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OSCILLATION AND NONOSCILLATION OF NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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Abstract. In this paper, the authors establish some sufficient conditions for oscillation and nonoscillation of the second order nonlinear neutral delay difference equation

$$\Delta^2(x_n - p_n x_{n-k}) + q_n f(x_{n-\ell}) = 0, \quad n \geq n_0$$

where $\{p_n\}$ and $\{q_n\}$ are non-negative sequences with $0 < p_n \leq 1$, and k and ℓ are positive integers.

1. Introduction

Consider the second order nonlinear neutral delay difference equation

$$\Delta^2(x_n - p_n x_{n-k}) + q_n f(x_{n-\ell}) = 0, \quad n \geq n_0 \in \mathbb{N} \tag{1}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, subject to the following conditions:

(c_1) $\{p_n\}$ and $\{q_n\}$ are non-negative real sequences with $\{q_n\}$ not identically zero for infinitely many values of n;

(c_2) $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing such that $uf(u) > 0$ for $u \neq 0$;

(c_3) there is a positive constant p such that $0 < p_n \leq p < 1$, and k and ℓ are positive integers.

For any real sequence $\{\phi_n\}$ defined in $n_0 - \theta \leq n \leq n_0$ where $\theta = \max\{k, \ell\}$, equation (1) has a solution $\{x_n\}$ defined for $n \geq n_0$ and satisfying the initial condition $x_n = \phi_n$ for $n_0 - \theta \leq n \leq n_0$. A solution $\{x_n\}$ of equation (1) is oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In several recent papers [3, 4, 7−20], the oscillatory and nonoscillatory behavior of solutions of equation (1) has been studied when $\{p_n\}$ is a non-positive real sequence. However in [14], the authors consider the case $\{p_n\}$ is non-negative and attempted to extend the known results in [1] on delay difference equation to neutral difference equation with $p_n \equiv p \in (0, 1)$. In fact the authors [14] proved the following two theorems:

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Theorem A. *Assume that* $0 < p < 1$, $\{q_n\}$ *is a nonnegative real sequence and* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing with* $uf(u) > 0$ *for* $u \neq 0$. *If*

$$0 < \int_{\varepsilon}^{\infty} \frac{dx}{f(x)}, \int_{-\infty}^{-\varepsilon} \frac{dx}{f(x)} < \infty \quad \text{for all} \quad \varepsilon > 0 \tag{2}$$

then every solution of the equation

$$\Delta^2(x_n - px_{n-k}) + q_n f(x_{n-\ell}) = 0, \quad n \geq n_0 \tag{3}$$

is oscillatory if and only if

$$\sum_{n=n_0}^{\infty} nq_n = \infty. \tag{4}$$

Theorem B. *Assume that* $0 < p < 1$, $\{q_n\}$ *is a nonegative real sequence and* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing with* $uf(u) > 0$ *for* $u \neq 0$. *If*

$$0 < \int_{0}^{\varepsilon} \frac{dx}{f(x)}, \int_{-\varepsilon}^{0} \frac{dx}{f(x)} < \infty \quad \text{for all} \quad \varepsilon > 0 \tag{5}$$

and

$$f(uv) \geq f(u)f(v) \quad \text{if} \quad uv > 0 \quad \text{and} \quad |v| \geq M \tag{6}$$

for some constant $M > 0$, *then every solution of equation* (3) *is oscillatory if and only if*

$$\sum_{n=n_0}^{\infty} f(n)q_n = \infty. \tag{7}$$

In the following we give an example which illustrates the sufficient part of Theorem A is false.

Let $k, \ell \geq 1$, $0 < p < 1$, $\alpha > 1$. Choose $\lambda > -\frac{1}{k}\log p$ and set $q_n = \frac{(pe^{\lambda k}-1)(e^{-\lambda}-1)^2 e^{(\alpha-1)\lambda n}}{e^{\lambda \ell k}}$. It is easy to see that $\{x_n\} = \{e^{-\lambda n}\}$ is a positive solution of the equation

$$\Delta^2(x_n - px_{n-k}) + q_n|x_{n-\ell}|^{\alpha-1}x_{n-\ell} = 0, \quad n \geq n_0 \tag{8}$$

even if (4) is satisfied. The error occurred in the proof is due to their false assertion that if $\{x_n\}$ is eventually positive solution of equation (3) then $z_n = x_n - px_{n-k}$ is also eventually positive. The same false assertion was also used in the proof of Theorem B and therefore the sufficient part of Theorem B may not be true. Therefore, so far there are hardly any results on the oscillatory behavior of solutions of equation (1) with $\{p_n\}$ is nonnegative.

