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ON OSTROWSKI-TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES ARE m-CONVEX AND (α,m)-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper we establish variant inequalities of Ostrowski's type for functions whose derivatives in absolute value are m-convex and (α,m)-convex. Applications to some special means are obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $\left|f'(x)\right| \le K$, then the following inequality,

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \le K(b-a)\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]$$

holds. This result is known in the literature as the *Ostrowski inequality*. For recent results and generalizations concerning Ostrowski's inequality see [1] – [3], [11, 12, 14, 15] and the references therein.

Recently in [1], Alomari et. al., have established some new inequalities for class of functions whose derivatives in absolute value are s-convex in the second sense by using the following lemma (see [7]):

Lemma 1 ([7])**.** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° where $a,b \in I$ with $a<b$. If $f' \in L[a,b]$, then the following equality holds:*

$$f(x)-\frac{1}{b-a}\int_a^b f(u)\,du = \frac{(x-a)^2}{b-a}\int_0^1 tf'(tx+(1-t)a)\,dt-\frac{(b-x)^2}{b-a}\int_0^1 tf'(tx+(1-t)b)\,dt$$

for each $x \in [a,b]$.

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Let $[0,b]$, where b is greater than 0, be an interval of the real line $\mathbb{R}$, and let $K(b)$ denote the class of all functions $f:[0,b]\to\mathbb{R}$ which are continuous and nonnegative on $[0,b]$ and such that $f(0)=0$. A function f is said to be convex on $[0,b]$ if

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)$$

holds for all $x,y\in[0,b]$ and $t\in[0,1]$. Let $K_C(b)$ denote the class of all functions $f\in K(b)$ convex on $[0,b]$, and let $K_F(b)$ be the class of all functions $f\in K(b)$ convex in mean on $[0,b]$, that is, the class of all functions $f\in K(b)$ for which $F\in K_C(b)$, where the mean function F of the function $f\in K(b)$ is defined by

$$F(x)=\begin{cases}\frac{1}{x}\int_0^x f(t)dt, & x\in(0,b]\\ 0, & x=0\end{cases}$$

Let $K_S(b)$ denote the class of all functions $f\in K(b)$ which are starshaped with respect to the origin on $[0,b]$, that is, the class of all functions f with the property that

$$f(tx)\le tf(x)$$

holds for all $x\in[0,b]$ and $t\in[0,1]$. In [4], Bruckner and Ostrow, among others, proved that

$$K_C(b)\subset K_F(b)\subset K_S(b).$$

In [18] G. Toader, (see also [5, Definition 1.1, Page 2]) defined m-convexity: another intermediate between the usual convexity and starshaped convexity.

Definition 1. The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be m-convex, where $m\in[0,1]$, if we have

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is m-concave if $-f$ is m-convex.

Denote by $K_m(b)$ the class of all m-convex functions on $[0,b]$ for which $f(0)\le 0$. Obviously, for $m=1$, m-convexity is the standard convexity of functions on $[0,b]$, and for $m=0$ the concept of starshaped functions. The following lemmas hold (see [18] see also [5, Lemma A & Lemma B, Page 2]).

Lemma 2. *If f is in the class $\mathrm{K}_m(b)$, then it is starshaped.*

Lemma 3. *If f is in the class $\mathrm{K}_m(b)$ and $0<n<m\le 1$, then f is in the class $\mathrm{K}_n(b)$.*

From Lemma 2 and Lemma 3 it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0,1)$. Note that in the class $K_1(b)$ are only convex functions $f:[0,b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is, $K_1(b)$ is a proper subclass of the class of convex functions on $[0,b]$. It is interesting to point out that for any $m \in (0,1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense (see [19]). The notion of m-convexity was further generalized by [13] in the following definition (see also [5, Definition 1.2, Page 3]).

Definition 2. The function $f:[0,b] \to \mathbb{R}$, $b>0$, is said to be (α,m)-convex, where $(\alpha,m) \in [0,1]^2$, if we have

$$f(tx+m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Denote by $K_m^{\alpha}(b)$ the class of all (α,m)-convex functions on $[0,b]$ for which $f(0) \leq 0$. It can be easily seen that for $(\alpha,m) \in \{(0,0),(\alpha,0),(1,0),(1,m),(1,1),(\alpha,1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex functions respectively. Note that in the class $K_1^1(b)$ are only convex functions $f:[0,b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is $K_1^1(b)$ is a proper subclass of the class of all convex functions on $[0,b]$. For further results on inequalities related to m-convex and (α,m)-convex functions we refer the readers [5].

