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A NEW OSCILLATION CRITERION FOR TWO-DIMENSIONAL DYNAMIC SYSTEMS ON TIME SCALES

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Abstract. Consider the linear dynamic system on time scales

$$u^{\Delta} = pv, \qquad v^{\Delta} = -qu^{\sigma} \tag{0.1}$$

where $p > 0$ and q are rd-continuous functions on a time scale $\mathbb{T}$ such that $\sup \mathbb{T} = \infty$. When $q(t)$ is allowed to take on negative values, we establish an oscillation criterion for system (0.1). Our result improves a main result of Fu and Lin [S. C. Fu and M. L. Lin, Oscillation and nonoscillation criteria for linear dynamic systems on time scales, Computers and Mathematics with Applications, 59(2010), 2552-2565].

1. Introduction

Consider the linear dynamic system on time scales

$$u^{\Delta} = pv, \qquad v^{\Delta} = -qu^{\sigma} \tag{1.1}$$

where $p(t) > 0$ and $q(t)$ are rd-continuous functions on a time scale $\mathbb{T}$ such that $\sup \mathbb{T} = \infty$. For convenience, we put $P(t) = \int_{t_0}^{t} p(s)\Delta s$. A solution $(x(t), y(t))$ of system (1.1) is called oscillatory if both $x(t)$ and $y(t)$ are oscillatory functions, and otherwise it will be called nonoscillatory. System (1.1) will be called oscillatory if its solutions are oscillatory. System (1) can be reduced to a single dynamic equation

$$\left(\frac{1}{p}u^{\Delta}\right)^{\Delta} + qu^{\sigma} = 0. \tag{1.2}$$

In [1], the following oscillation theorem is obtained.

Theorem 1.1. *Suppose that $p(t)$ and $q(t)$ are nonnegative and $\int_{t_0}^{\infty} p(s)\Delta s = \infty$, $\lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0$. Suppose also that there exists $\lambda \in (0,1)$ such that*

$$\int_{t_0}^{\infty} P^{\lambda}(t)q(t)\Delta t = \infty,$$

then system (1.1) is oscillatory.

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In this paper, we prove the above Theorem 1.1 when $q(t)$ is allowed to take on negative values. As applications, we get that the linear differential system

$$x' = t^{a-1}y, \qquad y' = -\left(\frac{b}{t^{\frac{a}{2}+1}} + \frac{\sin ct}{t^{\frac{a}{2}}}\right)x \tag{1.3}$$

and the linear difference system

$$\Delta x(n) = ny(n), \qquad \Delta y(n) = -\left(\frac{b}{n^2} + \frac{(-1)^n}{n}\right)x(n+1) \tag{1.4}$$

where $a > 0, b > 0, c \in R$, are oscillatory.

2. Some lemmas

Lemma 2.1. *If $(u(t), v(t))$ is a nonoscillatory solution of system (1.1), then the component $u(t)$ is also nonoscillatory.*

Proof. Assume that $(u(t), v(t))$ is a solution of system (1.1) and $u(t)$ is oscillatory, but $v(t)$ is nonoscillatory. Without loss of generality, we let $v(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. In view of the first equation of system (1.1), we have $u^\Delta(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Thus, $u(t) > 0$ or $u(t) < 0$ for all large t, which leads to a contradiction. So, the oscillation of $u(t)$ implies that of $v(t)$ as well. ☐

The following two lemmas may also be found in [3].

Lemma 2.2. *Assume $a \in \mathbb{T}$, let $\omega = \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If equation (1.2) is nonoscillatory on $[a,\omega)$, then there is a solution u_1, called a recessive solution at ω, such that for any second linearly independent solution u_2, called a dominant solution at ω, we have*

$$\lim_{t\to\omega^-} \frac{u_1(t)}{u_2(t)} = 0, \int_b^\omega \frac{p(t)}{u_1(t)u_1^\sigma(t)}\Delta t = \infty, \text{ and } \int_b^\omega \frac{p(t)}{u_2(t)u_2^\sigma(t)}\Delta t < \infty, \tag{2.1}$$

where $b < \omega$ is sufficiently close. Furthermore,

$$\frac{u_2^\Delta(t)}{p(t)u_2(t)} > \frac{u_1^\Delta(t)}{p(t)u_1(t)} \tag{2.2}$$

for $t < \omega$ sufficiently close.