In this paper, we study the oscillatory and nonoscillatory behavior of equation (1) with $0 \leq p_n < 1$ and the nonlinear function f is either supelinear or sublinear. In Section 2, we present a new sufficient condition for the oscillation of all solutions of equation (1) when f is superlinear and extend the necessary part of Theorem A to equation (1). Section 3 contains similar results for equation (1) when f is sublinear. For basic results on the oscillation theory of difference equations one can refer the recent monographs [1] and [2].

2. Oscillation results for superlinear case

In this section we shall investigate the oscillatory behavior of solutions of equation (1) when f is superlinear. The function f is said to be superlinear if there exists a constant $\alpha > 0$ such that

$$\lim_{x\to 0} \inf \Big(\frac{|f(x)|}{|x|^{\alpha}}\Big) > 0. \tag{9}$$

We need the following lemma given in [12] to prove our main result of this section.

Lemma 1. *Let $\{Q_n\}$ be a nonnegative real sequence, $f : \mathbb{R} \to \mathbb{R}$ be continuous with $uf(u) > 0$ for $u \neq 0$, and δ be a positive integer. Assume that there exist $\beta > 0$ and $\lambda > \frac{1}{\delta}\log\beta$ such that $\lim_{x\to 0}(\frac{|f(x)|}{|x|^{\beta}}) > 0$ and $\lim_{n\to\infty}\inf[Q_n \exp(-e^{\lambda n})] > 0$ then the following inequality*

$$\Delta x_n + Q_n f(x_{n-\delta}) \leq 0, \quad n \geq n_0,$$

has no eventually positive solutions.

Theorem 2. *With respect to the difference equation* (1) *assume that $\ell > k$, and condition* (9) *hold. If there exist a $\lambda > \frac{\log\alpha}{\ell - k}$ such that*

$$\lim_{n\to\infty} \inf q_n \exp(-e^{\lambda n}) > 0 \tag{10}$$

then every solution of equation (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1). We may assume without loss of generality that $x_n > 0$ and $0 < p_n < p$ for all $n \geq n_0$ for some integer $n_0 \in N$. Set

$$y_n = x_n - p_n x_{n-k}. \tag{11}$$

Then it follows from equation (1) that $\Delta^2 y_n \leq 0$ for all $n \geq n_0 + \theta$. This implies that $\{\Delta y_n\}$ is nonincreasing for all $n \geq n_0 + \theta$. Hence, there are two possible cases that $\Delta y_n > 0$ for all $n \geq n_0 + \theta$ or $\Delta y_n < 0$ for all $n \geq n_1$ for some integer $n_1 \geq n_0$. If the later case holds, then there exits a constant $c > 0$ and an integer $n_2 \geq n_1$ such that

$$x_n - p_n x_{n-k} \leq -c, \quad n \geq n_2,$$

which implies that

$$x_n \leq -c + p x_{n-k}, \quad n \geq n_2. \tag{12}$$

From (12), we have

$$\begin{aligned} x_{n_2+k} &\le -c + p x_{n_2} \\ x_{n_2+2k} &\le -c + p(x_{n_2+k}) \le -c - pc + p^2 x_{n_2} \\ x_{n_2+3k} &\le -c + p(x_{n_2+2k}) \le -c - pc - p^2 c + p^3 x_{n_2} \end{aligned}$$

and hence it follows that

$$x_{n_2+jk} \le -\sum_{i=0}^{j-1} cp^i + p^j x_{n_2},$$

and so $x_{n_2+jk} < 0$ for large j, which contradicts the fact that $x_n > 0$ for all $n \ge n_0$. Hence

$$\Delta y_n > 0 \quad \text{for all} \quad n \ge n_0 + \theta. \tag{13}$$

From (13), it follows that $\{y_n\}$ is increasing for all $n \ge n_0 + \theta$ and so there are two possible cases:

(i) $y_n < 0$ for $n \ge n_0 + \theta$ or
(ii) $y_n > 0$ for $n \ge n_3$ for integer $n_3 \ge n_2$.

