TAMKANG JOURNAL OF MATHEMATICS
Volume 42, Number 1, 69-77, Spring 2011
doi:10.5556/j.tkjm.42.2011.69-77

Available online at http://journals.math.tku.edu.tw/index.php/TJM

$\beta-$CONNECTEDNESS IN $L-$TOPOLOGICAL SPACES

BO CHEN

Abstract. In this paper, a new kind of connectivity called $\beta-$connectedness in $L-$topological spaces is introduced by means of $\beta-$closed $L-$sets. Some fundamental properties of $\beta-$connectedness are obtained. Especially, the famous K.Fan's Theorem can be extended to $L-$topological spaces for $\beta-$connectedness.

1. Introduction

As well known, connectivity occupies very important place in topology. Many authors have presented different kinds of connectivity in fuzzy setting([2], [6], [11], [13], [14]). In [3], Balasubramanian introduced the concepts of $\beta-$open sets in $L-$topological spaces. Consequently, Balasubramanian ([3],[4]) and Hanafy([8]) further developed different important topological concepts such as separation and compactness by means of fuzzy $\beta-$open $L-$set and fuzzy $\beta-$closed $L-$set.

In this paper, we shall introduce the concept of $\beta-$connectedness in $L-$ topological spaces by means of $\beta-$open $L-$sets. $\beta-$connectedness preserves many nice properties of connectedness in general topological spaces. Meanwhile, the famous K.Fan's Theorem can be generalized to $L-$topological spaces for $\beta-$connectedness.

2. Preliminaries

Throughout this paper, $(L, \bigvee, \bigwedge, ')$ will denote a completely distributive *De Morgan* algebra. For a nonempty set X, L^X denotes the set of all $L-$fuzzy sets ($L-$sets for short) on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$ respectively.

A non-null element a in L is called $\bigvee-$irreducible element if $a \leq b \bigvee c$ implies $a \leq b$ or $a \leq c$. The set of all $\bigvee-$irreducible elements in L is denoted by $M^*(L)$. It is clear that $M^*(L^X) = \{x_\alpha | \alpha \in M^*(L)\}$.

Received November 29, 2010.
2000 *Mathematics Subject Classification*. 54A40, 54D35.
Key words and phrases. $\beta-$open $L-$set,$\beta-$closed $L-$set, $\beta-$separated, $\beta-$connectedness, $L-$topological space.

An $L-$topological space is a pair (L^X,δ), where δ is a subfamily of L^X which contains $\underline{0}$ $\underline{1}$ and is closed for any suprema and infima. δ is called an $L-$topology on X. Every member of δ is called an open $L-$set and its quasicomplementation is called a closed $L-$set.

Definition 2.1.([3]) Let (L^X,δ) be an $L-$topological space, $A \in L^X$. Then A is said to be a

(1) β-open set if $A \leq cl(int(cl(A)))$.

(2) β-closed set if $A \geq int(cl(int(A)))$.

$\beta O(L^X)$ and $\beta C(L^X)$ will always denote the family of all β-open sets and β-closed sets respectively.

Obviously, $A \in \beta O(L^X)$ if and only if $A' \in \beta C(L^X)$.

Definition 2.2.([3]) Let (L^X,δ) be an $L-$topological space, $A, B \in L^X$. Let $\beta int(A) = \bigvee\{B \in L^X | B \leq A, B \in \beta O(L^X)\}$, $\beta cl(A) = \bigwedge\{B \in L^X | A \leq B, B \in \beta C(L^X)\}$. Then $\beta int(A)$ and $\beta cl(A)$ are called the $\beta-$interior and $\beta-$closure of A respectively.

By Definition 2.1 and Definition 2.2, the following proposition is clear.

Proposition 2.1. *Let (L^X,δ) be an $L-$topological space, $A \in L^X$. Then*

(1) *A is a β-open set if and only if $A = \beta int(A)$.*

(2) *A is a β-closed set if and only if $A = \beta cl(A)$.*

(3) $\beta cl(A') = (\beta int(A))', \beta int(A') = (\beta cl(A))'$.

Definition 2.3.([8]) Let (L^X,δ), (L^Y,μ) be two $L-$topological spaces. A mapping $f : (L^X,\delta) \rightarrow (L^Y,\mu)$ is called

(1) $\beta-$continuous if $f^{-1}(A) \in \beta O(L^X)$ for each $A \in \mu$.