Lemma 2.3. *(Picone's Identity). Assume $u(t)$ is a positive solution of (1.2), $z(t) = \frac{u^\Delta(t)}{p(t)u(t)}$ on $[a,\infty)$ and assume $h : \mathbb{T} \to \mathbb{R}$ is a continuously differentiable function. Then we have for all $t \in [a,\infty)$,*

$$[zh^2]^\Delta(t) = -q(t)h^2(\sigma(t)) + \frac{[h^\Delta(t)]^2}{p(t)} - \left[\frac{z(t)h^\sigma(t)}{\sqrt{\frac{1}{p(t)} + \mu(t)z(t)}} - \sqrt{\frac{1}{p(t)} + \mu(t)z(t)}\,h^\Delta(t)\right]^2.$$

The following theorem may be found in [6], Theorem 11, Page 17-18.

Theorem 2.4. *The equation* $(\frac{1}{p(t)}u'(t))' + q(t)u(t) = 0$ *is oscillatory on the interval* $[t_0,\infty)$, *if* $\int_{t_0}^{\infty} p(t)dt = \infty$ *and there exists a continuously differentiable function* $h(t) > 0$ *such that*

$$\int_{t_0}^{\infty}\left[q(t)h^2(t) - \frac{1}{p(t)}(h'(t))^2\right]dt = +\infty.$$

Analogous to the above theorem, we may obtain a corresponding time scales version which we state as follows:

Lemma 2.5. *If* $\int_{t_0}^{\infty} p(t)\Delta t = \infty$ *and there exists a function* $h \in C^1_{rd}[t_0,\infty)$ *such that* $h(t) > 0$ *on* $[t_0,\infty)$ *and*

$$\int_{t_0}^{\infty}\left[q(t)h^2(\sigma(t)) - \frac{1}{p(t)}[h^{\Delta}(t)]^2\right]\Delta t = +\infty. \tag{2.3}$$

Then the dynamic equation (1.2) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.2) is nonoscillatory. By Lemma 2.2, there is a dominant solution $u_2(t) > 0$ at ∞ such that for $t_1 > t_0$, sufficiently large,

$$\int_{t_1}^{\infty}\frac{p(t)}{u_2(t)u_2^{\sigma}(t)}\Delta t < \infty. \tag{2.4}$$

Let $z(t) = \frac{u_2^{\Delta}(t)}{p(t)u_2(t)}$. We have

$$z^{\Delta}(t) = -q(t) - \frac{z^2(t)}{\frac{1}{p(t)} + \mu(t)z(t)},$$

and

$$\frac{1}{p(t)} + \mu(t)z(t) = \frac{u_2(\sigma(t))}{p(t)u_2(t)} > 0.$$

From lemma 2.3, we get

$$\begin{aligned}
[zh^2]^{\Delta}(t) &= -q(t)h^2(\sigma(t)) + \frac{[h^{\Delta}(t)]^2}{p(t)} \\
&\quad - \left[\frac{z(t)h^{\sigma}(t)}{\sqrt{\frac{1}{p(t)} + \mu(t)z(t)}} - \sqrt{\frac{1}{p(t)} + \mu(t)z(t)}\,h^{\Delta}(t)\right]^2 \\
&\le -q(t)h^2(\sigma(t)) + \frac{[h^{\Delta}(t)]^2}{p(t)}.
\end{aligned}$$

Integrating from t_1 to t, we get

$$z(t)h^2(t) \le z(t_1)h^2(t_1) - \int_{t_1}^{t}\left[q(t)h^2(\sigma(t)) - \frac{[h^{\Delta}(t)]^2}{p(t)}\right]\Delta t.$$