2. Ostrowski's type inequalities for m-convex functions

In this section we establish Ostrowski type inequalities by using Lemma 1 for m-convex functions, we begin with the following result:

Theorem 1. *Let $f: I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I such that $f' \in L^1([a,b])$, where $0 \leq a < b < \infty$. If $\left|f'\right|$ is m-convex on $[a,b]$ for some fixed $m \in (0,1]$ and $\left|f'(x)\right| \leq K$, $x \in [a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)\,du \right| \leq \min\{M_1(x), M_2(x)\}, \tag{2.1}$$

where

$$M_1(x) = \frac{K}{3}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] + \frac{m}{6}\left[\frac{(x-a)^2\left|f'\left(\frac{a}{m}\right)\right| + (b-x)^2\left|f'\left(\frac{b}{m}\right)\right|}{b-a}\right]$$

and

$$M_2(x) = \left(\frac{K}{6} + \frac{m}{3}\left|f'\left(\frac{x}{m}\right)\right|\right)\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],\ x \in [a,b].$$

Proof. By Lemma 1, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f^{'}(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f^{'}(tx+(1-t)b)\right|dt. \tag{2.2}$$

Since $\left|f^{'}\right|$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$, for any $t\in[0,1]$, we have

$$\left|f^{'}(tx+(1-t)a)\right| = \left|f^{'}\left(tx+m(1-t)\frac{a}{m}\right)\right| \le t\left|f^{'}(x)\right| + m(1-t)\left|f^{'}\left(\frac{a}{m}\right)\right| \tag{2.3}$$

and

$$\left|f^{'}(tx+(1-t)b)\right| = \left|f^{'}\left(tx+m(1-t)\frac{b}{m}\right)\right| \le t\left|f^{'}(x)\right| + m(1-t)\left|f^{'}\left(\frac{b}{m}\right)\right|. \tag{2.4}$$

Using (2.3) and (2.4) in (2.2), we get that

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{K}{3}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] + \frac{m}{6}\left[\frac{(x-a)^2\left|f^{'}\left(\frac{a}{m}\right)\right| + (b-x)^2\left|f^{'}\left(\frac{b}{m}\right)\right|}{b-a}\right], \tag{2.5}$$

for all $x\in[a,b]$.

Analogously we obtain

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{K}{6}+\frac{m}{3}\left|f^{'}\left(\frac{x}{m}\right)\right|\right)\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] \tag{2.6}$$

for all $[a,b]$.

From (2.5) and (2.6), we get (2.1), and the proof is completed. □

Remark 1. For $m=1$, the m-convexity is the standard convexity, therefore (2.1) naturally reduces to (1.1).

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 2. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° where $a,b\in I$ with $0\le a<b<\infty$. If $\left|f^{'}\right|^q$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$, p, q, $\frac{1}{p}+\frac{1}{q}=1$ and $\left|f^{'}(x)\right|\le K$, $x\in[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \min\{N_1(x),N_2(x)\}, \tag{2.7}$$

where

$$N_1(x) = \frac{1}{(p+1)^{\frac{1}{p}}}\left[\frac{(x-a)^2}{b-a}\left(\frac{K^q}{2}+\frac{m}{2}\left|f^{'}\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}} + \frac{(b-x)^2}{b-a}\left(\frac{K^q}{2}+\frac{m}{2}\left|f^{'}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right]$$

and

$$N_2(x) = \frac{1}{(p+1)^{\frac{1}{p}}}\left(\frac{K^q}{2}+\frac{m}{2}\left|f^{'}\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],\ x\in[a,b].$$

Proof. Suppose $p>1$. From Lemma 1 and using Hölder inequality, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt \\
&\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \qquad (2.8)
\end{aligned}$$

