



WEAKLY PRIMAL GRADED SUPERIDEALS

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Abstract. Let G be an abelian group and let R be a commutative G -graded super-ring (briefly, graded super-ring) with unity $1 \neq 0$. We say that $a \in h(R)$, where $h(R)$ is the set of homogeneous elements in R , is *weakly prime* to a graded superideal I of R if $0 \neq ra \in I$, where $r \in h(R)$, then $r \in I$. If $v(I)$ is the set of homogeneous elements in R that are not weakly prime to I , then we define I to be weakly primal if $P = \bigoplus_{g \in G} (v(I) \cap R_g^0 + v(I) \cap R_g^1) \cup \{0\}$ forms a graded superideal of R . In this paper we study weakly primal graded superideals of R . Moreover, we classify the relationship among the families of weakly prime graded superideals, primal and weakly primal graded superideals of R .

1. Introduction

Weakly prime ideals in commutative rings have been introduced and studied by D. D. Anderson and E. Smith in [1], weakly prime graded ideals in commutative graded rings have been studied by many authors such as S. E. Atani in [3], and primal ideals in commutative rings have been introduced by Ladislav Fuchs in [6]. Graded super-rings are new important concept. They have been studied by many authors, includes Y. A. Bahturin in [4]. Because of the importance of the graded super-rings, we generalize the concept of primal, weakly primal and weakly prime to the graded supercase. Therefore, we introduce and study primal, weakly primal graded superideals and weakly prime graded superideals. Moreover, we classify the relationship among the families of weakly prime graded superideals, primal and weakly primal graded superideals. A number of results and examples concerning weakly primal graded superideals are given.

Let G be an abelian group and let R be any commutative ring with unity, then R is called a G -graded super-ring (for short graded super-ring), if $R = \bigoplus_{g \in G} (R_g^0 + R_g^1)$, such that if $a, b \in G$, then $R_a^\alpha R_b^\beta \subseteq R_{ab}^{\alpha+\beta}$, where $\alpha, \beta \in \mathbb{Z}_2$. Let $h_0(R) = \bigcup_{g \in G} R_g^0$ and $h_1(R) = \bigcup_{g \in G} R_g^1$, then $h(R) = h_0(R) \cup h_1(R)$ is the set of homogeneous elements in R and $1 \in R_e^0$, where e is the identity element in G .

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We say that an element $a \in h(R)$ is *weakly prime to a graded superideal* I of R if $0 \neq ra \in I$, where $r \in h(R)$, then $r \in I$. If I is a proper graded superideal of R and $v(I)$ is the set of homogeneous elements of R that are not weakly prime to I , then we define I to be *weakly primal* if the set $P = \bigoplus_{g \in G} (v(I) \cap R_g^0 + v(I) \cap R_g^1) \cup \{0\}$ forms a graded superideal of R . In this case we say that I is a *P-weakly primal* graded superideal of R . By the definition of weakly primal graded superideal we have that $I = \{0\}$ is a $\{0\}$ -weakly primal graded superideal of R . Moreover, it is easy to see that if I is a *P-weakly primal* graded superideal of R , then $v(I) = h(P) - \{0\}$.

We define a proper graded superideal P of R to be *weakly prime (prime)* if $0 \neq ab \in P$ ($ab \in P$) implies that $a \in P$ or $b \in P$, where $a, b \in h(R)$. Let I be a proper graded superideal of R , an element $a \in h(R)$ is called *prime to I* if $ar \in I$, where $r \in h(R)$, then $r \in I$. If I is a proper graded superideal of R and $\rho(I)$ is the set of homogeneous elements of R that are not prime to I , then we define I to be *primal* if the set $P = \bigoplus_{g \in G} (\rho(I) \cap R_g^0 + \rho(I) \cap R_g^1)$ forms a graded superideal of R . In this case we say that I is a *P-primal* graded superideal of R . Moreover, if I is a *P-primal* graded superideal of R , then it is easy to check that P is a prime graded superideal of R .

A graded superideal I of R is said to be *irreducible* if whenever I is the intersection of two graded superideals of R then I is one of them.

If I and J are graded superideals of R , then the graded superideal $\{r \in R : rJ \subseteq I\}$ is denoted by $(I : J)$. A homogeneous element in R is *regular* if it is a nonzero divisor in R . That is $a \in h(R)$ is regular if $(0 : a) = 0$. A commutative graded super-ring R with unity is called a *graded superdomain* if $h(R)$ has no divisors of zero.

In [2], S. E. Atani defined the prime subgroup P_g of R_g for $g \in G$, where P is a graded ideal in R . Throughout this work we always assume that if P is a graded superideal (ideal) of R , then P_e^1 is not a prime subgroup of R_e^1 , and P_g^α (P_g) is not a prime subgroup of R_g^α (R_g) for all $g \neq e \in G$ and $\alpha \in \mathbb{Z}_2$.

2. Weakly prime graded superideals

Let R be a commutative G -graded super-ring and let I be a graded superideal of R . Then every element prime to I is weakly prime to I , but the converse is not true. For example if $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$ with $u^2 = 0$, then R is a commutative super-ring with unity. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$, then I is a superideal of R and $\bar{6}$ is weakly prime to I , since if $\bar{a} \in h(R)$, then $\bar{a} \cdot \bar{6} \in I$ if and only if $\bar{a} \cdot \bar{6} = 0$ or $\bar{a} \in u\mathbb{Z}_{24} \subseteq I$. But $\bar{6}$ is not prime to I , since $\bar{4} \cdot \bar{6} \in I$ and $\bar{4} \notin I$.

Let R be a commutative G -graded super-ring which is not a graded superdomain. Then, by definition, $\{0\}$ is a $\{0\}$ -weakly primal graded superideal of R but not primal; hence weakly primal graded superideals need not to be primal graded superideals.