If case (i) holds, that is, $y_n < 0$ for all $n \ge n_0 + \theta$ then

$$x_{n-\ell} > -\frac{1}{p} y_{n+k-\ell}, \quad n \ge n_0 + 2\theta, \tag{14}$$

and

$$\Delta^2 y_n + q_n f\Big(-\frac{1}{p} y_{n+k-\ell}\Big) \le 0, \quad n \ge n_0 + 2\theta. \tag{15}$$

Summing the inequality (15) from $n \ge n_0 + 2\theta$ to ∞, we find

$$-\Delta y_n + \sum_{s=n}^{\infty} q_s f\Big(-\frac{1}{p} y_{s+k-\ell}\Big) \le 0, \quad n \ge n_0 + 2\theta. \tag{16}$$

From the assumption $\lambda > \frac{\log \alpha}{\ell - k}$, we can choose an integer m such that $1 \le m \le \ell - k$ and

$$\alpha e^{-\lambda(\ell-k-m)} < 1. \tag{17}$$

Note that $-\Delta y_n$ is decreasing for all $n \ge n_0 + \theta$, it follows from (16) that

$$-\Delta y_n + \Big(\sum_{s=n}^{n+m} q_s\Big) f\Big(-\frac{1}{p} y_{n+k-\ell+m}\Big) \le 0, \quad n \ge n_0 + 2\theta. \tag{18}$$

Set

$$z_n = -\frac{1}{p}\Delta y_n, \quad \delta = \ell - k - m, \quad Q_n = \frac{1}{p}\sum_{s=n}^{n+m} q_s.$$

Then (18) can be written as

$$\Delta z_n + Q_n f(z_{n-\delta}) \le 0, \quad n \ge n_0 + 2\theta. \tag{19}$$

This shows that (19) has an eventually positive solution $\{z_n\}$. On the other hand, by (10),

$$\lim_{n\to\infty} \inf[Q_n \exp(-e^{\lambda n})] \geq \frac{(m+1)}{p} \lim_{n\to\infty} \inf\left[\left(\min_{n\leq s\leq n+m} q_s\right)\exp(-e^{\lambda n})\right] > 0. \quad (20)$$

In view of (17) and (20), Lemma 1 implies that the inequality (19) has no eventually positive solutions. This contradiction shows that case (i) is impossible.

If case (ii) holds, that is , $y_n > 0$ for all $n \geq n_3$, then it follows from equation (1) that

$$\Delta^2 y_n + q_n f(y_{n-\ell}) \leq 0, \quad n \geq n_3 + \theta. \quad (21)$$

Summing (21) from $n_4 = n_3 + \theta$ to n and then taking $n \to \infty$, we find

$$\sum_{n=n_4}^{\infty} q_n f(y_{n-\ell}) \leq \Delta y_{n_4}. \quad (22)$$

Since $f(y_n)$ is nondecreasing for all $n \geq n_4$, it follows from (22) that

$$f(y_{n_3}) \sum_{s=n}^{\infty} q_s \leq \Delta y_{n_4} < \infty,$$

which contradicts (10) and so case (ii) is also impossible. This completes the proof of the theorem.

In the following theorem, we extend the necessary part of Theorem A to equation (1) without assuming that f is non-decreasing or satisfies Lipschitz condition on the given interval as in [14].