Since $\left|f'\right|^q$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$ and $\left|f'(x)\right|\le K$, $x\in[a,b]$, we get that

$$\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\le\frac{K^q}{2}+\frac{m}{2}\left|f'\left(\frac{a}{m}\right)\right|^q$$

and

$$\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\le\frac{K^q}{2}+\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Therefore (2.8) reduces to

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
&\le\frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{K^q}{2}+\frac{m}{2}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}+\left(\frac{K^q}{2}+\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]. \qquad (2.9)
\end{aligned}$$

Analogously we also have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\le\frac{1}{(p+1)^{\frac{1}{p}}}\left(\frac{K^q}{2}+\frac{m}{2}\left|f'\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \qquad (2.10)$$

From (2.9) and (2.10), we get (2.7), which completes the proof. □

Theorem 3. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° where $a,b\in I$ with $0\le a<b<\infty$. If $\left|f'\right|^q$ is m-convex on $[a,b]$ for some fixed $m\in(0,1]$, $q\ge1$ and $\left|f'(x)\right|\le K$, $x\in[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\le\min\{S_1(x),S_2(x)\}, \qquad (2.11)$$

where

$$S_1(x)=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{K^q}{3}+\frac{m}{6}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}+\left(\frac{K^q}{3}+\frac{m}{6}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]$$

and

$$S_2(x)=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{K^q}{6}+\frac{m}{3}\left|f'\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],\quad x\in[a,b].$$

Proof. Suppose $q \geq 1$. From Lemma 1 and using power mean inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq \frac{(x-a)^2}{b-a}\int_0^1 t\left|f^{'}(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f^{'}(tx+(1-t)b)\right|dt \\
&\leq \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f^{'}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f^{'}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \qquad (2.12)
\end{aligned}$$

Since $\left|f^{'}\right|^q$ is m-convex on $[a,b]$, for some fixed $m \in (0,1]$ and $\left|f^{'}(x)\right| \leq K$, $x \in [a,b]$, we get that

$$\int_0^1 t\left|f^{'}(tx+(1-t)a)\right|^q dt \leq \frac{K^q}{3} + \frac{m}{6}\left|f^{'}\left(\frac{a}{m}\right)\right|^q$$

and

$$\int_0^1 t\left|f^{'}(tx+(1-t)b)\right|^q dt \leq \frac{K^q}{3} + \frac{m}{6}\left|f^{'}\left(\frac{b}{m}\right)\right|^q.$$

Therefore (2.8) reduces to

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \\
&\leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{K^q}{3} + \frac{m}{6}\left|f^{'}\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a} + \left(\frac{K^q}{3} + \frac{m}{6}\left|f^{'}\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]. \qquad (2.13)
\end{aligned}$$

Analogously we also have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{K^q}{6} + \frac{m}{3}\left|f^{'}\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \qquad (2.14)$$

The inequalities (2.13) and (2.14) give (2.10), which completes the proof. □

Remark 2. For any $p > 1$, $(1+p)^{\frac{1}{p}} < 2$, therefore (2.10) gives better result than (2.7) for any $m \in (0,1)$ and any $K > 0$. This also reveals that the approach via power mean inequality gives better result than that of the results obtained via Hölder inequality.

Corollary 1. *Under the assumptions of Theorem 2 and Theorem 3, For $m = 1$, we get the following results:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{K}{(p+1)^{\frac{1}{p}}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \ x \in [a,b], \qquad (2.15)$$

and

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{K}{2}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \ x \in [a,b]. \qquad (2.16)$$

Proof. It is a direct consequence of Theorem 2 and Theorem 3. □

Remark 3. In all the above inequalities one can obtain midpoint inequalities by setting $x = \frac{a+b}{2}$ and we omit the details for the interested readers.