(2) $M\beta-$continuous if $f^{-1}(A) \in \beta O(L^X)$ for each $A \in \beta O(L^X)$.

Definition 2.4.([10],[15]) Let (L^X,δ) be an $L-$topological space, $A, B \in L^X$. Then A, B are said to be separated if $cl(A) \bigwedge B = B \bigwedge cl(A) = \underline{0}$.

Definition 2.5.([10],[15]) Let (L^X,δ) be an $L-$topological space, $A \in L^X$. A is called connected if A can not be represented as a union of two non-null separated $L-$sets. (L^X,δ) is said to be a connected $L-$topological space if $A = \underline{1}$ is connected.

3. $\beta-$connectivity

Definition 3.1. Let (L^X,δ) be an $L-$topological space, $A, B \in L^X$. Then A, B are said to be $\beta-$separated if $\beta cl(A) \bigwedge B = B \bigwedge \beta cl(A) = \underline{0}$.

By Definition 3.1, the following proposition is obvious.

Proposition 3.1. *Let (L^X,δ) be an $L-$topological space, $A,B\in L^X$. If A,B are $\beta-$separated and $C\le A, D\le B$, then C,D are $\beta-$separated.*

Definition 3.2. Let (L^X,δ) be an $L-$topological space, $A\in L^X$. A is called $\beta-$connected if A can not be represented as a union of two non-null $\beta-$separated $L-$sets. If $A=\underline{1}$ is $\beta-$connected, we call (L^X,δ) a $\beta-$connected $L-$topological space.

Remark. It is obvious that a $\beta-$connected $L-$set must be connected, but the converse is not true as the following example.

Example 3.1. Let $X=\{x,y\}, L=\{0,a,b,1\}$, where $0^{'}=1, 1^{'}=0, a^{'}=a, b^{'}=b, 0<a<1, 0<b<1, a\bigwedge b=\underline{0}, a\bigvee b=\underline{1}$, a and b are incomparable.Take $A,B,C,D\in L^X$ as

$$A(x)=1,\quad A(y)=0;$$
$$B(x)=a,\quad B(y)=b;$$
$$C(x)=a,\quad C(y)=0;$$
$$D(x)=0,\quad D(y)=b.$$

Let (L^X,δ) be an $L-$topological space where $\delta=\{\underline{0},A,\underline{1}\}$. Then B is a connected $L-$set. In fact, B can only be expressed as the union of disjoint non-null $L-$sets C and D, i.e., $A=C\bigvee D, C\bigwedge D=\underline{0}, C\neq\underline{0}, D\neq\underline{0}$. By simple computation we can see that $cl(C)=\underline{1}$ and so $cl(C)\bigwedge D\neq\underline{0}$, i.e., C and D are not separated. Then B is connected. On the other hand, C and D are $\beta-$closed $L-$sets, i.e., $\beta cl(C)=C$ and $\beta cl(D)=D$ following from

$$C\ge int(cl(int(C)))=\underline{0}, D\ge int(cl(int(D)))=\underline{0}.$$

Hence $\beta cl(C)\bigwedge D=C\bigwedge\beta cl(D)=\underline{0}$, i.e., C and D are $\beta-$separated. Thus,B is not $\beta-$connected.

Theorem 3.1. *Let (L^X,δ) be an $L-$topological space, $A\in L^X$. Then the following statements are equivalent:*

(1) *A is $\beta-$connected.*

(2) *If C,D are $\beta-$separated in (L^X,δ) and $A\le C\bigvee D$, then $A\bigwedge C=\underline{0}$ or $A\bigwedge D=\underline{0}$.*

(3) *If C,D are $\beta-$separated in (L^X,δ) and $A\le C\bigvee D$, then $A\le C$ or $A\le D$.*

Proof. (1) $\Rightarrow$ (2) If C,D are $\beta-$separated in (L^X,δ) and $A\le C\bigvee D$, then $A\bigwedge C$ and $A\bigwedge D$ are $\beta-$separated by Proposition 3.1. From $A=A\bigwedge(C\bigvee D)=(A\bigwedge C)\bigvee(A\bigwedge D)$ and A is $\beta-$connected, one of $A\bigwedge C$ and $A\bigwedge D$ is equal to $\underline{0}$.