Theorem 3. *With respcet to the difference equation* (1) *assume that*

$$\sum_{n=n_0}^{\infty} (n+1) q_n < \infty. \quad (23)$$

Then equation (1) *has a bounded nonoscillatory solution.*

Proof. Set $M = \max\{f(x) : \frac{2}{3}(1-p) \leq x \leq \frac{4}{3}\}$. By (23), we can choose an integer $N > n_0$ sufficiently large such that $M \sum_{n=N}^{\infty}(n+1)q_n < \frac{1-p}{3}$. Let $\mathcal{B}$ be the set of all real sequences $x = \{x_n\}_{n=N}^{\infty}$ with the norm $\|x\| = \sup_{n\geq N} |x_n| < \infty$. Then $\mathcal{B}$ is a Banach space. We define a closed, bounded and convex subset $\mathcal{S}$ of $\mathcal{B}$ as follows:

$$\mathcal{S} = \left\{ x = \{x_n\} \in \mathcal{B} : \frac{2(1-p)}{3} \leq x_n \leq \frac{4}{3},\ n \geq N \right\}.$$

Define two maps $\mathcal{T}_1$ and $\mathcal{T}_2 : \mathcal{S} \to \mathcal{B}$ as follows:

$$\mathcal{T}_1 x_n = \begin{cases} 1 - p + p_n x_{n-k}, & n \geq N + \theta \\ \mathcal{T}_1 x_{N+\theta}, & N \leq n \leq N + \theta \end{cases}$$

$$\mathcal{T}_2 x_n = \begin{cases} -\sum_{s=n}^{\infty}(s-n+1)q_s f(x_{s-\ell}), & n \geq N+\theta \\ \mathcal{T}_2 x_{N+\theta}, & N \leq n \leq N+\theta. \end{cases}$$

First we show that for any $x, y \in \mathcal{S}$, $\mathcal{T}_1 x + \mathcal{T}_2 y \in \mathcal{S}$. Infact, for every $x, y \in \mathcal{S}$ and $n \geq N+\theta$, we have

$$\mathcal{T}_1 x_n + \mathcal{T}_2 y_n \leq 1 - p - \frac{4}{3}p + \frac{1-p}{3} = \frac{4}{3}$$

and

$$\mathcal{T}_1 x_n + \mathcal{T}_2 y_n \geq 1 - p - \frac{1-p}{3} = \frac{2(1-p)}{3}.$$

Hence

$$\frac{2(1-p)}{3} \leq \mathcal{T}_1 x_n + \mathcal{T}_2 y_n \leq \frac{4}{3} \quad \text{for all} \quad n \geq N.$$

Thus, we have proved that $\mathcal{T}_1 x + \mathcal{T}_2 y \in \mathcal{S}$ for any $x, y \in \mathcal{S}$.

Next we shall show that $\mathcal{T}_1$ is a contraction mapping on $\mathcal{S}$. Indeed for any $x, y \in \mathcal{S}$ and $n \geq N+\theta$, we have

$$|\mathcal{T}_1 x_n - \mathcal{T}_1 y_n| \leq p_n |x_{n-k} - y_{n-k}| \leq p\|x - y\|.$$

This implies that

$$\|\mathcal{T}_1 x - \mathcal{T}_1 y\| \leq p\|x - y\|.$$

Since $p \in (0,1)$, we conclude that $\mathcal{T}_1$ is a contraction mapping on $\mathcal{S}$.

Now we show that $\mathcal{T}_2$ is completely continuous. First we will show that $\mathcal{T}_2$ is continuous. Let $x^{(i)} = \{x_n^{(i)}\} \in \mathcal{S}$ be such that $x_n^{(i)} \to x_n$ as $i \to \infty$. Because $\mathcal{S}$ is closed $x = \{x_n\} \in \mathcal{S}$. For $n \geq N+\theta$, we have

$$|\mathcal{T}_2 x_n^{(i)} - \mathcal{T}_2 x_n| \leq \sum_{s=N+\theta}^{\infty} (s-n+1)q_s \left| f\left(x_{s-\ell}^{(i)}\right) - f(x_{s-\ell}) \right|.$$

Since

$$q_s(s-n+1)\left| f\left(x_{s-\ell}^{(i)}\right) - f(x_{s-\ell}) \right| \leq 2M(s+1)q_s$$

and $|f(x_{s-\ell}^{(i)}) - f(x_{s-\ell})| \to 0$ as $i \to \infty$, in view of (23), and applying the Lebesgue dominated convergence theorem, we conclude that $\lim_{i\to\infty} \|\mathcal{T}_2 x^{(i)} - \mathcal{T}_2 x\| = 0$. This means that $\mathcal{T}_2$ is continuous.