Let R be a G -graded super-ring (briefly graded super-ring) with unity. We start this section by giving two characterizations of weakly primal graded superideals of R .

Theorem 2.1. *Let I and P be proper graded superideals of R . Then the following statements are equivalent.*

- (1) I is a P -weakly primal graded superideal of R .
- (2) For $x \notin h(P) - \{0\}$, $h((I : x)) = h(I) \cup h((0 : x))$ and for $x \neq 0 \in h(P)$, $h(I) \cup h((0 : x)) \subsetneq h((I : x))$.

Proof. (1) \Rightarrow (2) Let I be a P -weakly primal graded superideal of R . Then $v(I) = h(P) - \{0\}$. If $x \notin v(I)$, then x is weakly prime to I . Let $r \in h((I : x))$. If $0 \neq rx \in I$, then $r \in I$, since x is weakly prime to I . If $rx = 0$, then $r \in (0 : x)$. Therefore we have $h((I : x)) \subseteq h(I) \cup h((0 : x))$ but also we clearly have that $I \subseteq (I : x)$ and $(0 : x) \subseteq (I : x)$; hence $h((I : x)) = h(I) \cup h((0 : x))$.

Now, assume that $x \neq 0 \in h(P)$ (i.e. $x \in v(I)$). Then x is not weakly prime to I ; hence there exists $r \in h(R) - h(I)$ such that $0 \neq rx \in I$. Thus $r \in h((I : x))$, but $r \notin h(I) \cup h((0 : x))$. Therefore, $h(I) \cup h((0 : x)) \subsetneq h((I : x))$.

(2) \Rightarrow (1) By (2) we have $v(I) = h(P) - \{0\}$. Thus I is a P -weakly primal graded superideal of R . \square

We remark that if A, B, C are graded superideals of R with $h(A) = h(B) \cup h(C)$ then $A = B + C$, but the converse need not to be true. For example $2\mathbb{Z} = 4\mathbb{Z} + 10\mathbb{Z}$ but $2 \notin 4\mathbb{Z} \cup 10\mathbb{Z}$.

Lemma 2.2. *Let I be a proper graded superideal of R . Then the following hold.*

- (1) If I is a P -weakly primal graded superideal of R , then $I \subseteq P$.
- (2) I is a $\{0\}$ -weakly primal graded superideal of R if and only if $I = \{0\}$.

Proof. (1) If $0 \neq a \in h(I)$, then $1.a \in I$ and $1 \notin I$, so a is not weakly prime to I . Thus $a \in P$ and hence $I \subseteq P$.

(2) This follows from (1) and the definition of weakly primal graded superideal. \square

The next two results are very important facts about weakly primal graded superideals, and will be used throughout the rest of the paper.

Theorem 2.3. *Every weakly prime graded superideal of R is weakly primal.*

Proof. Let P be a weakly prime graded superideal of R . Then $P \neq \{0\}$. It suffices to show that $v(P) = h(P) - \{0\}$. Let $0 \neq a \in h(P)$. Then $a = 1.a \in P$ with $1 \notin P$, so a is not weakly prime to P and hence $a \in v(P)$. On the other hand, if $a \notin h(P) - \{0\}$, then because P is weakly prime, we have $h((P : a)) = h(P) \cup h((0 : a))$, so if $r \in h(R)$ with $0 \neq ra \in P$, then $r \in h(P)$ and hence a is weakly prime to P , so $a \notin v(P)$. Thus $v(P) = h(P) - \{0\}$; hence P is a weakly primal graded superideal of R . \square

Proposition 2.4. *If I is a P -weakly primal graded superideal of R , then P is a weakly prime graded superideal of R .*

Proof. Suppose that $a, b \in h(R) - h(P)$ we show that $ab = 0$ or $ab \notin h(P)$. Assume that $ab \neq 0$ and let $0 \neq rab \in I$ for some $r \in h(R)$. Then by Theorem 2.1 we have $0 \neq ra \in h((I : b)) = h(I) \cup h((0 : b))$, but $ra \notin h((0 : b))$; hence $0 \neq ra \in h(I)$. Then, from Theorem 2.1, $0 \neq r \in h((I : a)) = h(I) \cup h((0 : a))$, where $ra \neq 0$. So, $r \in h(I)$ and hence ab is weakly prime to I . Thus $ab \notin h(P)$. \square

Now we give an example of a weakly primal superideal of R that is not weakly prime.

Example 2.5. Let $R = \mathbb{Z}_8 + u\mathbb{Z}_8$ where $u^2 = 0$. Then R is a commutative super-ring with unity. If $I = 4\mathbb{Z}_8 + u\mathbb{Z}_8$ then I is not a weakly prime superideal of R , since $\bar{2} \cdot \bar{2} \neq 0 \in I$, but $\bar{2} \notin I$. Let $P = 2\mathbb{Z}_8 + u\mathbb{Z}_8$, we show that I is a P -weakly primal superideal of R . It is enough to show that $v(I) = h(P) - \{0\}$. Let $0 \neq \bar{a} \in h(P)$, if $\bar{a} \in 2\mathbb{Z}_8$ then $\bar{a} = 2k \in \mathbb{Z}_8$. If k is an odd number, then $0 \neq \bar{2}\bar{a} \in I$, but $\bar{2} \notin I$, and if k is an even number $0 \neq \bar{1}\bar{a} \in I$ with $\bar{1} \notin I$; hence $\bar{a} \in v(I)$. If $\bar{a} \in u\mathbb{Z}_8$ then $\bar{a} \in I \subseteq v(I)$. On the other hand, if $\bar{a} \in h(R) - h(P)$ then \bar{a} is an odd number in \mathbb{Z}_8 . If $0 \neq \bar{a}\bar{m} \in I$ for some $\bar{m} \in \mathbb{Z}_8$ then 4 divides am and so, 4 divides m since $(4, a) = 1$; hence $\bar{m} \in I$. Thus I is a P -weakly primal superideal of R .

