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EXISTENCE OF SOLUTIONS FOR NEUTRAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper, by using fractional power of operators and Sadovskii's fixed point theorem, we study the existence of mild and strong solutions of nonlinear neutral functional integrodifferential equations. The results we obtained are a generalization and continuation of the recent results on this issue.

1. Introduction

In this paper, we study the existence of solutions for nonlinear neutral functional integrodifferential equations. More precisely, we consider the following Cauchy problem:

$$\begin{aligned}&\frac{d}{dt}[x(t)+F(t,x(t),x(b_1(t)),\ldots,x(b_m(t))]+Ax(t)\\&\qquad=G(t,x(t),x(a_1(t)),\ldots,x(a_n(t)))+K(t,x(t),\int_0^t k(t,s,x(s))ds),\qquad 0\le t\le a\\&x(0)=x_0 \qquad (1)\end{aligned}$$

where $-A$ generates an analytic semigroup, and F, G, K, and k are given functions to be specified later. The Cauchy problem was considered by Byszewski [4] and the importance of local conditions in different fields has been discussed in [4] and the references therein. In the past several years theorems about existence, uniqueness and stability of differential and functional differential abstract evolution Cauchy problem have been studied by Byszewski and Lakshmikantham [7], Byszewski and Akca [6], Byszewski [4, 5], Balachandran and Chandrasekaran [1], Balachandran and Murugesu [2, 3] Ntouyas and Tsamatos [11] and Lin and Liu [9].

In this paper, we extend this problem to neutral functional integrodifferential equations and discuss the existence results of solutions for (1) by using Sadovskii's fixed point theorem [13]. The result obtained is a generalization and a continuation of some results reported in [1, 2, 3, 4, 5, 6, 7, 9, 11]. Particularly in paper [6], when discussing the

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existence of a classical solution the authors required the condition that the mild solution $x(b(\cdot))$ is Lipschitz continuous. But this condition is very difficult to verify and so is almost impossible to apply. To take away this unsatisfactory condition, we directly construct the bounded closed convex set $B = \{x \in E : \|x\| \leq k, \|x(t) - x(s)\| \leq L^*\|t - s\|, t, s \in [0, a]\}$ and then prove that the operator P has a fixed poinht on this set. In addition, our result can also be regarded as an extension of the corresponding results on classical Cauchy problem in [8, 12].

2. Preliminaries

Let $-A$ be the infinitesimal generator of a compact analytic semigroup of uniformly bounded linear operators $T(t)$ defined in the Banach space X. Let $0 \in \rho(A)$, then define the fractionl power A^α, for $0 \leq \alpha \leq 1$, as a closed linear operator on its domain $D(A^\alpha)$ which is dense in X. Further $D(A)$ is a Banach space under the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha)$$

which we denote by X_α. Then for each $0 < \alpha \leq 1$, $X_\alpha \to X_\beta$ for $0 < \beta < \alpha \leq 1$ and the imbedding is compact whenever the resolvent operator of A is compact. We assume that

(a) there is a $M \geq 1$ such that $\|T(t)\| \leq M$, for all $0 \leq t \leq a$;

(b) for any $a > 0$, there exists a positive constant C_α such that

$$\|A_\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad 0 < t \leq a \tag{2}$$

For our convenience let us take $F(0, x(0), x(b_1(0)), \ldots, x(b_m(0)) = 0$. Let $M_0 = \|A^{-\beta}\|$ assume the following conditions:

(H_1) $F : [0, a] \times X^{m+1} \to X$ is a continuous functions and there exists a $\beta \in (0, 1)$ and $L, L_1 > 0$ such that the function $A^\beta F$ satisfies the Lipschitz condition:

$$\begin{aligned}&\|A^\beta F(s_1, x_0, x_1, \ldots, x_m) - A^\beta F(s_2, \bar{x_0}, \bar{x_1}, \ldots, \bar{x_m})\\ &\quad \leq L(|s_1 - s_2| + \max_{i=0,\ldots,m} \|x_i - \bar{x_i}\|)\end{aligned}$$

for any $0 \leq s_1, s_2 \leq a$, $x_i, \bar{x_i} \in X$, $i = 0, 1, \ldots, m$ and the inequality

$$\|A^\beta F(t, x_0, x_1, \ldots, x_m)\| \leq L_1(\max\{\|x_i\| : i = 0, 1, \ldots, m\} + 1) \tag{3}$$

holds for any $(t, x_0, x_1, \ldots, x_m) \in [0, a] \times X^{m+1}$

(H_2) The function $G : [0, a] \times X^{n+1} \to X$ satisfies the following conditions:

(i) For each $t \in [0, a]$, the function $G(t, \cdot) : \times X^{n+1} \to X$ is continuous and for each $(x_0, x_1, \ldots, x_n) \in X^{n+1}$ the function $G(\cdot, x_0, x_1, \ldots, x_n) : [0, a] \to X$ is strongly measurable.