Next, we shall show that $\mathcal{T}_2\mathcal{S}$ is relatively compact. For any given $\varepsilon > 0$, by (23) there exists an integer $N_1 \geq N+\theta$ such that

$$M \sum_{s=N_1}^{\infty} (s+1)q_s < \frac{\varepsilon}{2}.$$

Then for any $x = \{x_n\} \in \mathcal{S}$ and $j, n \geq N_1$,

$$\begin{aligned}|\mathcal{T}_2 x_j - \mathcal{T}_2 x_n| &\leq \sum_{s=j}^{\infty}(s-j+1)q_s|f(x_{s-\ell})| + \sum_{s=n}^{\infty}(s-n+1)q_s|f(x_{s-\ell})| \\ &\leq M\sum_{s=j}^{\infty}(s+1)q_s + M\sum_{s=n}^{\infty}(s+1)q_s \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This means that $\mathcal{T}_2\mathcal{S}$ is uniformly Cauchy. Hence by [5], $\mathcal{T}_2\mathcal{S}$ is relatively compact. By Krasonselskii fixed point theorem [6], there is a $x = \{x_n\} \in \mathcal{S}$ such that $\mathcal{T}_1 x + \mathcal{T}_2 x = x$. Clearly $x = \{x_n\}$ is a bounded positive solution of equation (1). This completes the proof.

3. Oscillation results for sublinear case

In this section we establish conditions for the oscillation and non oscillation of equation (1) when the nonlinear function f is sublinear. The function f is said to be sublinear if f satisfies condition (5).

Theorem 4. *With respect to the difference equation* (1) *assume* $\ell > k$ *and condition* (5) *hold. If*

$$\sum_{n=n_0}^{\infty} q_n = \infty, \tag{24}$$

then every solution of equation (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1). We may assme without loss of generality that $x_n > 0$ and $0 < p_n \leq p$ for all $n \geq N$ for some integer $N > n_0$. Set y_n in (11). Using the same argument as in the proof of Theorem 2, one can consider two possible cases:

(i) $\Delta^2 y_n \leq 0$, $\Delta y_n > 0$, $y_n < 0$ for $n \geq n_1 \geq N + \theta$

(ii) $\Delta^2 y_n \leq 0$, $\Delta y_n > 0$, $y_n > 0$ for $n \geq n_2 \geq N + \theta$.

If case (i) holds, then

$$x_{n-\ell} > -\frac{1}{p}y_{n+k-\ell}, \quad n \geq n_1.$$

Substituting this into equation (1) and using the nondecreasing nature of $f(x)$, we obtain

$$\Delta^2 y_n + q_n f\Big(-\frac{1}{p}y_{n+k-\ell}\Big) \leq 0, \quad n \geq n_1.$$

Summing the last inequality from $n \geq n_1$ to ∞, we find

$$-\Delta y_n + \sum_{s=n}^{\infty} q_s f\Big(-\frac{1}{p}y_{s+k-\ell}\Big) \leq 0. \tag{25}$$

Since $-y_n$ in decreasing for $n \geq n_1$, we have from (25)

$$-\Delta y_n + \Big(\sum_{s=n}^{n+\ell-k} q_s\Big) f\Big(-\frac{1}{p} y_n\Big) \leq 0. \tag{26}$$

Set $z_n = -\frac{1}{p} y_n$. Then (26) can be written as

$$\Delta z_n + \frac{1}{p}\Big(\sum_{s=n}^{n+\ell-k} q_s\Big) f(z_n) \leq 0, \quad n \geq n_1.$$

From the last inequality, it follows that

$$\frac{\Delta z_n}{f(z_n)} + \frac{1}{p}\Big(\sum_{s=n}^{n+\ell-k} q_s\Big) \leq 0, \quad n \geq n_1. \tag{27}$$

Summing (27) from n_1 to N and using sublinear condition (5), we have

$$\begin{aligned}\frac{1}{p}\sum_{s=n_1}^{N}\Big(\sum_{t=s}^{s+\ell-k} q_t\Big) &\leq \sum_{s=n_1}^{N} \frac{-\Delta z_s}{f(z_s)} \\ &\leq \sum_{s=n_1}^{N} \int_{z_{s-1}}^{z_s} \frac{du}{f(u)} \leq \int_0^{z_{n_3}} \frac{du}{f(u)}.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\infty > \sum_{s=n_1}^{\infty}\Big(\sum_{t=s}^{s+\ell-k} q_t\Big) \geq (\ell - k) \sum_{s=n_1+\ell-k}^{\infty} q_s$$

which contradicts condition (24) and so case (i) is impossible.