By (2.3), we obtain

$$\lim_{t\to\infty} z(t)h^2(t) = -\infty.$$

So there exists a large $t_2 > t_1$ such that for $t > t_2$, we have

$$z(t) = \frac{u_2^{\Delta}(t)}{p(t)u_2(t)} < 0.$$

This implies $u_2^{\Delta}(t) < 0$ for $t > t_2$. Hence

$$\begin{aligned}\int_{t_2}^{\infty} p(t)\Delta t &= u_2(t_2)u(\sigma(t_2))\int_{t_2}^{\infty}\frac{p(t)}{u_2(t_2)u_2(\sigma(t_2))}\Delta t\\ &\le u_2(t_2)u(\sigma(t_2))\int_{t_2}^{\infty}\frac{p(t)}{u_2(t)u_2(\sigma(t))}\Delta t\\ &< \infty.\end{aligned}$$

which is a contradiction. □

From Lemma 2.1 and Lemma 2.5, we can get the following

Lemma 2.6. *Under the assumption of Lemma 2.5, the system (1.1) is oscillatory.*

Let $\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following (see [4] and [5]).

Condition (C). We say that $\mathbb{T}$ satisfies condition C if there is an $M > 0$ such that

$$\chi(t) \le M\mu(t), \quad t \in \mathbb{T}.$$

We note that if $\mathbb{T}$ satisfies condition (C), then the set

$$\check{\mathbb{T}} = \{t \in \mathbb{T} \mid t > 0 \text{ is isolated or right-scattered or left-scattered}\}$$

is necessarily countable.

Lemma 2.7. *Assume that $\mathbb{T}$ satisfies condition C. Then for all $\lambda \in [0,1)$, we have*

$$\int_{t_0}^{t}\frac{p(s)}{P^{2-\lambda}(\sigma(s))}\Delta s \le \frac{[P(t_0)]^{-1+\lambda}}{1-\lambda}.$$

Proof. For any $t \in \mathbb{T}$, if $t = t' < t'' = \sigma(t)$, from $P^{\Delta}(s) = p(s) > 0$, we get that

$$\int_{t}^{\sigma(t)}\frac{p(s)}{P^{2-\lambda}(\sigma(s))}\Delta s = \frac{\mu(t)P^{\Delta}(t)}{P^{2-\lambda}(\sigma(t))}$$

$$= \frac{P(\sigma(t)) - P(t)}{P^{2-\lambda}(\sigma(t))}$$
$$\le \int_{P(t)}^{P(\sigma(t))} \frac{1}{v^{2-\lambda}} dv. \tag{2.5}$$

If the real interval $[t', t''] \subset \mathbb{T}$, then

$$\int_{t'}^{t''} \frac{p(s)}{P^{2-\lambda}(\sigma(s))} \Delta s = \int_{t'}^{t''} \frac{p(s)}{P^{2-\lambda}(s)} ds = \int_{P(t')}^{P(t'')} \frac{1}{v^{2-\lambda}} dv. \tag{2.6}$$

Note that since $\mathbb{T}$ satisfies condition C, we have from (2.5), (2.6) and the additivity of the integral that

$$\int_{t_0}^{t} \frac{p(s)}{P^{2-\lambda}(\sigma(s))} \Delta s \le \int_{P(t_0)}^{P(t)} \frac{1}{v^{2-\lambda}} dv$$
$$= \frac{[P(t)]^{-1+\lambda} - [P(t_0)]^{-1+\lambda}}{-1+\lambda}$$
$$\le \frac{[P(t_0)]^{-1+\lambda}}{1-\lambda}.$$