(2) $\Rightarrow$ (3) Suppose that $A \wedge C = \underline{0}$, then $A = A \wedge (C \vee D) = (A \wedge C) \vee (A \wedge D) = A \wedge D$. Therefore, $A \leq D$. Similarly, if $A \wedge D = \underline{0}$, then $A \leq C$.

(3) $\Rightarrow$ (1) Suppose C, D are $\beta-$separated in (L^X, δ) and $A = C \vee D$. Then $A \leq C$ or $A \leq D$ by (3). If $A \leq C$, then $D = D \wedge A \leq D \wedge C \leq D \wedge \beta cl(C) = \underline{0}$ since C, D are $\beta-$separated. Similarly, $A \leq D$ implies $C = \underline{0}$. This shows that one of C and D equals to $\underline{0}$. So A can not be represented as a union of two non-null $\beta-$separated $L-$sets. Hence, A is $\beta-$connected. □

Corollary 3.1. *For each $e \in M^*(L^X)$, e is $\beta-$connected.*

Theorem 3.2. *Let (L^X, δ) be an $L-$topological space, $A \in L^X$. Then the following statements are equivalent:*

(1) *A is $\beta-$connected.*

(2) *There do not exist two $\beta-$closed $L-$sets C, D such that*

$$C \bigwedge A \neq \underline{0}, D \bigwedge A \neq \underline{0}, A \leq C \bigvee D \text{ and } C \bigwedge D \bigwedge A = \underline{0}.$$

(3) *There do not exist two $\beta-$closed $L-$sets C, D such that*

$$A \nleq C, A \nleq D, A \leq C \bigvee D \text{ and } C \bigwedge D \bigwedge A = \underline{0}.$$

Proof. (1) $\Rightarrow$ (2) Suppose that A is $\beta-$connected and there exist two $\beta-$closed $L-$subsets C, D such that

$$C \bigwedge A \neq \underline{0}, D \bigwedge A \neq \underline{0}, A \leq C \bigvee D \text{ and } C \bigwedge D \bigwedge A = \underline{0}.$$

Then $A = A \wedge (C \vee D) = (A \wedge C) \vee (A \wedge D)$. We will prove $A \wedge C$ and $A \wedge D$ are $\beta-$separated which shows that A is not $\beta-$connected. In fact, we can get the conclusion from

$$\beta cl(A \bigwedge C) \bigwedge (A \bigwedge D) \leq \beta cl(C) \bigwedge (A \bigwedge D) = C \bigwedge D \bigwedge A = \underline{0}$$

and

$$A \bigwedge C \bigwedge \beta cl(A \bigwedge D) \leq A \bigwedge C \bigwedge \beta cl(D) = C \bigwedge D \bigwedge A = \underline{0}.$$

(2) $\Rightarrow$ (3) Suppose that there exist two $\beta-$closed $L-$subsets C, D such that $A \nleq C, A \nleq D, A \leq C \vee D$ and $C \wedge D \wedge A = \underline{0}$. We can easily get that $C \wedge A \neq \underline{0}, D \wedge A \neq \underline{0}$ which is a contradiction.

(3) $\Rightarrow$ (1) Suppose A is not $\beta-$connected. Then there exist two non-null $\beta-$separated $L-$subsets E and F such that $A = E \vee F$. Put $C = \beta cl(E), D = \beta cl(F)$, then C and D are two $\beta-$closed $L-$sets satisfying

$$A = E \bigvee F \leq \beta cl(E) \bigvee \beta cl(F) = C \bigvee D$$

and

$$\begin{aligned} C\bigwedge D\bigwedge A &= \beta cl(E)\bigwedge \beta cl(F)\bigwedge A \\ &= \beta cl(E)\bigwedge \beta cl(F)\bigwedge (E\bigvee F) \\ &= (\beta cl(E)\bigwedge \beta cl(F)\bigwedge E)\bigvee (\beta cl(E)\bigwedge \beta cl(F)\bigwedge F) \\ &= (\beta cl(F)\bigwedge E)\bigvee (F\bigwedge \beta cl(E)) \\ &= \underline{0}\bigvee \underline{0} \\ &= \underline{0}. \end{aligned}$$