3. Ostrowski's type inequalities for (α, m)-convex functions

We start with the following result:

Theorem 4. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° where $a,b \in I$ with $0 \le a < b < \infty$. If $\left|f'\right|$ is (α,m)-convex on $[a,b]$ for some fixed α, $m \in (0,1]$ and $\left|f'(x)\right| \le K$, $x \in [a,b]$, then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \min\left\{M_1'(x), M_2'(x)\right\}, \tag{3.1}$$

where

$$M_1'(x) = \frac{K}{\alpha+2}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] + \frac{m\alpha}{2(\alpha+2)}\left[\frac{(x-a)^2\left|f'\left(\frac{a}{m}\right)\right| + (b-x)^2\left|f'\left(\frac{b}{m}\right)\right|}{b-a}\right]$$

and

$$M_2'(x) = \frac{1}{\alpha+2}\left(\frac{\alpha K}{2} + m\left|f'\left(\frac{x}{m}\right)\right|\right)\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \; x \in [a,b].$$

Proof. By Lemma 1, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt + \frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt. \tag{3.2}$$

Since $\left|f'\right|$ is (α,m)-convex on $[a,b]$ for some fixed α, $m \in (0,1]$, for any $t \in [0,1]$, we have

$$\left|f'(tx+(1-t)a)\right| = \left|f'\left(tx + m(1-t)\frac{a}{m}\right)\right| \le t^\alpha\left|f'(x)\right| + m(1-t^\alpha)\left|f'\left(\frac{a}{m}\right)\right| \tag{3.3}$$

and

$$\left|f'(tx+(1-t)b)\right| = \left|f'\left(tx + m(1-t)\frac{b}{m}\right)\right| \le t^\alpha\left|f'(x)\right| + m(1-t^\alpha)\left|f'\left(\frac{b}{m}\right)\right|. \tag{3.4}$$

Using (3.3) and (3.4) in (3.2), we get that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{K}{\alpha+2}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] + \frac{m\alpha}{2(\alpha+2)}\left[\frac{(x-a)^2\left|f'\left(\frac{a}{m}\right)\right| + (b-x)^2\left|f'\left(\frac{b}{m}\right)\right|}{b-a}\right]. \tag{3.5}$$

for all $x \in [a,b]$.

Analogously we obtain

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{1}{\alpha+2}\left(\frac{\alpha K}{2}+m\left|f'\left(\frac{x}{m}\right)\right|\right)\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \tag{3.6}$$

for all $x \in [a,b]$, so that, from (3.5) and (3.6), we get (3.1), which completes the proof. □

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

Theorem 5. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° where $a,b \in I$ with $0 \le a < b < \infty$. If $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$ for some fixed $\alpha,\ m \in (0,1]$, $p,\ q,\ \frac{1}{p}+\frac{1}{q}=1$ and $\left|f'(x)\right| \le K$, $x \in [a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \min\left\{N_1'(x), N_2'(x)\right\} \tag{3.7}$$

where

$$N_1'(x)=\frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{K^q}{\alpha+1}+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}+\left(\frac{K^q}{\alpha+1}+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]$$

and

$$N_2'(x)=\frac{1}{(p+1)^{\frac{1}{p}}}\left(\frac{\alpha K^q}{\alpha+1}+\frac{m}{\alpha+1}\left|f'\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],\ x \in [a,b].$$

Proof. Suppose $p > 1$. From Lemma 1 and using Hölder inequality, we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| &\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt\\ &\le \frac{(x-a)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{3.8}$$

Since $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$ for some fixed $\alpha,\ m \in (0,1]$ and $\left|f'(x)\right| \le K$, $x \in [a,b]$, we get that

$$\int_0^1\left|f'(tx+(1-t)a)\right|^q dt \le \frac{K^q}{\alpha+1}+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a}{m}\right)\right|^q$$

and

$$\int_0^1 \left|f'(tx+(1-t)b)\right|^q dt \le \frac{K^q}{\alpha+1}+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Therefore (3.8) reduces to

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{1}{(p+1)^{\frac{1}{p}}}\left[\left(\frac{K^q}{\alpha+1}+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}+\left(\frac{K^q}{\alpha+1}+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]. \tag{3.9}$$

Analogously we also have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \frac{1}{(p+1)^{\frac{1}{p}}}\left(\frac{\alpha K^q}{\alpha+1}+\frac{m}{\alpha+1}\left|f'\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \tag{3.10}$$

The inequalities (3.9) and (3.10) give (3.7), and thus the proof is established. □

Another approach yields to the following result.