The following corollary depends on the fact if (R, P) is a quasilocal graded super-ring with $P^2 = \{0\}$, then every nontrivial proper graded superideal of R is a weakly prime graded superideal.

Corollary 2.6. *If (R, P) is a quasilocal graded super-ring with $P^2 = \{0\}$, then every nontrivial proper graded superideal of R is a weakly primal graded superideal.*

We recall that if P is a weakly prime graded superideal in R , and P_e^0 is not a prime ideal in R_e^0 , where e is the identity element in G , then $(P_e^0)^2 = \{0\}$ (See [1, Theorem 3]). Next we have the following results about graded superideals.

Theorem 2.7. *Let P be a graded superideal in R , where R is a graded super-ring and where P_e^0 is not a prime ideal in R_e^0 . Then $(P_g)^4 = \{0\}$ for all $g \in G$.*

Proof. Let $a = a^0 + a^1 \in P_g$. Then $a^2 = (a^0)^2 + (a^1)^2 + 2a^0a^1 = 2a^0a^1$ which implies that $a^4 = 4(a^0)^2(a^1)^2 = 0$. Thus $(P_g)^4 = \{0\}$. \square

The next result on weakly prime graded superideals depends on Theorem 2.7.

Theorem 2.8. *Assume that G is a finite abelian group of order $n \geq 2$. If P is a graded superideal of R , where P_e^0 is not a prime ideal in R_e^0 . Then $P^{4n} = \{0\}$.*

Proof. Let $c = a \in P_g$ then, by Theorem 2.7, $c^4 = 0$. Suppose for $c = a_1 + \cdots + a_k \in P$, where $a_i \in P_{g_i}$ for all $i = 1, 2, \dots, k$ and $k \leq n-1$, $c^{4k} = 0$. We show that it is true for n . Let $c = a_1 + \cdots + a_n \in P$, where $a_i \in P_{g_i}$ for all i , then $c = a_1 + x$ where $x = a_2 + \cdots + a_n$ and

$$c^{4(n-1)} = m_{4n-5} a_1^{4n-5} x + \cdots + m_1 a_1 x^{4n-5} = m_3 a_1^3 x^{4n-7} + m_2 a_1^2 x^{4n-6} + m_1 a_1 x^{4n-5}$$

where $m_i \in \mathbb{Z}^+$; hence

$$c^{4n} = c^{4(n-1)} c^4 = (m_3 a_1^3 x^{4n-7} + m_2 a_1^2 x^{4n-6} + m_1 a_1 x^{4n-5})(m' a_1 x^3 + m'' x^4) = 0,$$

since $a_1^4 = 0$ or $x^{4(n-1)} = 0$. Thus, $c^{4n} = 0$ which implies that $P^{4n} = \{0\}$. \square

Easy computations give us the following corollary of Theorem 2.8.

Corollary 2.9. *Let P be a weakly prime graded superideal of R , where R is a graded super-ring, and where P_e^0 is not a prime ideal in R_e^0 . Then $P^{4n-1} \sqrt{0} = \{0\}$.*

Now we have the following known result in the ungraded case.

Proposition 2.10. *Let R be a commutative ring with unity. If P and Q are weakly prime ideals of R , but not prime ideals then $PQ = \{0\}$.*

Proof. See [1, Corollary 5]. \square

Following the same proof used in Theorem 2.8 we have the following result about graded ideals.

Corollary 2.11.

- (1) *Let P be a weakly prime graded ideal of R , where P_e is not a prime ideal in R_e , and let $|G| = n$, then $P^{2n} = \{0\}$.*
- (2) *If $|G| = 1$, then $P^2 = \{0\}$.*

Depending on Theorem 2.7, [1, Theorem 3], Proposition 2.10 and [2, Theorem 2.6], we have the following remark, where P and Q are weakly prime graded superideals (ideals) of R and where P_e^0 (P_e) and Q_e^0 (Q_e) are not prime ideals in R_e^0 (R_e).

Remark 2.12.

- (1) Let P and Q be weakly prime graded ideals of R . Then by [2, Theorem 2.6], $P + Q$ is a weakly prime graded ideal of R .
 - (a) If $(P_g + Q_g)^2 = \{0\}$, then $P_g Q_g = \{0\}$ for all $g \neq e \in G$.
 - (b) $(P_g Q_h)^2 = \{0\}$ for all $g, h \in G$.

- (c) If $P_e + Q_e$ is not a prime ideal in R_e , then $(P_e + Q_e)^2 = \{0\}$; hence $P_e Q_e = \{0\}$.
- (2) Let P and Q be weakly prime graded superideals of R . Then by [2, Theorem 2.6], $P + Q$ is a weakly prime graded superideal of R .
- (a) If $(P_g + Q_g)^4 = \{0\}$ for all $g \neq e \in G$, then $P_g Q_g (P_g + Q_g)^2 = \{0\}$.
- (b) $(P_g Q_h)^4 = \{0\}$ for all $g, h \in G$.
- (c) If $(P_e^1 + Q_e^1)^2 = \{0\}$ and $(P_e^0 + Q_e^0)$ is not a prime ideal in R_e^0 , then $(P_e + Q_e)^4 = \{0\}$.

Now depending on [1, Theorem 3], Theorem 2.8 and Corollary 2.9 we have the following result.