Then $C\bigwedge D\bigwedge A = \underline{0}$. Moreover we have that $A \not\leq C$ and $A \not\leq D$. In fact, if $A \leq C$, then $D\bigwedge A = D\bigwedge (A\bigwedge C) = \underline{0}$. So, $F = F\bigwedge A \leq \beta cl(F)\bigwedge A = D\bigwedge A = \underline{0}$ which is a contradiction. Similarly, we can get $A \not\leq D$. But the above results contradict (3). □

Corollary 3.2. *Let (L^X,δ) be an $L-$topological space. Then the following statements are equivalent:*

(1) *(L^X,δ) is $\beta-$connected.*

(2) *If C, D are $\beta-$open $L-$sets such that $C\bigvee D = \underline{1}$ and $C\bigwedge D = \underline{0}$, then $C = \underline{0}$ or $D = \underline{0}$.*

(3) *If C, D are $\beta-$closed $L-$sets such that $C\bigvee D = \underline{1}$ and $C\bigwedge D = \underline{0}$, then $C = \underline{0}$ or $D = \underline{0}$.*

Theorem 3.3. *Let (L^X,δ) be an $L-$topological space, $A \in L^X$. Then A is $\beta-$connected if and only if for any pair of non-null $\bigvee-$irreducible elements a, b in A, there exists a $\beta-$connected $L-$set B such that $a, b \leq B \leq A$.*

Proof. The necessity is obvious if we take $B = A$.

Conversely, suppose that A is not $\beta-$connected, then there exist two $\beta-$closed $L-$sets C, D such that

$$A \not\leq C, A \not\leq D, A \leq C\bigvee D \text{ and } C\bigwedge D\bigwedge A = \underline{0}.$$

Take two non-null $\bigvee-$irreducible elements a, b in A such that $a \not\leq C$ and $b \not\leq D$. Then for each $L-$set B such that $a, b \leq B \leq A$, we get that

$$B \not\leq C, B \not\leq D, B \leq C\bigvee D \text{ and } C\bigwedge D\bigwedge B = \underline{0}.$$

This shows that B is not $\beta-$connected by Theorem 3.2 which is a contradiction. □

Theorem 3.4. *Let (L^X,δ) be an $L-$topological space, $A \in L^X$ is $\beta-$connected. If $A \leq B \leq \beta cl(A)$, then B is $\beta-$connected.*

Proof. Suppose there exist $C, D \in L^X$ such that $B = C \bigvee D$ and $\beta cl(C) \bigwedge D = C \bigwedge \beta cl(D) = \underline{0}$. Put $E = A \bigwedge C, F = A \bigwedge D$. Then $A = E \bigvee F$ and E, F are $\beta-$ separated by Proposition 3.1. It follows that $E = \underline{0}$ or $F = \underline{0}$ for A is $\beta-$connected. If $E = \underline{0}$, then $A = F = A \bigwedge D \leq D$. Therefore, $C = C \bigwedge \beta cl(A) \leq C \bigwedge \beta cl(D) = \underline{0}$, i.e., $C = \underline{0}$. Analogously, $F = \underline{0}$ will imply $D = \underline{0}$. This shows that A is not $\beta-$connected in L^X.

Corollary 3.3. *Let (L^X, δ) be an $L-$topological space. If $A \in L^X$ is $\beta-$connected, then so is $\beta cl(A)$.*

Theorem 3.5. *Let (L^X, δ) be an $L-$topological space, $\{A_t\}_{t \in T}$ be a family of $\beta-$connected $L-$sets. If there exists $s \in T$ such that A_s and A_t are not $\beta-$separated for each $t \neq s$, then $A = \bigvee_{t \in T} A_t$ is $\beta-$connected.*

Proof. Suppose $A = \bigvee_{t \in T} A_t$ is not $\beta-$connected. Then there exist $C, D \in L^X$ such that $A = C \bigvee D$ and $\beta cl(C) \bigwedge D = C \bigwedge \beta cl(D) = \underline{0}$. Let $C_t = A_t \bigwedge C$ and $D_t = A_t \bigwedge D$ for each $t \in T$. Then $A_t = C_t \bigvee D_t$ and $\beta cl(C_t) \bigwedge D_t = C_t \bigwedge \beta cl(D_t) = \underline{0}$ for each $t \in T$. Since A_t is $\beta-$connected, $C_t = \underline{0}$ or $D_t = \underline{0}$, therefore, $A_t = C_t \leq C$ or $A_t = D_t \leq D$. Especially, $A_s = C_s \leq C$ or $A_s = D_s \leq D$. Without loss of generality, we may assume that $A_s = C_s \leq C$. Then for each $t \neq s, A_t \leq C$. In fact, if $A_t \nleq C$, then $A_t \leq D$ and

$$A_t \bigwedge \beta cl(A_s) = A_t \bigwedge \beta cl(C_s) \leq D \bigwedge \beta cl(C) = \underline{0},$$

$$\beta cl(A_t) \bigwedge A_s = \beta cl(A_t) \bigwedge C_s \leq \beta cl(D) \bigwedge C = \underline{0}.$$

