}(i = 1, \ldots, n)$ and $a_i \ge b_i$. Then the function $\Phi(x)$ is logarithmically completely monotonic on (m,∞).*

Theorem 2. *Let a_i and b_i be real numbers such that $0 \le a_1 \le \cdots \le a_n$, $0 \le b_1 \le \cdots \le b_n$, $m = \max\{a_i, b_i\}(i = 1, \ldots, n)$ and $\sum_{i=1}^k a_i \ge \sum_{i=1}^k b_i$ for $k = 1, \ldots, n$. Then the function $\Phi(x)$ is logarithmically completely monotonic on (m,∞).*

Corollary 1. *The function*

$$P(x; a, b) ={}_2 F_1(a, b, x; 1) = \frac{\Gamma(x)\Gamma(x - a - b)}{\Gamma(x - a)\Gamma(x - b)} \qquad (a, b > 0)$$

is strictly logarithmically completely monotonic on $(a + b,\infty)$.

In 1995, L. Maligranda et al. [21] concluded that the function

$$x \to \Gamma(x)^{n-1}\Gamma\Big(x + \sum_{i=1}^{n} a_i\Big) / \prod_{i=1}^{n} \Gamma(x + a_i)$$

$(a_i > 0; i = 1, \ldots, n)$ is decreasing on $(0,\infty)$. Subsequently, Alzer extended this result in [2, p.385], and obtained the necessary and sufficient condition of the statement that the function is strictly completely monotonic. The following theorem provides a slight extension of Alzer' result.

Theorem 3. *Let α be a real number, a_i $(i = 1, \ldots, n; n \ge 2)$ be positive real numbers and $m = \sum_{i=1}^n a_i$. The function*

$$x \to \Gamma(x)^{\alpha}\Gamma\Big(x - \sum_{i=1}^{n} a_i\Big) / \prod_{i=1}^{n} \Gamma(x - a_i)$$

is strictly logarithmically completely monotonic on (m,∞) if and only if $\alpha = n - 1$.

In [2], Alzer proved that the function $F_r(x) = x^r(e/x)^x\Gamma(x)$ is decreasing on $(0,\infty)$ if and only if $r \le 1/2$. Moreover, Alzer obtained that the function $g(x)(f_1(x)-c)$ is strictly completely monotonic on $(0,\infty)$ if and only if $c \le 1/2$, where $g(x)$ is a strictly completely monotonic function and $f_1(x) = x(\ln(x) - \psi(x))$. This extended a result of E. Muldoon

[23], who proved the complete monotonicity for the special case $g(x) = 1/x$. In 2006, Alzer and Berg [4] established the completely monotonicity of function $[x^a(e/x)^x\Gamma(x)]^b$ for $a, b \in \mathbb{R}$ and $b \neq 0$.

Motivated by the results above, we establish several functions involving gamma and polygamma functions, and investigate their logarithmically completely monotonic properties in the following theorems.

Lemma 1.[[2, p.374]] *Let α be a real number. The function*

$$f_\alpha(x) = x^\alpha(\ln(x) - \psi(x))$$

is strictly completely monotonic on $(0, \infty)$ if and only if $\alpha \leq 1$.

Theorem 4. *Let r be real number. The function*

$$F_r(x) = x^r(e/x)^x\Gamma(x)$$

is strictly logarithmically completely monotonic on $(0, \infty)$ if and only if $r \leq \frac{1}{2}$; The function $(F_r(x))^{-1}$ is strictly logarithmically completely monotonic on $(0, \infty)$ for $r \geq 1$.

Corollary 2. *The function*

$$F_{r,\alpha}(x) = \left[x^r(e/x)^x\Gamma(x)\right]^\alpha$$

is logarithmically completely monotonic on $(0, \infty)$ if and only if $r \leq \frac{1}{2}$ and $\alpha > 0$.

Finally, we study the problem of characterizing $\Gamma(x)$ by means of the logarithmically completely monotonic functions related to $\Gamma(x)$. From Theorem 4, we obtain the results as follows.