Theorem 2.13. *Let I be a P -weakly primal graded superideal of R and J a Q -weakly primal graded superideal of R , where P_e^0 and Q_e^0 are not prime ideals in R_e^0 and where $|G| = n$. Then the following statements are hold.*

- (1) $I^{4n} = \{0\}$, $IP^{4n-1} = \{0\}$, $I^{4n-1}\sqrt{0} = \{0\}$ and $\sqrt{I} = \sqrt{\{0\}}$.
- (2) $(IJ)^{4n} = \{0\}$.

Proof. (1) By Lemma 2.2, Proposition 2.4, [1, Theorem 3] and Theorem 2.8 we get $I^{4n} \subseteq P^{4n} = \{0\}$, $IP^{4n-1} \subseteq P^{4n} = \{0\}$ and $I^{4n-1}\sqrt{0} \subseteq P^{4n-1}\sqrt{0} = \{0\}$. Finally, since $I^{4n} = \{0\}$, we get $I \subseteq \sqrt{0}$; hence $\sqrt{I} \subseteq \sqrt{0}$, but $\sqrt{0} \subseteq \sqrt{I}$ is trivial, so $\sqrt{I} = \sqrt{\{0\}}$.

(2) By Lemma 2.2, Proposition 2.4, [1, Theorem 3] and Theorem 2.8 we have $(IJ)^{4n} \subseteq P^{4n}Q^{4n} = \{0\}$. Thus, $(IJ)^{4n} = \{0\}$. \square

In Theorem 2.8 and Corollary 2.11 we showed that if P is a weakly prime graded superideal (ideal) of R , where P_e^0 (P_e) is not a prime ideal in R_e^0 (R_e), then $P^{4n} = \{0\}$ ($P^{2n} = \{0\}$), where $n = |G|$. In the next result we will use an important fact for finite abelian groups to improve Theorem 2.8 and Corollary 2.11. Compare the next result with Theorem 2.8 and Corollary 2.11.

Theorem 2.14.

- (1) Let $|G| = 2k$ or $2k + 1$, and let P be weakly prime graded ideal of R where P_e is not a prime ideal in R_e . Then $P^{k+2} = \{0\}$.
- (2) Let $|G| = 2k$ or $2k + 1$, and let P be a weakly prime graded superideal of R where P_e^0 is not a prime ideal in R_e^0 . Then $P^{|G|+2} = \{0\}$.

Proof. If $|G| = 1$ or $|G| = 2$ and P is a weakly prime graded ideal of R where P_e is not a prime ideal in R_e then by [1, Theorem 3], $(P_e)^2 = \{0\}$. This implies that $P^2 = \{0\}$ if $|G| = 1$, or $P^3 = \{0\}$ if $|G| = 2$. Similarly, if $|G| = 1$ or $|G| = 2$ and P is a weakly prime graded superideal of R where

P_e^0 is not a prime ideal in R_e^0 then by [1, Theorem 3], $(P_e^0)^2 = \{0\}$. This implies that $P^3 = \{0\}$ if $|G| = 1$, or $P^4 = \{0\}$ if $|G| = 2$.

Next suppose that G is a finite abelian group of order $2k$ or $2k+1$ where $k \geq 2$. We prove (1) and (2) by using the following fact about G . Let $S \subseteq G$ with $|S| = k+2$. For any $s \neq e \in S$, let $T = S - \{s\}$. Then $T \cap sT \neq \emptyset$. Because, if $T \cap sT = \emptyset$ then $T \cup sT \subseteq G$ and $|T \cup sT| = 2(k+1) = 2k+2 > |G|$, a contradiction. Therefore, $T \cap sT \neq \emptyset$. Let $x \in T \cap sT$, then there exists $y \in T$ with $sy = x$. Since $s \neq e$, we have $x \neq y$. Moreover, $x \neq s$ and $y \neq s$, since $x, y \in T = S - \{s\}$.

(1) Now, since P_e is weakly prime ideal in R_e but not a prime ideal we have that $(P_e)^2 = \{0\}$. So, to show that $P^{k+2} = \{0\}$, then it is enough to prove that for any $p_1, p_2, \dots, p_{k+2} \in h(P)$ we have $p_1 \cdot p_2 \cdot \dots \cdot p_{k+2} = 0$. Let $p_1, p_2, \dots, p_{k+2} \in P$, where each $p_i \in P_{g_i}$ for all $i = 1, 2, \dots, k+2$ and $g_i \neq g_j$ for all $i \neq j$. If $g_i = g_j$ for some i, j between 1 and $k+2$, then $p_1 \cdot p_2 \cdot \dots \cdot p_{k+2} \in (P_{g_i})^2 = \{0\}$. Therefore we may assume that $g_i \neq g_j$ for all $i \neq j$. Let $g_i \neq e$ for some i between 1 and $k+2$ then by the fact above if $T = \{g_1, g_2, \dots, g_{k+2}\} - \{g_i\}$ then there exist $n \neq m$ between 1 and $k+2$ such that $g_n, g_m \in T$ and $g_i = g_n g_m$. Thus $p_1, p_2, \dots, p_{k+2} \in (P_{g_i})^2 = \{0\}$; hence $P^{k+2} = \{0\}$.

(2) Since P_e^0 is not a prime ideal in R_e^0 we have $(P_e^0)^2 = \{0\}$.

In part (2), we may assume that P is a weakly prime graded ideal of R over the new abelian group $G' = \mathbb{Z}_2 \times G$. So, by using part (1) we get that $P^m = \{0\}$ for any $m \geq |G| + 2$. \square

We end this section by the following result. Compare the next result with Proposition 2.10.