This shows that A_s and A_t are $\beta-$separated for each $t \neq s$ which is a contradiction. So $\bigvee_{t \in T} A_t \leq C$ and $D = D \bigwedge (\bigvee_{t \in T} A_t) \leq D \bigwedge C = \underline{0}$. We get that $A = \bigvee_{t \in T} A_t$ is $\beta-$connected.

Corollary 3.4. *Let (L^X, δ) be an $L-$topological space, $\{A_t\}_{t \in T}$ be a family of $\beta-$connected $L-$sets. If $\bigwedge_{t \in T} A_t \neq \underline{0}$, then $A = \bigvee_{t \in T} A_t$ is $\beta-$connected.*

Definition 3.3. Let (L^X, δ) be an $L-$topological space, $A \in L^X$. A is called a $\beta-$connected component of (L^X, δ) if A is a maximal $\beta-$connected $L-$set, i.e., $A = B$ for each $\beta-$connected $L-$set B in (L^X, δ) such that $A \leq B$.

Theorem 3.6. *Let (L^X, δ) be an $L-$topological space, then*

(1) *Every element $e \in M^*(L^X)$ is contained in a $\beta-$connected component of (L^X, δ).*

(2) *The join of all the $\beta-$connected components of (L^X, δ) equals to $\underline{1}$.*

(3) *The intersection of different $\beta-$connected components of (L^X,δ) is empty.*

(4) *Each $\beta-$connected component of (L^X,δ) is a $\beta-$closed $L-$set.*

Proof. (1) For each $e \in M^*(L^X)$, define $\mathscr{A} = \{A(e) \in L^X | A(e)$ is $\beta-$connected such that $e \le A(e)\}$. Then $\mathscr{A} \ne \emptyset$ by Corollary 3.1. Let $A = \bigvee \mathscr{A}$, then A is $\beta-$connected. Clearly, A is a $\beta-$connected component.

(2) The proof follows from (1) and the fact that $\bigvee M^*(L^X) = \underline{1}$.

(3) Suppose A and B are different $\beta-$connected components and $A \bigwedge B \ne \emptyset$. Then $A \bigvee B$ is $\beta-$connected by Corollary 3.4 which is a contradiction.

(4) Suppose A is a $\beta-$connected component of (L^X,δ), then $\beta cl(A)$ is $\beta-$connected and $A \le \beta cl(A)$. Therefore, $A = \beta cl(A)$ by Definition 3.3, that is, A is $\beta-$closed. □

Theorem 3.7. *Let $(L^X,\delta), (L^Y,\mu)$ be two $L-$topological spaces and $f : L^X \to L^Y$ be an $M\beta-$continuous order homomorphism. If A is $\beta-$connected in (L^X,δ), then $f(A)$ is $\beta-$connected in (L^Y,μ).*

Proof. Suppose $f(A)$ is not $\beta-$connected. Then there are two $\beta-$closed $L-$sets $C, D \in L^Y$ such that

$$f(A) \not\le C, f(A) \not\le D, f(A) \le C \bigvee D \text{ and } C \bigwedge D \bigwedge f(A) = \underline{0}.$$

So,

$$A \not\le f^{-1}(C), A \not\le f^{-1}(D), A \le f^{-1}(C \bigvee D) = f^{-1}(C) \bigvee f^{-1}(D)$$

and

$$\begin{aligned} f^{-1}(C) \bigwedge f^{-1}(D) \bigwedge A &\le f^{-1}(C) \bigwedge f^{-1}(D) \bigwedge f^{-1}(f(A)) \\ &= f^{-1}(C \bigwedge D \bigwedge f(A)) \\ &= \underline{0}. \end{aligned}$$

Since f is $M\beta-$continuous, $f^{-1}(C)$ and $f^{-1}(D)$ are two $\beta-$ closed $L-$sets in (L^X,δ). This shows that A is not $\beta-$connected which is a contradiction. Therefore $f(A)$ is $\beta-$connected in (L^Y,μ).