□

3. Main Theorem

Theorem 3.1. *Assume that $\mathbb{T}$ satisfies condition C and* $\lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0$, $\int_{t_0}^{\infty} p(t)\Delta t = \infty$. *Suppose also that there exists* $\lambda \in [0, 1)$ *such that*

$$\int_{t_0}^{\infty} P^{\lambda}(\sigma(t)) q(t) \Delta t = \infty. \tag{3.1}$$

Then system (0.1) *is oscillatory.*

Proof. Take $h(t) = [P(t)]^{\frac{\lambda}{2}}$. Using the Pötzsche chain rule [3, Theorem 1.90], we get

$$h^{\Delta}(t) = \frac{\lambda}{2} \int_0^1 [(1-\tau)P(t) + \tau P(\sigma(t))]^{\frac{\lambda}{2}-1} d\tau\, p(t). \tag{3.2}$$

Since $\lim_{t\to\infty} \frac{\mu(t)p(t)}{P(t)} = 0$, it follows that $\lim_{t\to\infty} \frac{P(\sigma(t))}{P(t)} = 1$ so given ϵ with $0 < \epsilon < 1$, there exists t_1 sufficiently large so that for $t \ge t_1$ we have

$$P(t) \ge (1-\epsilon) P(\sigma(t)). \tag{3.3}$$

From (3.2), (3.3) and Lemma 2.7, we get

$$\int_{t_1}^{\infty} \frac{[h^{\Delta}(t)]^2}{p(t)} \Delta t \le \left(\frac{\lambda}{2}\right)^2 \int_{t_1}^{\infty} p(t) [P(\sigma(t))]^{\lambda-2} \left\{ \int_0^1 [(1-\tau)(1-\epsilon) + \tau]^{\frac{\lambda}{2}-1} d\tau \right\}^2 \Delta t$$
$$= \frac{1}{\epsilon^2} [1 - (1-\epsilon)^{\frac{\lambda}{2}}]^2 \int_{t_1}^{\infty} p(t) [P(\sigma(t))]^{\lambda-2} \Delta t$$

$$< +\infty. \tag{3.4}$$

From (3.1) and (3.4), we obtain

$$\int_{t_1}^{\infty}\left[q(t)h^2(\sigma(t)) - \frac{1}{p(t)}[h^{\Delta}(t)]^2\right]\Delta t = +\infty. \tag{3.5}$$

From (3.5) and Lemma 2.6, the system (1.1) is oscillatory. □

Remark 3.2. From the proof of the Theorem 3.1, it is easy to see that the assumption " $\lim\limits_{t\to\infty}\frac{\mu(t)p(t)}{P(t)} = 0$" in Theorem 3.1 can be replaced by "$\frac{P(\sigma(t))}{P(t)}$ is bounded for all $t \in \mathbb{T}$".

Let $\mathbb{T} = \mathbb{N}$. Consider the linear difference system

$$\Delta x(n) = 2^n y(n), \qquad \Delta y(n) = -\frac{\alpha + \beta(-1)^n}{[2^{n+1}-1]^{\frac{1}{2}}}x(n+1) \tag{3.6}$$

where $\beta > \alpha > 0$. It is easy to see that $P(t) = \int_0^t p(t)\Delta t = 2^t - 1$. $\lim_{t\to\infty}\frac{p(t)}{P(t)} = 1 \neq 0$. $\frac{P(\sigma(t))}{P(t)} = \frac{2^{t+1}-1}{2^t-1}$ is bounded for $t \geq 1$. Choose $\lambda = 1/2$. We have

$$\int_0^{\infty} P^{\frac{1}{2}}(\sigma(t))q(t)\Delta t = \sum_{n=1}^{\infty}(\alpha + \beta(-1)^n) = \infty.$$

So by Remark 3.2, (3.6) is oscillatory, but do not follow from Theorem 3.1.

4. Example

Example 4.1. Consider the linear differential system

$$x' = t^{a-1}y, \qquad y' = -\left(\frac{b}{t^{\frac{a}{2}+1}} + \frac{\sin ct}{t^{\frac{a}{2}}}\right)x \tag{4.1}$$

where $a > 0, b > 0, c \in R$.