Theorem 2.15. *Let P and Q be weakly prime graded superideals (ideals) of R , where $|G| = n$, and let P_e^0 (P_e) and Q_e^0 (Q_e) be not prime ideals in R_e^0 (R_e).*

(1) *If P and Q are weakly prime graded ideals of R , then $(PQ)^2 = \{0\}$.*

(2) *If P and Q be weakly prime graded superideals of R , then $(PQ)^2 = \{0\}$.*

Proof. (1) We prove it by induction on $k \leq n$. Let $k = 1$ and let $a = a_1 \in h(PQ)$, then $a = p_g q_h$, where $p_g \in P_g$ and $q_h \in Q_h$. Then $a^2 = p_g^2 q_h^2 = 0$. Suppose $a^2 = 0$ for any $a = a_1 + \dots + a_k$, where $k \leq n-1$ and $a_i \in h(PQ)$ for all $i = 1, \dots, k$. Now, let $a = a_1 + a_2 + \dots + a_n$, where $a_i \in h(PQ)$ for all $i = 1, \dots, n$. Then $a = a_1 + x$, where $x = a_2 + \dots + a_n$, and by induction hypothesis $a^2 = a_1^2 + 2a_1x + x^2 = 0 + 2a_1x + 0$. Thus, $a^2 = 2a_1x$. Now, $a_1x = a_1a_2 + \dots + a_1a_n$. For any $j = 2, 3, \dots, n$ then by induction hypothesis we have that $(a_1 + a_j)^2 = 0$, so $(a_1 + a_j)^2 = 0 = a_1^2 + a_j^2 + 2a_1a_j$. Since $a_1^2 = a_j^2 = 0$ we get that $2a_1a_j = 0$; hence $a^2 = 2a_1x = 0$. Therefore, $(PQ)^2 = \{0\}$.

(2) The proof is similar to that of (1). \square

3. Weakly primal graded superideals

We start this section by recalling that there is a one-to-one correspondence between weakly prime graded superideals of R containing the weakly prime graded superideal J of R and weakly prime graded superideals of R/J . Therefore, we have the following result.

Theorem 3.1. *Let I be a proper graded superideal of R , and let J be a weakly prime graded superideal of R with $J \subseteq I$. Then I is a weakly primal graded superideal of R if and only if I/J is a weakly primal graded superideal of R/J .*

Proof. Suppose that I is a P -weakly primal graded superideal of R with $J \subseteq I$. Then by Lemma 2.2(1) and Proposition 2.4, P is a weakly prime graded superideal of R containing J . Therefore, P/J is a weakly prime graded superideal of R/J . It suffices to show that $v(I/J) = h(P/J) - \{0\}$. Let $0 \neq a + J \in h(P/J)$. Then $0 \neq a$ is not weakly prime to I ; hence there exists $r \in h(R) - (I)$ such that $0 \neq ra \in I$. If $ra \in J$ then, because J is a weakly prime graded superideal of R and $a \notin J$, $r \in J$ which is a contradiction since $J \subseteq I$ and $r \notin I$. It follows that $0 \neq (r + J)(a + J) \in I/J$ with $r + J \notin h(I/J)$, therefore, $a + J$ is not weakly prime to I/J . Now assume that $b + J \in h(R/J)$ is not weakly prime to I/J . Then there exists $c + J \in h(R/J) - (I/J)$ such that $0 \neq cb + J \in I/J$; hence $0 \neq cb \in I$ with $c \in h(R) - h(I)$. So, $b \neq 0$ is not weakly prime to I ; hence $0 \neq b \in h(P)$ and therefore, $b + J \in h(P/J) - \{0\}$. Thus, $v(I/J) = h(P/J) - \{0\}$.

Conversely, suppose that I/J is a P/J -weakly primal graded superideal of R/J ; we show that I is a P -weakly primal graded superideal of R . By Proposition 2.4, P/J is a weakly prime graded superideal of R/J and hence P is a weakly prime graded superideal of R containing J . It is enough to show that $v(I) = h(P) - \{0\}$. Let $0 \neq a \in h(P)$. By Lemma 2.2(1), we can assume that $a \notin I$. Therefore, $0 \neq a + J \in P/J$. As I/J is a P/J -weakly primal graded superideal of R/J , there exists $r + J \in h(R/J) - h(I/J)$ such that $0 \neq ra + J \in I/J$; hence $0 \neq ra \in I$ with $r \in h(R) - h(I)$. Thus a is not weakly prime to I that is $a \in v(I)$. Now assume that $a \in v(I)$. So, $a \neq 0$. If $a \in h(I)$ then $a \in h(P)$, so we can assume that $a \notin h(I)$. Then there is an element $r \in h(R) - h(I)$ with $0 \neq ra \in I$. Because J is a weakly prime graded superideal of R and $a \notin h(J)$ and $r \notin J$ we get that $ra \notin J$. Therefore, $0 \neq ra + J = (r + J)(a + J) \in I/J$ with $r + J \in h(R/J) - h(I/J)$; hence $a + J \in h(P/J) - \{0\}$, since I/J is a P/J -weakly primal graded superideal of R/J . Therefore, $0 \neq a \in h(P)$ which implies that $v(I) = h(P) - \{0\}$. \square

Let S be a subset of $h(R) - \{0\}$, where R is a graded super-ring. Then S is called a multiplicatively closed subset of R if for any $a, b \in S$ we have $ab \in S$.

Lemma 3.2. *Let S be a multiplicatively closed subset of R . Then the following hold.*

- (1) *If I is a P -weakly primal graded superideal of R with $h(P) \cap S = \emptyset$ and $0 \neq a/s \in S^{-1}I$, then $a \in I$.*

(2) If Q is a weakly prime graded superideal of R with $h(Q) \cap S = \emptyset$ and $0 \neq a/s \in S^{-1}Q$, then $a \in Q$.

Proof. First assume that $0 \neq a/s \in S^{-1}I$ but $a \in h(R) - h(I)$. Then $a/s = r/t$ for $r \in h(I)$ and $t \in S$, so there exists $u \in S$ such that $0 \neq uta = usr \in I$ with $a \notin I$. Hence $ut \in S$ is not weakly prime to I which is a contradiction. Thus, $a \in h(I)$.