Theorem 6. *Let $f: I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° where $a,b\in I$ with $0\le a<b<\infty$. If $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$ for some fixed α, $m\in(0,1]$, $q\ge1$ and $\left|f'(x)\right|\le K$, $x\in[a,b]$, then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\le \min\left\{S_1'(x),S_2'(x)\right\}, \tag{3.11}$$

where

$$S_1'(x)=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{K^q}{\alpha+2}+\frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}+\left(\frac{K^q}{\alpha+2}+\frac{m\alpha}{\alpha+2}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]$$

and

$$S_2'(x)=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\alpha K^q}{2(\alpha+2)}+\frac{m}{\alpha+2}\left|f'\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right],\ x\in[a,b].$$

Proof. Suppose $q\ge1$. From Lemma 1 and using power mean inequality, we have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| &\le \frac{(x-a)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)a)\right|dt+\frac{(b-x)^2}{b-a}\int_0^1 t\left|f'(tx+(1-t)b)\right|dt\\ &\le \frac{(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\ &\quad+\frac{(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.\end{aligned} \tag{3.12}$$

Since $\left|f'\right|^q$ is (α,m)-convex on $[a,b]$ for some fixed α, $m\in(0,1]$ and $\left|f'(x)\right|\le K$, $x\in[a,b]$, we get that

$$\int_0^1 t\left|f'(tx+(1-t)a)\right|^q dt \le \frac{K^q}{\alpha+2}+\frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a}{m}\right)\right|^q$$

and

$$\int_0^1 t\left|f'(tx+(1-t)b)\right|^q dt \le \frac{K^q}{\alpha+2}+\frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{b}{m}\right)\right|^q.$$

Therefore (3.12) reduces to

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{K^q}{\alpha+2}+\frac{m\alpha}{2(\alpha+2)}\left|f'\left(\frac{a}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(x-a)^2}{b-a}\right.$$
$$\left.+\left(\frac{K^q}{\alpha+2}+\frac{m\alpha}{\alpha+2}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\frac{(b-x)^2}{b-a}\right]. \tag{3.13}$$

Analogously we also have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \le \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\alpha K^q}{2(\alpha+2)}+\frac{m}{\alpha+2}\left|f'\left(\frac{x}{m}\right)\right|^q\right)^{\frac{1}{q}}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]. \tag{3.14}$$

From (3.13) and (3.14), we obtain (3.11) and this completes the proof of the Theorem. □

Remark 4. For any $p>1$, $(1+p)^{\frac{1}{p}}<2$, therefore (3.11) gives better result than (3.7) for any α, $m\in(0,1]$ and any $K>0$. This also reveals that the approach via power mean inequality gives better result than that of obtain via Hölder inequality.

Remark 5. 1. In all the above inequalities one can obtain midpoint inequalities by setting $x=\frac{a+b}{2}$ and we omit the details for the interested readers.

2. For $\alpha=1$, we get the same inequalities as we obtained for m-convex functions, $m\in(0,1]$.

4. Applications to special means

We shall consider the means for arbitrary real numbers α, β $(\alpha\ne\beta)$. We take

1. The arithmetic mean:
$$A=:A(\alpha,\beta)=\frac{\alpha+\beta}{2},\ \ \alpha,\beta\ge 0.$$
2. The geometric mean
$$G=:G(\alpha,\beta)=\sqrt{\alpha\beta},\ \ \alpha,\beta\ge 0.$$
3. The Harmonic mean:
$$H=H(\alpha,\beta):=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},\alpha,\beta>0.$$

4. The identric mean:

$$I =: (\alpha,\beta) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}, \ \alpha,\beta > 0.$$

5. The logarithmic mean:

$$L =: L(\alpha,\beta) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}, \ \alpha,\beta > 0.$$

6. The p-logarithmic mean:

$$L_p := L_p(\alpha,\beta) = \begin{cases} \left(\frac{\beta^{p+1}-\alpha^{p+1}}{(p+1)(\beta-\alpha)}\right)^{\frac{1}{b-a}}, & \alpha \neq \beta \\ \alpha & \alpha = \beta \end{cases}, \ p \in \mathbb{R}\setminus\{-1,0\}, \ \alpha,\beta > 0.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} = L$ and $L_0 := I$. In particular, we have the following inequality $L \leq A$.

Now, using the results of Section 4, we give some applications to special means of real numbers. In the following we obtain some error estimates for some special means.