Theorem 5. *If function $x^{1/2}(e/x)^x f(x)$ is logarithmically completely monotonic on $(0, \infty)$ and that $f(x_k) = \Gamma(x_k)$ for each point x_k in an increasing sequence $\{x_k\} \subset (0, \infty)$ for which $\sum(1/x_k)$ diverges, then $f(x) = \Gamma(x)$, $0 < x < \infty$.*

Corollary 3. *If function $(f(x))^{-1}$ is logarithmically completely monotonic on $(0, \infty)$ for $n = 2, 3, \ldots$ and that $f(x_k) = \Gamma(x_k)$ for each point x_k in an increasing sequence $\{x_k\} \subset (0, \infty)$ for which $\sum(1/x_k)$ diverges, then $f(x) = \Gamma(x)$, $0 < x < \infty$.*

Remark 1. The function $(\Gamma(x))^{-1}$ is logarithmically completely monotonic on $(0, \infty)$ for $n = 2, 3, \ldots$.

Moreover, the function $\Gamma(x)$ can be characterized in the following way [5, p.14]: $f(x) = \Gamma(x)$ $(0 < x < \infty)$ if and only if

1. $f(1) = 1, f(x + 1) = xf(x), \quad 0 < x < \infty$;

2. $f(x)$ is defined and logarithmically convex for $0 < x < \infty$.

Requirement (2) can be modified by logarithmically completely monotonicity properties. The result is as follows.

Theorem 6. *Suppose that*

1. $f(1) = 1, f(x+1) = xf(x), \quad 0 < x < \infty;$
2. $f(x)$ *is logarithmically completely monotonic on* $(0, \infty)$.

Then $f(x) = \Gamma(x)$, $0 < x < \infty$.

2. Proofs of Theorems

Proof of Theorem 1. Taking logarithm and differentiation yields

$$
\begin{aligned}
(\ln \Phi(x))' &= \sum_{i=1}^{n} \psi(x - a_i) - \sum_{i=1}^{n} \psi(x - b_i) \\
&= \int_0^{\infty} \frac{\sum_{i=1}^{n} e^{-(x-b_i)t} - \sum_{i=1}^{n} e^{-(x-a_i)t}}{1 - e^{-t}} \mathrm{d}t \\
&= \int_0^{\infty} \frac{\sum_{i=1}^{n} (e^{b_i t} - e^{a_i t})}{1 - e^{-t}} e^{-xt} \mathrm{d}t \qquad (7)
\end{aligned}
$$

Applying power series expansion of e^x to (7), we get

$$
(\ln \Phi(x))' = \int_0^{\infty} \frac{\sum_{i=1}^{n} \left(\sum_{k=1}^{\infty} (b_i^k - a_i^k) \frac{t^k}{k!} \right)}{1 - e^{-t}} e^{-xt} \mathrm{d}t \le 0 \qquad (8)
$$

By (3), we have

$$
\begin{aligned}
(-1)^m (\ln \Phi(x))^{(m)} &= (-1)^m \Big(\sum_{i=1}^{n} \psi^{(m-1)}(x - a_i) - \sum_{i=1}^{n} \psi^{(m-1)}(x - b_i) \Big) \\
&= \int_0^{\infty} \frac{\sum_{i=1}^{n} (e^{a_i t} - e^{b_i t})}{1 - e^{-t}} t^{m-1} e^{-xt} \mathrm{d}t \qquad (9) \\
&= \int_0^{\infty} \frac{\sum_{i=1}^{n} \left(\sum_{k=1}^{\infty} (a_i^k - b_i^k) \frac{t^k}{k!} \right)}{1 - e^{-t}} t^{m-1} e^{-xt} \mathrm{d}t \\
&\ge 0
\end{aligned}
$$

The proof is complete.

In order to prove Theorem 2 we need the following lemma [22, p.10].

Lemma 2. *Let* a_i *and* b_i $(i = 1 \ldots, n)$ *be real numbers such that* $a_1 \le \cdots \le a_n$, $b_1 \le \cdots \le b_n$, *and* $\sum_{i=1}^{k} b_i \le \sum_{i=1}^{k} a_i$ *for* $k = 1, \ldots, n$. *If the function* f *is increasing and convex on* $\mathbb{R}$, *then*

$$
\sum_{i=1}^{n} f(b_i) \le \sum_{i=1}^{n} f(a_i).
$$

Proof of Theorem 2. Since the function $x \to e^{xt} (t > 0)$ is increasing and convex on $\mathbb{R}$, we conclude from Lemma 2 that $\sum_{i=1}^{n}(e^{a_i t} - e^{b_i t}) \geq 0$. Therefore (9) implies

$$(-1)^m (\ln \Phi(x))^{(m)} \geq 0 \qquad (m = 1, 2, \ldots)$$

for $x > 0$, and the function $\Phi(x)$ is logarithmically completely monotonic on $(0, \infty)$.