Take $\lambda = \frac{1}{2}$. It is easy to see that

$$P(t) = \int_1^t p(s)ds = \frac{1}{a}(t^a - 1),$$

$$\int_1^{\infty} P^{\frac{1}{2}}(t)q(t)dt = \frac{b}{\sqrt{a}}\int_1^{\infty}\left[\frac{t^a-1}{t^{a+2}}\right]^{\frac{1}{2}}dt + \frac{1}{\sqrt{a}}\int_1^{\infty}(1-t^{-a})^{\frac{1}{2}}\sin ct\,dt.$$

By using Taylor's expansion, there exists a positive integer m such that $ma > 1$ and

$$(1-t^{-a})^{\frac{1}{2}} = 1 - 2^{-1}t^{-a} + \frac{2^{-1}(2^{-1}-1)}{2!}t^{-2a} + \cdots$$
$$+ (-1)^m\frac{2^{-1}(2^{-1}-1)\cdots(2^{-1}-m+1)}{m!}t^{-ma} + o(t^{-ma}).$$

Note that for $k = 1, 2, \cdots, m-1$, $\int_1^{\infty} t^{-ka}\sin ct\,dt$ and $\int_1^{\infty} o(t^{-ma})dt$ are convergent. It is obvious that $\int_1^{\infty}\left[\frac{t^a-1}{t^{a+2}}\right]^{\frac{1}{2}}dt = +\infty$. So we have $\int_1^{\infty} P^{\frac{1}{2}}(t)q(t)dt = +\infty$. By Theorem 3.1, the system (4.1) is oscillatory.

Example 4.2. Consider the linear difference system

$$\Delta x(n) = n y(n), \qquad \Delta y(n) = -\left(\frac{b}{n^2} + \frac{(-1)^n}{n}\right) x(n+1) \tag{4.2}$$

where $b > 0$.

Take $\lambda = \frac{1}{2}$. It is easy to see that $P(n) = \frac{n(n-1)}{2}$,

$$\int_1^\infty P^{\frac{1}{2}}(t)q(t)\Delta t = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} [n(n-1)]^{\frac{1}{2}} \left(\frac{b}{n^2} + \frac{(-1)^n}{n}\right).$$

Using Taylor's formula, we have

$$\left(1 - \frac{1}{n}\right)^{\frac{1}{2}} = 1 - \frac{1}{2n} + \frac{2^{-1}(2^{-1}-1)}{2} \frac{1}{n^2} + o(n^{-2}).$$

So for any m, $\sum_{n=1}^{m} (-1)^n (1 - \frac{1}{n})^{\frac{1}{2}}$ is bounded. Therefore $\int_1^\infty P^{\frac{1}{2}}(t)q(t)dt = +\infty$. By Theorem 3.1, the system (4.2) is oscillatory.

Example 4.3. Consider the q-difference system

$$x^\Delta(t) = t^{a-1} y(t), \qquad y^\Delta(t) = -\left(\frac{b + (-1)^n}{t^{\frac{a}{2}}}\right) x(qt). \tag{4.3}$$

where $a > 0, 0 < b < 1$, $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}, q > 1$.

Take $\lambda = \frac{1}{2}$. It is easy to see that $P(t) = \frac{(q-1)(t^a-1)}{q^a-1}$, $P(\sigma(t)) = \frac{(q-1)[(qt)^a-1]}{q^a-1}$. So $\lim_{t\to\infty} \frac{P(\sigma(t))}{P(t)} = q^a$, which implies $\frac{P(\sigma(t))}{P(t)}$ is bounded.

$$\int_1^\infty P^{\frac{1}{2}}(t)q(t)\Delta t = \left(\frac{q-1}{q^a-1}\right)^{\frac{1}{2}} \sum_{i=1}^{\infty} \left(\frac{q^{(i-1)a}-1}{q^{(i-1)a}}\right)^{\frac{1}{2}} [b + (-1)^i] = +\infty.$$

By Theorem 3.1 and Remark 3.2, the system (4.3) is oscillatory.

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