Consider $f : [a,b] \to \mathbb{R}$, $(0 < a < b)$, $f(x) = x^r$, $r \in \mathbb{R}\setminus\{-1,0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x) = L_r^r(a,b).$$

Using the inequality (3.1), we get

$$\left|x^r - L_r^r(a,b)\right| \leq \min\left\{M_1'(x), M_2'(x)\right\}, \ x \in [a,b],$$

where

$$\mu_r(a,b) = \begin{cases} r b^{r-1}, & r \geq 1 \\ |r|\, a^{r-1}, & r \in (-\infty,0)\cup(0,1)\setminus\{-1\} \end{cases},$$

$$M_1'(x) = \frac{\mu_r(a,b)}{\alpha+2}\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right] + \frac{m\alpha}{2(\alpha+2)}\left[\frac{(x-a)^2\left|f'\left(\frac{a}{m}\right)\right| + (b-x)^2\left|f'\left(\frac{b}{m}\right)\right|}{b-a}\right]$$

and

$$M_2'(x) = \frac{1}{\alpha+2}\left(\frac{\alpha\mu_r(a,b)}{2} + m\left|f'\left(\frac{x}{m}\right)\right|\right)\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right], \ x \in [a,b].$$

For instance, if one chooses, $x = A, G, H, I, L$, then we deduce some inequalities for the mentioned means, and we omit the details.

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References

[1] M. Alomari, M. Darus, S.S. Dragomir and P. Cerone, Ostrowski's inequalities for functions whose derivatives are s-convex in the second sense, *Appl. Math. Lett.*, Volume **23** (9) (2010), 1071–1076.

[2] M. Alomari, M. Darus, Some Ostrowski type inequalities for convex functions with applications, *RGMIA* **13** (1) (2010) article No. 3. Preprint.

[3] M. Alomari, M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, *RGMIA* **13** (2) (2010) article No. 3. Preprint.

[4] A. M. Bruckner and E. Ostrow, Some function classes related to the class of convex functions, *Pacific J. Math.*, **12** (1962), 1203–1215.

[5] M. K. Bakula, M. E. Özdemir and J. Pečarić, Hadamard type inequalities for m-convex and (α, m)-convex functions, *J. Inequal. Pure & Appl. Math.*, **9**(2008), Article 96. [ONLINE:htpp://jipam.vu.edu.au]

[6] M. K Bakula, J. Pečarić, and M. Ribičić, Companion inequalities to Jensen's inequality for m-convex and (α, m)-convex functions convex functions, *J. Inequal. Pure & Appl. Math.*, **7**(2006), Article 194. [ONLINE:htpp://jipam.vu.edu.au]

[7] P. Cerone and S.S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.*, **37** (2004), 299–308

[8] S. S. Dragomir and G. Toader, Some inequalities for m-convex functions, *Studia Univ. Babeş-Bolyai Math.*, **38**(1993), 21–28.

[9] S. S. Dragomir , Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.*, **167** (1992), 49–56.

[10] S.S. Dragomir and Th. M. Rassias, (Eds) *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.

[11] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html].

[12] Havva Kavurmacı, M. Emin Özdemir and Merve Avcı, New Ostrowski type inequalityes for m-convex functions and applications, *Hacettepe Journal of Mathematics and Statistics*, Volume **40** (2) (2011), 135–145

[13] V.G. Miheşan, A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca (Romania) (1993).

[14] M. E. Özdemir M. Avci and H. Kavurmaci, Hermite-Hadamard-type inequalities via (α, m)-convexity, *Computers & Mathematics with Applications*, Volume **61** (2011), 2614–2620.

[15] M. E Ödemir, H. Kavurmac and E. Set, Ostrowski's type inequalities for (α, m)-onvex functions, *Kyungpook Math. J.*, **50**(2010), 371–378.

[16] J.E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press Inc., 1992, p.137

[17] G. H. Toader, Some generalizations of the convexity, *Proc. Colloq. Approx. Optim*, Cluj-Napoca(Romania), 1984, 329–338.

[18] G. Toader, On a generalization of the convexity, *Mathematica*, **30** (53) (1988), 83–87.

[19] S. Toader, The order of a star-convex function, *Bull. Applied & Comp. Math.*, **85-B** (1998), BAM-1473, 347–350.

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