Proof of Corollary 1. With analogous proof method as Theorem 1, we get

$$\begin{aligned}
&(-1)^n (\ln P(x; a, b))^{(n)} \\
&= (-1)^n \big(\psi^{(n-1)}(x) + \psi^{(n-1)}(x - a - b) - \psi^{(n-1)}(x - a) - \psi^{(n-1)}(x - b)\big) \\
&= \int_0^\infty \frac{t^{n-1} e^{-xt}}{1 - e^{-t}} \big(e^{(a+b)t} + 1 - e^{at} - e^{bt}\big) \mathrm{d}t \\
&= \int_0^\infty \frac{t^{n-1} e^{-xt}}{1 - e^{-t}} \Big(\sum_{k=2}^{\infty} \big((a + b)^k - a^k - b^k\big) \frac{t^k}{k!}\Big) \mathrm{d}t > 0 \qquad (10)
\end{aligned}$$

Now we provide another method to prove Corollary 1.

For $n \geq 0$, we have that

$$\begin{aligned}
(\ln P(x; a, b))^{(n+1)} &= \psi^{(n)}(x) + \psi^{(n)}(x - a - b) - \psi^{(n)}(x - a) - \psi^{(n)}(x - b) \\
&= a\bigg(\frac{\psi^{(n)}(x) - \psi^{(n)}(x - a)}{a} - \frac{\psi^{(n)}(x - b) - \psi^{(n)}(x - b - a)}{a}\bigg). \qquad (11)
\end{aligned}$$

By (4), $y \longmapsto \psi^{(n)}(y)$ is strictly convex for odd n, the ratio

$$\frac{\psi^{(n)}(x) - \psi^{(n)}(x - a)}{a} \qquad (12)$$

is increasing with $y \in (a, \infty)$. For even n, the function $\psi^{(n)}(x)$ is concave, and the ratio (12) is decreasing. Thus, by (11) we conclude that the sign of $(\ln P(x; a, b))^{(n+1)}$ is $(-1)^{n+1}$, for $n \geq 0$ and $x \in (a + b, \infty)$.

Proof of Theorem 3. Let

$$p_\alpha(x) = \Gamma(x)^\alpha \Gamma\Big(x - \sum_{i=1}^{n} a_i\Big) \Big/ \prod_{i=1}^{n} \Gamma(x - a_i).$$

It is obvious that $p_{n-1}(x)$ is strictly logarithmically completely monotonic on (m, ∞) from Theorem 2.

Next, we assume that $p_\alpha(x)$ is strictly logarithmically completely monotonic on (m, ∞). Then, we get

$$\frac{\partial}{\partial x} \ln p_\alpha(x) = \alpha\psi(x) + \psi(x - m) - \sum_{i=1}^{n} \psi(x - a_i) \leq 0. \qquad (13)$$

This implies for all sufficiently large x:

$$\alpha \leq \sum_{i=1}^{n} \frac{\psi(x-a_i)}{\psi(x)} - \frac{\psi(x-m)}{\psi(x)}. \tag{14}$$

Since p_α is completely monotonic on (m, ∞), we obtain

$$\begin{aligned} 0 &\leq (p_\alpha(x))^{-2}\left[p_\alpha(x)\frac{\partial^2 p_\alpha(x)}{\partial x^2} - \Big(\frac{\partial p_\alpha(x)}{\partial x}\Big)^2\right] \\ &= \alpha\psi^{'}(x) + \psi^{'}(x-m) - \sum_{i=1}^{n}\psi^{'}(x-a_i). \end{aligned}$$

Hence, we have for $x > m$:

$$\sum_{i=1}^{n} \frac{\psi^{'}(x-a_i)}{\psi^{'}(x)} - \frac{\psi^{'}(x-m)}{\psi^{'}(x)} \leq \alpha. \tag{15}$$

Since

$$\lim_{x\to\infty} \psi(x-A)/\psi(x) = \lim_{x\to\infty} \psi^{'}(x-A)/\psi^{'}(x) = 1 \quad (A > 0),$$

we conclude from (14) and (15) that $\alpha = n - 1$.