If case (ii) holds, then $x_n \geq y_n$ for $n \geq n_2$. Substituting this into equation (1) and using the fact thet $f(u)$ is nondecreasing in u, we obtain

$$\Delta^2 y_n + q_n f(y_{n-\ell}) \leq 0, \quad n \geq n_2 + \theta.$$

Summing the last inequality from $n_3 = n_2 + \theta$ to ∞, we find

$$\sum_{n=n_3}^{\infty} q_n f(y_{n-\ell}) \leq \Delta y_{n_3}. \tag{28}$$

Since f is nondecreasing, it follows from (28) that

$$f(y_{n_2}) \sum_{n=n_3}^{\infty} q_n < \Delta y_{n_3}.$$

which contradicts (23) and so case (ii) is also impossible. This completes the proof of the theorem.

Theorem 5. *With respect to the difference equation* (1) *assume that*

$$\sum_{n=n_0}^{\infty} f(n)q_n < \infty. \tag{29}$$

holds. Then equation (1) *has an eventually positive solution which tends to infinity as* $n \to \infty$.

Proof. Choose an integer $N_0 > \theta + \frac{k}{1-p}$ sufficiently large such that

$$\sum_{n=N_0}^{\infty} f(n)q_n < \frac{1-p}{2}. \tag{30}$$

Choose an integer $m > 0$ such taht $mk \geq \theta$ and $N_0 > (m+1)k$. Set

$$a = \frac{(1-p)(N_0 - mk)}{N_0 - mk - p_{N_0 - mk}(N_0 - mk - k)}.$$

Then

$$1 - p = \frac{(1-p)(N_0 - mk)}{N_0 - mk} \leq a \leq \frac{(1-p)(N_0 - mk)}{(1-p)(N_0 - mk)} = 1.$$

Define the sequence $\{y_n\}$ as follows:

$$y_n = \begin{cases} an, & N_0 - (m+1)k \leq n \leq N_0 - mk \\ p_n y_{n-k} + (1-p)n, & N_0 - mk \leq n \leq N_0 \\ p_n y_{n-k} + (1-p)n + \displaystyle\sum_{s=N_0}^{n-1} (s - n + 1) q_s f(y_{s-\ell}), & n \geq N_0. \end{cases} \tag{31}$$

It is easy to see that

$$(1-p)n \leq y_n < n \tag{32}$$

for $N_0 - (m+1)k \leq n \leq N_0$. In the sequel, we prove that

$$\frac{1}{2}(1-p)n < y_n < n, \quad n \geq N_0 - (m+1)k. \tag{33}$$

If (33) is not true, then there exists an integer $n_1 \geq N_0$ such that

$$y_{n_1} \leq \frac{1}{2}(1-p)n_1$$

and

$$\frac{1}{2}(1-p)n < y_n < n, \quad N_0 - (m+1)k \leq n < n_1 \tag{34}$$

or

$$y_{n_1} \geq n_1 \quad \text{and} \quad \frac{1}{2}(1-p)n < y_n < n, \qquad N_0 - (m+1)k \leq n < n_1. \tag{35}$$

If (34) holds, then from (30), (31) and (34), we have

$$\begin{aligned} y_{n_1} &= p_{n_1} y_{n_1-k} + (1-p)n_1 + \sum_{s=N_0}^{n_1-1} (s-n_1+1) q_s f(y_{s-\ell}) \\ &\geq (1-p)N_0 + (n_1 - N_0)\Big[1 - p - \sum_{s=N_0}^{n_1-1} q_s f(y_{s-\ell})\Big] \\ &\geq (1-p)N_0 + (n_1 - N_0)\Big[1 - p - \sum_{s=N_0}^{n-1} q_s f(y_{s-\ell})\Big] \\ &> (1-p)n_0 + (n_1 - N_0)\Big[1 - p - \frac{1-p}{2}\Big] \\ &> \frac{1}{2}(1-p)n_1. \end{aligned}$$