Corollary 3.5. *Let $(L^X,\delta), (L^Y,\mu)$ be two $L-$topological spaces and $f : L^X \to L^Y$ be an onto $M\beta-$continuous order homomorphism. If (L^X,δ) is a $\beta-$connected $L-$topological space, then so is (L^Y,μ).*

Now, we will extend the K. Fan's Theorem to $L-$topological space for $\beta-$connectedness.

Definition 3.4. Let (L^X,δ) be an $L-$topological space and $e\in M^*(L^X)$. Then $P\in\beta C(L^X)$ is called $\beta-$closed remote neighborhood of e if $e\not\leq P$. The set of all $\beta-$closed remote neighborhood of e will be denoted by $\beta\eta(e)$.

Theorem 3.8. *Let (L^X,δ) be an $L-$topological space and $A\in L^X$. Then A is $\beta-$connected if and only if for each pair a,b in $M^*(A)$ and each $\beta-$closed remote neighborhood mapping $P:M^*(A)\to\bigcup\{\beta\eta(e)|e\in M^*(A)\}$ where $P(e)\in\beta\eta(e)$ for each $e\in M^*(A)$, there exists finite number of points $e_1=a,e_2,\ldots,e_n=b$ in $M^*(A)$ such that $A\not\leq P(e_i)\bigvee P(e_{i+1}), i=1,2,\ldots,n-1$.*

Proof. Sufficiency. Suppose that A is not $\beta-$connected. Then there exist two non-null $\beta-$separated $L-$sets B,C such that $A=B\bigvee C$. Define the mapping $P:M^*(A)\to\bigcup\{\beta\eta(e)|e\in M^*(A)\}$ as the following:

$$P(e)=\begin{cases}\beta cl(C),\ \text{if } e\leq B,\\ \beta cl(B),\ \text{if } e\leq C.\end{cases}$$

We have $e\not\leq P(e)$ since $\beta cl(B)\bigwedge C=B\bigwedge\beta cl(C)=\underline{0}$. For $P(e)$ is a $\beta-$closed $L-$set, $P(e)\in\beta\eta(e)$ for each $e\in M^*(A)$. Take the points $a,b\in M^*(A)$ such that $a\leq B, b\leq C$. Since for arbitrary finite points $e_1=a,e_2,\cdots,e_n=b$ there is only one of $e_i\leq B$ and $e_i\leq C$ hold, we have $P(e_i)=\beta cl(B)$ or $P(e_i)=\beta cl(C)$. But $P(e_1)=\beta cl(C)$ and $P(e_n)=\beta cl(B)$, hence there exists $j(1\leq j\leq n-1)$ such that $P(e_j)=\beta cl(C)$ and $P(e_{j+1})=\beta cl(B)$. This shows that $A=B\bigvee C\leq P(e_j)\bigvee P(e_{j+1})$ which is a contradiction.

Necessity. Suppose that condition of theorem is not true, i.e., there are two points $a,b\in M^*(A)$ and a $\beta-$closed remote neighborhood mapping $P:M^*(A)\to\bigcup\{\beta\eta(e)|e\in M^*(A)\}$ such that $A\not\leq P(e_i)\bigvee P(e_{i+1}),(i=1,2,\ldots,n-1)$ is not true for arbitrary finite points $e_1,\ldots,e_n\in M^*(A)$. For the sake of convenience, we follow the agreement that a and b are $\beta-$ linked if there exist finite points $e_1=a,e_2,\ldots,e_n=b$ in $M^*(A)$ such that $A\not\leq P(e_i)\bigvee P(e_{i+1}), i=1,2,\ldots,n-1$. Otherwise, a and b are not $\beta-$ linked. Let

$$\Phi=\{e\in M^*(A)|a \text{ and } e \text{ are } \beta-\text{linked}\},$$

$$\Psi=\{e\in M^*(A)|a \text{ and } e \text{ are not } \beta-\text{linked}\},$$

$$B=\bigvee\Phi, C=\bigvee\Psi.$$

Obviously, a and a are $\beta-$linked for that $a\not\leq P(a)$ imply $A\not\leq P(a)$. So, $a\in\Phi, a\leq B$. By the hypothesis, a and b are not $\beta-$ linked, then $b\in\Psi$ and $b\leq C$. Hence, $B\neq\underline{0}, C\neq\underline{0}$. Since for each $e\in M^*(A)$, $e\in\Phi$ or $e\in\Psi$, we have $A=B\bigvee C$. We will prove $\beta cl(B)\bigwedge C=B\bigwedge\beta cl(C)=\underline{0}$. Hence, A is not $\beta-$connected which is a contradiction.