Proof of Theorem 4. Using Binet's formula [14, p.18, (22)], we get

$$\begin{aligned} \big(\ln F_r(x)\big)^{'} &= \frac{r}{x} - \ln x + \psi(x) \\ &= \frac{r}{x} - \ln x + \ln x - \frac{1}{2x} + \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right) e^{-xt}\mathrm{d}t \\ &= \frac{r-1/2}{x} + \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right) e^{-xt}\mathrm{d}t \\ &< 0. \end{aligned} \tag{16}$$

(16) follows from $r \leq \frac{1}{2}$ and the inequality

$$\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}} = \frac{2-t-(2+t)e^{-t}}{2t(1-e^{-t})} < 0 \qquad (0 < t < \infty). \tag{17}$$

Indeed, let $g(t) = 2 - t - (2+t)e^{-t}$, we can get $g^{''}(t) = -te^{-t} < 0$ and $\lim_{t\to 0} g^{'}(t) = 0$. This implies that $g(t)$ is decreasing on $(0, \infty)$. Since $\lim_{t\to 0} g(t) = 0$, (17) holds.

Taking the nth derivative of $\ln F_r(x)$, we obtain

$$\begin{aligned}
&(-1)^n\big(\ln F_r(x)\big)^{(n)}\\
&= (-1)^n\left[\Big(\frac{r-1/2}{x}\Big)^{(n-1)} + \left(\int_0^\infty \Big(\frac{1}{2}+\frac{1}{t}-\frac{1}{1-e^{-t}}\Big)e^{-xt}\mathrm{d}t\right)^{(n-1)}\right]\\
&= -\frac{(r-1/2)(n-1)!}{x^n} - \int_0^\infty \left(\frac{1}{2}+\frac{1}{t}-\frac{1}{1-e^{-t}}\right)t^{n-1}e^{-xt}\mathrm{d}t\\
&> 0 \qquad (r \le \frac{1}{2}).
\end{aligned} \tag{18}$$

Next, it is clear from (16) that

$$r < x(\ln x - \psi(x)). \tag{19}$$

By Lemma 1, we obtain that $f_1(x) = x(\ln x - \psi(x))$ is strictly decreasing on $(0,\infty)$. Moreover,

$$\lim_{x\to\infty} f_1(x) = \frac{1}{2}.$$

It follows from the representations

$$f_1(x) = x\ln x - x\psi(x+1) + 1$$

and

$$f_1(x) = \frac{1}{2} + \frac{1}{12x} - \frac{\theta}{120x^3} \qquad (0<\theta<1);$$

see [16, p.824]. Therefore, we conclude $r \le \frac{1}{2}$.

To prove the second part, a simple calculation shows that

$$\big(\ln(F_r(x))^{-1}\big)' = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \alpha\right)e^{-xt}\mathrm{d}t < 0$$

and

$$(-1)^n\big(\ln(F_r(x))^{-1}\big)^{(n)} = -\int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \alpha\right)t^{n-1}e^{-xt}\mathrm{d}t > 0$$

for $r \ge 1$. Since we write

$$f(t) = \frac{1}{1-e^{-t}} - \frac{1}{t},$$

we have $f(0+) = \frac{1}{2}$, $f(\infty) = 1$.

This completes the whole proof.

Proof of Theorem 5. It is shown by W. Feller [15, p.671] that two functions which are completely monotonic on $(0,\infty)$ must be identical if they coincide at the points of an increasing unbounded sequence $\{x_k\}$ where $\sum(1/x_k)$ diverges.

Since the functions $x^{1/2}(e/x)^x f(x)$ and $x^{1/2}(e/x)^x\Gamma(x)$ are both (logarithmically) completely monotonic on $(0,\infty)$, we get $f(x) = \Gamma(x)$ easily.

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