(ii) For each positive number $k \in N$, there is a positive function $g_k \in L^1([0,a])$ such that

$$\sup_{\|x_0\|,\ldots,\|x_n\| \le k} \|G(t, x_0, x_1, \ldots, x_n)\| \le g_k(t)$$

and

$$\lim_{k\to\infty} \frac{1}{k} \int_0^a g_k(s)ds = \gamma < \infty$$

(H_3) The function $K : [0,a] \times X \times X \to X$ satisfies the following conditions:

(i) For each $t \in [0,a]$, the function $K(t,\cdot,\cdot) : X \times X \to X$ and for each $x, y \in X$, $K(\cdot, x, y) : [0,a] \to X$ is strongly measurable.

(ii) For each positive number $r \in N$, there exists a positive function $q_r \in L^1([0,a])$ such that

$$\sup_{\|x\| \le r} \|K(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau\| \le q_r(s)$$

and

$$\lim_{r\to\infty} \frac{1}{r} \int_0^a q_r(s)ds = \gamma_1 < \infty$$

(H_4) $a_i, b_j \in C([0,a];[0,a])$, $i = 1, \cdots, n, j = 1, \cdots, m$. $g \in C(E;X)$, here and hereafter $E = C([0,a];X)$ and g is completely continuous.

3. Existence of mild solution

Definition 3.1. A continuous function $x(\cdot) : [0,a] \to X$ is said to be a mild solution of the Cauchy problem (1), if the function $AT(t-s)F(s, x(s), x(b_1(s)), \ldots, x(b_m(s)))$, $s \in [0,a)$ is integrable on $[0,a)$ and the integral equation

$$\begin{aligned} x(t) = T(t)[x_0] &- F(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) \\ &+ \int_0^t AT(t-s)F(s, x(s), x(b_1(s)), \ldots, x(b_m(s)))ds \\ &+ \int_0^t T(t-s)G(s, x(s), x(a_1(s)), \ldots, x(a_m(s)))ds \\ &+ \int_0^t T(t-s)[K(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau)]ds \end{aligned} \tag{4}$$

Theorem 3.1. *If assumptions* (H_1)–(H_3) *are sastified and* $x_0 \in X$, *then the Cauchy problem* (1) *has a mild solution provided that*

$$L_0 := L[M_0 + \frac{1}{\beta} C_{1-\beta} a^\beta] < 1 \tag{5}$$

and

$$(\gamma+\gamma_1)M + M_0 L_1 + \frac{1}{\beta} C_{1-\beta} a^{\beta} L_1 < 1 \tag{6}$$

where $M_0 = \|A^{-\beta}\|$.

Proof. For the sake of brevity, we rewrite that

$$(t, x(t), x(b_1(t)), \dots, x(b_m(t))) = (t, v(t))$$

and

$$(t, x(t), x(a_1(t)), \dots, x(a_n(t))) = (t, u(t))$$

Define the operator P on E by the formula

$$\begin{aligned}(Px)(t) &= T(t)[x_0] - F(t, v(t)) \\ &\quad + \int_0^t AT(t-s)F(s, v(s))ds \\ &\quad + \int_0^t T(t-s)G(s, u(s))ds \\ &\quad + \int_0^t T(t-s)[K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)]ds, \quad 0 \le t \le a.\end{aligned}$$

For each positive integer k, let

$$B_k = \{x \in E : \|x(t)\| \le k, 0 \le t \le a\}$$

Then for each k, B_k is clearly a bounded closed convex set in E. Since by (2) and (3) the following relation holds:

$$\begin{aligned}\|AT(t-s)F(s, v(s))\| &\le \|A^{1-\beta}T(t-s)A^{\beta}F(s, v(s))\| \\ &\le \frac{C^{1-\beta}}{(t-s)^{1-\beta}} L_1(k+1)\end{aligned}$$

then from Bocher's theorem [10] it follows that $AT(t-s)F(s, v(s))$ is integrable on $[0, a]$, so P is well defined on B_k. We claim that there exists a positive integer k such that $PB_k \subseteq B_k$. If it is not true, then for each positive integer k, there is a function $x_k(\cdot) \in B_k$, but $Px_k \notin B_k$, that is $\|Px_k(t)\| > k$ for some $t(k) \in [0, a]$, where $t(k)$ denotes t is dependent of k. However, on the other hand, we have

$$\begin{aligned}k &< \|Px_k(t)\| \\ &= \|T(t)[x_0] - F(t, v_k(t)) + \int_0^t AT(t-s)F(s, v_k(s))ds \\ &\quad + \int_0^t T(t-s)G(s, u_k(s))ds + \int_0^t T(t-s)[K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)]ds\|\end{aligned}$$

$$\le \|T(t)[x_0]\| - \|A^{-\beta}A^{\beta}F(t, v_k(t))\| + \|\int_0^t A^{1-\beta}T(t-s)A^{\beta}F(s, v_k(s))ds\|$$
$$+\|\int_0^t T(t-s)G(s, u_k(s))ds\| + \|\int_0^t T(t-s)[K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)]ds\|$$
$$\le M[\|x_0\|] + M_0L_1(k+1) + \int_0^t \frac{C^{1-\beta}}{(t-s)^{1-\beta}}L_1(k+1)ds$$
$$+M\int_0^a g_k(s)ds + M\int_0^a q_r(s)ds$$

Dividing on both sides by k and taking the lower limit as $k \to +\infty$, we get

$$(\gamma + \gamma_1)M + M_0L_1 + \frac{1}{\beta}C_{1-\beta}a^{\beta}L_1 \ge 1.$$

This contradicts (6). Hence for some positive integer k, $PB_k \subseteq B_k$.