(2) This follows from (1) and Theorem 2.3. \square

Recall that $a \in h(R)$ is a regular element in R , if a is a nonzero divisor; that is $a \in h(R)$ is a regular element in R , if $(0 : a) = \{0\}$. In the rest of this section, we always assume that if S is a multiplicatively closed subset of R , then S consists of regular elements.

Proposition 3.3. *Let S be a multiplicatively closed subset of R and let I be a P -weakly graded primal superideal of R such that $h(P) \cap S = \emptyset$. Then the following hold.*

- (1) $S^{-1}I$ is a $S^{-1}P$ -weakly primal graded superideal of $S^{-1}R$.
- (2) $I = S^{-1}I \cap R$.

Proof. (1) We will show that $v(S^{-1}I) = h(S^{-1}P) - \{0\}$. Let $0 \neq \frac{a}{s} \in h(S^{-1}P)$. Then, by Lemma 3.2(2), $0 \neq a \in h(P)$; hence there exists $r \in h(R) - h(I)$ such that $0 \neq ra \in I$. Because S consists of regular elements in R , we have $\frac{ra}{s} \neq 0 \in S^{-1}I$. Thus, $0 \neq \frac{ra}{s} = (\frac{r}{1})(\frac{a}{s}) \in S^{-1}I$ where, by Lemma 3.2(1), $\frac{r}{1} \notin S^{-1}I$. Hence $\frac{a}{s} \in v(S^{-1}I)$. On the other hand, suppose that $\frac{a}{s} \in v(S^{-1}I)$. Then there exists $\frac{r}{t} \in h(S^{-1}R) - h(S^{-1}I)$ such that $0 \neq (\frac{r}{t})(\frac{a}{s}) \in S^{-1}I$. Because S consists of regular elements in R , $ra \neq 0 \in I$ with $r \notin h(I)$; hence $0 \neq a \in h(P)$. Thus, $\frac{a}{s} \in h(S^{-1}P) - \{0\}$.

(2) Since it is clear that $I \subseteq S^{-1}I \cap R$; we will prove the reverse inclusion. Let $a \in S^{-1}I \cap h(R)$. Then, $\frac{a}{1} \in S^{-1}I$, so, $a \in h(I)$ by Lemma 3.2(1). \square

Proposition 3.4. *Let S be a multiplicatively closed subset of R and let Q be a weakly graded prime superideal of $S^{-1}R$. Then $Q \cap R$ is a weakly prime graded superideal of R .*

Proof. Suppose that $0 \neq ab \in Q \cap R$. Then $0 \neq \frac{ab}{1} \in Q$. Since S consists of regular elements, $0 \neq (\frac{a}{1})(\frac{b}{1}) \in Q$. Since Q is weakly prime, we have $\frac{a}{1} \in Q$ or $\frac{b}{1} \in Q$; hence $a \in Q \cap R$ or $b \in Q \cap R$. \square

Proposition 3.5. *Let S be a multiplicatively closed subset of R . If I is a Q -weakly primal graded superideal of $S^{-1}R$, then $I \cap R$ is a $Q \cap R$ -weakly primal graded superideal of R .*

Proof. By Proposition 3.4, $P = Q \cap R$ is a weakly prime graded superideal of R . We will show that $h(P) - \{0\} = v(I \cap R)$. Suppose that $0 \neq a \in h(P)$. Then $0 \neq \frac{a}{1} \in Q$, so there exists $\frac{r}{s} \in$

$h(S^{-1}R) - h(I)$ such that $0 \neq (\frac{r}{s})(\frac{a}{1}) \in I$. Hence $0 \neq ra \in I \cap R$ with $r \in h(R) - h(I \cap R)$. It follows that $a \in \nu(I \cap R)$. On the other hand assume that $b \in \nu(I \cap R)$, where $b \neq 0$. Then there is an element $s \in h(R) - h(I \cap R)$ with $0 \neq sb \in I \cap R$, where $\frac{sb}{1} \neq 0 \in I$ since S consists of regular elements. So, $(\frac{s}{1})(\frac{b}{1}) \neq 0 \in I$ with $\frac{s}{1} \notin I$ implies that $\frac{b}{1} \in Q$. Thus, $b \in h(P)$. \square

Let P be a weakly prime graded superideal of R with $h(P) \cap S = \emptyset$, where S is a multiplicatively closed subset of R consists of regular elements. Then $S^{-1}P$ is a weakly prime graded superideal of $S^{-1}R$. On the other hand if N is a proper weakly prime graded superideal of $S^{-1}R$, then $P = N \cap R$ is a weakly prime graded superideal of R with $h(P) \cap S = \emptyset$, since if $a \in h(P) \cap S$ then $\frac{a}{1} \in N$ which implies that $1 \in N$; hence $N = S^{-1}R$, a contradiction. Therefore, we can say that there is a one-to-one correspondence between weakly prime graded superideals P of R with $h(P) \cap S = \emptyset$ and weakly prime graded superideals of $S^{-1}R$, where S is a multiplicatively closed subset of R consists of regular elements. Now, using this fact, we are able to prove the following result.

Theorem 3.6. *Let S be a multiplicatively closed subset of R , and let P be a weakly prime graded superideal of R with $h(P) \cap S = \emptyset$. Then there exists a one-to-one correspondence between the P -weakly primal graded superideals of R and the $S^{-1}P$ -weakly primal graded superideals of $S^{-1}R$.*

Proof. This follows from Propositions 3.3, 3.4, 3.5 and the fact that there is a one-to-one correspondence between weakly prime graded superideals of R and weakly prime graded superideals of $S^{-1}R$, where S is a multiplicatively closed subset of R consists of regular elements. \square

4. Primal graded superideals

Let R be a commutative graded super-ring which is not a graded superdomain. We recall that 0 is a 0 -weakly primal graded superideal of R , but it is not primal. Now, in this section we investigate when a weakly primal graded superideal of a commutative graded super-ring is primal. But first we start by the following example which shows that a primal graded superideal of R need not to be weakly primal.