In fact, suppose $\beta cl(B)\bigwedge C\neq \underline{0}$ and take point $d\leq \beta cl(B)\bigwedge C$. By $d\leq \beta cl(B)$, we have $d\not\leq P(d)$ and $B\not\leq P(d)$. So there is $e\in\Phi$ such that $e\not\leq P(d)$. Hence $e\not\leq P(d)\bigvee P(e)$ and $e\leq B\leq A$. Thus, $A\not\leq P(d)\bigvee P(e)$. For e and a are $\beta-$ linked, then a and d are $\beta-$ linked. On the other hand, by $d\leq C$, we have $C\not\leq P(d)$. There exists $\lambda\in\Psi$ such that $\lambda\not\leq P(d)$. Hence $\lambda\not\leq P(d)\bigvee P(\lambda)$ and $\lambda\leq C\leq A$. Therefore, $A\not\leq P(d)\bigvee P(\lambda)$. By d and a are $\beta-$linked, we have a and λ are $\beta-$linked. This contradicts that $\lambda\in\Psi$. Thus, $\beta cl(B)\bigwedge C=\underline{0}$. Similarly, we can prove $B\bigwedge \beta cl(C)=\underline{0}$.

References

[1] M. E. Abd, El-Monsef, S. N. El-Deeb and R. A. Mahmould, *$\beta-$open sets and $\beta-$continous mappings*, Bull. Fac. Sci,. Assiut Univ, **12** (1983), 77–90.

[2] S. Z. Bai and W. L. Wang, *I type of strong connectivity in $L-$fuzzy topological spaces*, Fuzzy Sets and Systems, **99** (1998), 357–362.

[3] G. Balasubramanian, *On fuzzy $\beta-$compact spaces and fuzzy $\beta-$extremlly disconnected spaces*, Kybernetika, **33**(1997), 271–277.

[4] G. Balasubramanian, *Fuzzy $\beta-$open sets and fuzzy $\beta-$separation axioms*, Kybernetika, *35* (1999), 215–223.

[5] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24**(1968), 182–190.

[6] B. Ghosh, *Semicontinuous and semiclosed mappings and semi-connectedness in fuzzy setting*, Fuzzy Sets and Systems, **35** (1990), 345–355.

[7] G. Gierz, et al., *A compendium of continuous Lattices*, Springer Verlag, Berlin, 1980.

[8] I. M. Hanafy, *Fuzzy $\beta-$compactness and fuzzy $\beta-$closed spaces*, Turk. J. Math., **28** (2004), 281–293.

[9] S. G. Li, *Connectedness in $L-$ topological spaces*, Fuzzy Sets and Systems, **116** (2000), 361–368.

[10] Y. M. Liu and M. K. Luo, Fuzzy Topology, World Scientific, Singapore, 1997.

[11] R. Lowen and L. S. Xu, *Connectedness in FTS from different points of view*, J. Fuzzy. Math., **7** (1999), 51–66.

[12] S. Malakar, *On fuzzy semi-irresolute and strongly irresolute functions*, Fuzzy Sets and Systems, **45** (1992), 239–244.

[13] F. G. Shi and C. Y. Zheng, *Connectedness in fuzzy topological molecular lattices*, Fuzzy Sets and Systems, **29**(1989), 363–370.

[14] N. Turnli and D. Coker, *On some types of fuzzy connectedness in fuzzy topological spaces*, Fuzzy Sets and Systems, **60**(1993), 97–102.

[15] G. J. Wang, *Theory of $L-$fuzzy topological spaces*, Shaanxi Normal University Press, Xi'an, 1988.

[16] G. J. Wang, *Order homomorphisms on fuzzes*, Fuzzy Sets and Systems, **12**(1984), 281–288.

School of Mathematics and Statistics, Southwest University, Chongqing 400715, P.R.China.

E-mail: bobochen@swu.edu.cn