This contradiction implies that (34) is not true. If (35) holds then from (30), (31) and (35) we have

$$\begin{aligned} y_{n_1} &= p_{n_1} y_{n_1-k} + (1-p)n_1 + \sum_{s=N_0}^{n_1-1} (s-n_1+1) q_s f(y_{s-\ell}) \\ &\geq p(n_1 - k) + (1-p)n_1 = n_1 - pk < n_1. \end{aligned}$$

This is also a contradiction and so (35) is not true. Therefore (33) holds. It is easy to see that $\{y_n\}$ satisfies the equation

$$\Delta^2(y_n - p_n y_{n-k}) + q_n f(y_{n-\ell}) = 0, \quad n \geq N_0.$$

This shows that $\{y_n\}$ is a positive solution of equation (1) with the desired asymptotic behavior. The proof is now complete.

References

[1] R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, S. R. Grace and D. O' Regan, Oscillation Theory for Difference and Functional Differenctial Equations, Kluwer Academic, 2000.

[3] R. P. Agarwal, M. M. S. Manuel and E. Thandapani, *Oscillatory and nonoscillatory behavior of second order neutral delay difference equations*, Math. Comput. Modelling **24**(1996), 5–11.

[4] M. Budericenic, *Oscillation of a second order neutral difference equation*, Bull Cl. Sci, Math. Nat. Sci. Math. **22**(1994), 1–8.

[5] S. S. Cheng and W. T. Patula, *An existence theorem for a nonlinear difference equation*, Nonlinear Anal. **20**(1993), 193–203.

[6] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.

[7] S. R. Grace and B. S. Lalli, *Oscillation theorem for second order delay and neutral difference equations*, Utilitaes Math. **45**(1994), 197–211.

[8] W. T. Li, *Oscillation of higher order neutral nonlinear difference equations*, Appl. Math. Lett. **11**(1998), 1–8.

[9] H. J. Li and C. C. Yeh, *Oscillation criteria for second order neutral delay difference equations*, Comput. Math. Appl. **36**(1998), 123–132.

[10] Z. Liu, S. Wu and Z. Zhang, *Oscillation of solutions for even order nonlinear difference equations with nonlinear neutral term*, Indian J. Pure Appl. Math. **34**(2003), 1585–1598.

[11] A. Sternal and B. Szmanda, *Asymptotic and oscillatory behavior of certain difference equations*, Le Mat. **51**(1996), 77–86.

[12] X. H. Tang and Y. J. Liu, *Oscillations for nonlinear delay difference equations*, Tamkang J. Math. **32**(2001), 275–280.

[13] E. Thandapani and R. Arul, *Oscillation properties of second order nonlinear neutral delay difference equations*, Indian J. Pure Appl. Math. **28**(1997), 1567–1571.

[14] E. Thandapani and K. Mahalingam, *Necessary and sufficient conditions for oscillation of second order neutral difference equations*, Tamkang J. Math. **34**(2003), 137–145.

[15] E. Thandapani, P. Sundaram, J. R. Graef and P. W. Spikes, *Asymptotic properties of solutions of nonlinear second order neutral delay difference equations*, Dymanic Sys. Appl. **4**(1995), 125–136.

[16] A. Zafer and R. S. Dahiya, *Oscillation of a neutral delay difference equations*, Appl. Math. Lett. **6**(1993), 71–74.

[17] G. Zhang, *Oscillation for nonlinear neutral difference equations*, Applied Math. E- Notes **2**(2002), 22–24.

[18] B. G. Zhang and S. H. Saker, *Kamenev-type oscillation criteria for nonlinear neutral delay difference equations*, Indian J. Pure Appl. Math. **34**(2003), 1571–1584.

[19] Z. Zhang, J. Chen and C. Zhang, *Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term*, Comput. Math. Appl. **41**(2001), 1487–1494.

[20] G. Zhang, S. S. Cheng and Y. Gao, *Classification schemes for positive solutions of a second order nonlinear difference equations*, J. Comp. Appl. Math. **101**(1999), 39–51.

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