Example 4.1. Let $R = (\mathbb{Z}_{24} + u\mathbb{Z}_{24}) + v(\mathbb{Z}_{24} + u\mathbb{Z}_{24})$ where $uv = vu$ and $u^2 = v^2 = 0$. Then R is a commutative \mathbb{Z}_2 -graded super-ring. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24} + v(\mathbb{Z}_{24} + u\mathbb{Z}_{24})$.

- (1) Since $0 \neq \bar{2} \cdot \bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get that $\bar{2}$ and $\bar{4}$ are not weakly prime to I . Easy computations imply that $\bar{2} + \bar{4} = \bar{6}$ is weakly prime to I . Thus we obtain that I is not a weakly primal \mathbb{Z}_2 -graded superideal of R .

(2) Set $P = 2\mathbb{Z}_{24} + u\mathbb{Z}_{24} + v(\mathbb{Z}_{24} + u\mathbb{Z}_{24})$. We show that I is a P -primal \mathbb{Z}_2 -graded superideal of R . It is easy to check that every element of $h(P)$ is not prime to I . Conversely, assume that $\bar{a} \in h(R) - h(P)$, then $\bar{a} \in \mathbb{Z}_{24}$ with $(a, 8) = 1$. If $\bar{a} \cdot \bar{n} \in I$ for some $\bar{n} \in \mathbb{Z}_{24}$ then 8 divides n ; hence $\bar{n} \in I$. Therefore, $h(P)$ is exactly the set of elements in $h(R)$ which are not prime to I . Thus I is a P -primal \mathbb{Z}_2 -graded superideal of R .

Proposition 4.2. *Let R be a commutative super-ring and let I be a P -weakly primal superideal of R with $I_0 I_\alpha \neq \{0\}$ for each $\alpha \in \mathbb{Z}_2$. If P is a prime superideal of R then I is P -primal.*

Proof. It is enough to show that $\rho(I) = h(P)$. If $a \in h(P)$, then a is not prime to I . So, $a \in \rho(I)$. Assume that $a \in \rho(I)$ we show that $a \in h(P)$. Let $r_\alpha \in R_\alpha - I_\alpha$ such that $r_\alpha a \in I$. If $0 \neq r_\alpha a \in I$, then $a \in h(P)$ since I is a P -weakly primal superideal of R . So we may assume that $r_\alpha a = 0$.

Suppose $a \in R_\beta$.

If $a I_\alpha \neq \{0\}$, then there exists $s \in I_\alpha$ with $as \neq 0$. So, $r_\alpha + s \notin I_\alpha$ and $0 \neq a(r_\alpha + s) = as \in I$. Hence $a \in h(P)$. Therefore, we may assume that $a I_\alpha = \{0\}$. If $r_\alpha I_\beta \neq \{0\}$, then there exists $s_\beta \in I_\beta$ with $r_\alpha s_\beta \neq 0$, so $0 \neq (a + s_\beta)r_\alpha = s_\beta r_\alpha \in I$ which implies that $a \in h(P)$. Thus we may assume that $r_\alpha I_\beta = \{0\}$. Since $I_0 I_\alpha \neq \{0\}$, let $c \in I_0, d \in I_\alpha$ with $cd \neq 0$. If $\beta = 0$, then $0 \neq (a + c)(r_\alpha + d) = cd \in I_0$ implies that $a \in h(P)$. If $\beta = 1$, then $0 \neq (a^2 + c)(r_\alpha + d) = cd \in I_0$ implies that $a^2 \in h(P)$ and hence $a \in h(P)$, since P is a prime. \square

Now we generalize Proposition 4.2.

Theorem 4.3. *Let G be a finite abelian group of order n , and let R be a commutative graded super-ring. Suppose I is a P -weakly primal graded superideal of R with $(I_e^0)(I_g^\alpha) \neq \{0\}$ for all $g \in G$ and $\alpha \in \mathbb{Z}_2$. If P is a prime graded superideal of R then I is P -primal.*

Proof. It is enough to show that $v(I) = h(P)$. If $a \in h(P)$, then a is not prime to I . So, $a \in v(I)$. Assume that $a \in v(I)$. We show that $a \in h(P)$. Suppose $a \in R_x^\beta$ where $x \in G$ and $\beta \in \mathbb{Z}_2$. Then there exists $r_g^\alpha \in h(R) - h(I)$ with $r_g^\alpha a \in I$. If $r_g^\alpha a \neq 0$, then $a \in h(P)$ since I is a P -weakly primal graded superideal of R . Therefore we may assume that $r_g^\alpha a = 0$. If $a I_g^\alpha \neq \{0\}$, then there exists $r' \in I_g^\alpha$ with $ar' \neq 0$. Thus, $0 \neq a(r_g^\alpha + r') \in I$ and $r_g^\alpha + r' \notin I$; hence $a \in h(P)$. Therefore we may assume that $a I_g^\alpha = \{0\}$. Let m be the smallest positive integer with $x^m = e$, where e is the identity element in G . Then $a^{2m} \in R_e^0$, since $2m\beta = 0 \in \mathbb{Z}_2$. Now, if $r_g^\alpha I_e^0 \neq \{0\}$, then there exists $s \in I_e^0$ with $r_g^\alpha s \neq 0$. So, $0 \neq (a^{2m} + s)r_g^\alpha \in I$ which implies that $a^{2m} \in P$ and hence $a \in h(P)$, since P is a prime graded superideal of R . Therefore we may assume that $r_g^\alpha I_e^0 = \{0\}$. Since $(I_e^0)(I_g^\alpha) \neq \{0\}$, there exists $c \in I_e^0$ and $d \in I_g^\alpha$ with $cd \neq 0$. Now, $0 \neq cd = (a^{2m} + c)(r_g^\alpha + d) \in I_g^\alpha$ and $r_g^\alpha + d \notin I_g^\alpha$ implies that $a^{2m} + c \in P_e^0$. So, $a^{2m} \in P_e^0 \subseteq h(P)$; hence $a \in h(P)$. \square

Let G be a group of order n . We recall that if P is a prime graded superideal of R and I is a P -weakly primal graded superideal of R but not P -primal, then, by Theorem 4.3, there exists

$g \in G$ and $\alpha \in \mathbb{Z}_2$ such that $(I_g^0)(I_g^\alpha) = \{0\}$. Therefore if $g = e$ and $\alpha = 0$, then $(I_e^0)^2 = \{0\}$. Hence, by Theorem 2.13(1), $I^{4n} = \{0\}$ which implies that $\sqrt{I} = \sqrt{\{0\}}$.

Definition 4.4. Let I be a graded superideal of R , and let P be the smallest graded superideal of R satisfying the following condition: if $r, a \in h(R)$ with $ra \in I$, then $r \in I$ or $a \in P$. Then, I is called a P -primary graded superideal of R .

Remark 4.5.

- (1) If I is a P -primary graded superideal of R , then $\sqrt{I} \subseteq P$.
- (2) I is a primary graded superideal of R if and only if $\sqrt{I} = P$.

Now we have the following result about primary graded superideals.

Proposition 4.6. Every primary graded superideal of R is primal.

Proof. Let I be a primary graded superideal of R . We prove that $\rho(I) = h(\sqrt{I})$.

Let $r \in \rho(I)$, then there exists $a \in h(R) - h(I)$ with $ra \in I$. Because I is a primary graded superideal of R , then $r \in h(\sqrt{I})$.

Conversely, assume that $b \in h(\sqrt{I})$. If $b \in I$, then b is not prime to I ; hence $b \in \rho(I)$. So, we may assume that $b \notin I$. Let n be the smallest positive integer such that $b^n \in I$. Then $n \geq 2$ and $b^{n-1} \notin I$. Now, $b^n = bb^{n-1} \in I$ and $b^{n-1} \notin I$ implies that $b \in \rho(I)$. Thus, $\rho(I) = h(\sqrt{I})$. \square

The next result is a generalization of [6, Theorem 1] to the graded supercase.

Proposition 4.7. Every irreducible graded superideal of R is primal.

Proof. Let I be an irreducible graded superideal of R and let $b_1, b_2 \in h(R)$ be not prime to I . Then $I \subsetneq (I : b_1)$ and $I \subsetneq (I : b_2)$. Hence

$$I \subsetneq (I : b_1) \cap (I : b_2) = (I : \langle b_1 \rangle + \langle b_2 \rangle),$$

since I is irreducible, where $\langle a \rangle = aR$ for $a \in h(R)$. One can easily check that

$$(I : \langle b_1 \rangle + \langle b_2 \rangle) = (I : \langle b_1 \rangle + \langle b_1 - b_2 \rangle),$$

which implies that

$$I \subsetneq (I : b_1) \cap (I : b_1 - b_2).$$

Therefore, $I \subsetneq (I : b_1 - b_2)$; hence $b_1 - b_2$ cannot be prime to I . Moreover, $I \subsetneq (I : b_1)$ implies that rb_1 cannot be prime to I for any $r \in h(R)$. So, I is a primal graded superideal of R . \square

Recall that R is a graded superdomain if R is a commutative graded super-ring with unity and $h(R)$ has no divisors of zero.

Proposition 4.8. *A graded superideal of a graded superdomain is primal if and only if it is weakly primal.*

Proof. We may assume that $I \neq \{0\}$. Suppose that I is a P -primal graded superideal of R . We show that I is a weakly P -primal graded superideal of R . Let $a \in h(P) - \{0\}$. If $a \in I$ then a is not weakly prime to I . So we may assume that $a \notin I$. Because I is P -primal, there exists $b \in h(R) - h(I)$ with $0 \neq ab \in I$ since R is a graded superdomain. Thus a is not weakly prime to I . On the other hand, if a is not weakly prime to I then $a \neq 0$ and a is not prime to I since R is a graded superdomain. Thus, $a \in h(P) - \{0\}$.

Conversely, suppose that I is a weakly P -primal graded superideal of R . We show that I is P -primal. By Proposition 2.4, P is a weakly prime graded superideal of R . Hence it is a prime graded superideal of R , since R is a graded superdomain. It is enough to show that $h(P)$ is exactly the set of elements of $h(R)$ that are not prime to I . Clearly, $0 \in h(P)$. Let $0 \neq a \in h(P)$. Then a is not weakly prime to I . So it is not prime to I . On the other hand, suppose that $a \in h(R)$ is not prime to I . If $a = 0$ then $a \in h(P)$, so we may assume that $a \neq 0$. Then there exists $r \in h(R) - h(I)$ with $ra \in I$. Since R a graded superdomain we have that $0 \neq ar \in I$. So a is not weakly prime to I ; hence $a \in h(P)$. \square

By using Proposition 4.6, Proposition 4.7 and Proposition 4.8, we have the following result.

Corollary 4.9. *Let R be a graded superdomain. Then the following hold.*

- (1) *Every primary graded superideal of R is weakly primal.*
- (2) *Every irreducible graded superideal of R is weakly primal.*

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