Next we will show that the operator P has a fixed point on B_k, which implies Eq. (1) has a mild solution. To this end, we decompose P as $P = P_1 + P_2$, where the operators P_1, P_2 are defined on B_k, respectively, by

$$(P_1x)(t) = -F(t, v(t)) + \int_0^t AT(t-s)F(s, v(s))ds$$

and

$$(P_2x)(t) = T(t)[x_0] + \int_0^t T(t-s)G(s, u(s))ds$$
$$+\int_0^t T(t-s)[K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)]ds$$

for $0 \le t \le a$, and we will verify that P_1 is a contraction while P_2 is a compact operator.

To prove that P_1 is a contraction, we take x_1, $x_2 \in B_k$. Then for each $t \in [0, a]$ and by condition (H_1) and (5), we have

$$\|(P_1x_1)(t) - (P_1x_2)(t)\|$$
$$\le \|F(t, v_1(t)) - F(t, v_2(t))\| + \|\int_0^t AT(t-s)[F(s, v_1(s)) - F(s, v_2(s))]ds\|$$
$$\le M_0L \sup_{0\le s\le a}\|x_1(s) - x_2(s)\| + \int_0^t \frac{C^{1-\beta}}{(t-s)^{1-\beta}}Lds \sup_{0\le s\le a}\|x_1(s) - x_2(s)\|$$
$$\le L[(M_0 + \frac{1}{\beta}C_{1-\beta}a^{\beta}] \sup_{0\le s\le a}\|x_1(s) - x_2(s)\|$$
$$= L_0 \sup_{0\le s\le a}\|x_1(s) - x_2(s)\|$$

Thus

$$\|Px_1 - Px_2\| = L_0\|x_1 - x_2\|$$

So by assumption $0 < L_0 < 1$, we see that P_1 is a contraction.

To prove that P_2 is compact, firstly we prove that P_2 is continuous on B_k. Let $\{x_n\} \subseteq B_k$ with $x_n \to x$ in B_k, then by $(H_2)(i)$, we have

$$G(s, u_n(s)) \to G(s, u(s)), \quad n \to \infty.$$

$$K(t, x_n(t), \int_0^t k(t, s, x_n(s))ds) \to K(t, x(t), \int_0^t k(t, s, x(s))ds) \quad as \quad n \to \infty$$

Since

$$\|G(s, u_n(s)) - G(s, u(s))\| \leq 2g_k(s),$$

$$\|K(t, x_n(t), \int_0^t k(t, s, x_n(s))ds) - K(t, x(t), \int_0^t k(t, s, x(s))ds)\| \leq 2q_r(s),$$

by the dominated convergence theorem, we have

$$\begin{aligned} \|P_2x_n - P_2x\| = \sup_{0\leq t\leq a} &\|T(t) + \int_0^t T(t-s)[G(s, u_n(s)) - G(s, u(s))]ds \\ &+ \int_0^t T(t-s)[K(s, x_n(s), \int_0^s k(s, \tau, x(\tau))d\tau) \\ &- K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)]ds\| \\ &\to 0 \quad as \quad n \quad \to \infty \end{aligned}$$

i.e. P_2 is continuous.

Next, we prove that $\{P_2x : x \in B_k\}$ is a family of equicontinuous functions. To see this we fix $t_1 > 0$ and let $t_2 > t_1$ and $\epsilon > 0$ be enough small. Then

$$\begin{aligned} &\|(P_2x)(t_2) - (P_2x)(t_1)\| \\ &\quad \leq \|T(t_2) - T(t_1)\|\|x_0\| \\ &\qquad + \int_0^{t_1-\epsilon} \|T(t_2 - s) - T(t_1 - s)\|\|G(s, u(s))\|ds \\ &\qquad + \int_{t_1-\epsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\|G(s, u(s))\|ds \\ &\qquad + \int_{t_1}^{t_2} \|T(t_2 - s)\|\|G(s, u(s))\|ds \\ &\qquad + \int_0^{t_1-\epsilon} \|T(t_2 - s) - T(t_1 - s)\|\|K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)\|ds \\ &\qquad + \int_{t_1-\epsilon}^{t_1} \|T(t_2 - s) - T(t_1 - s)\|\|K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)\|ds \end{aligned}$$

$$+\int_{t_1}^{t_2} \|T(t_2-s)\| \|K(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)\| ds$$

Noting that $\|G(s,u(s)\| \le g_k(s)$ and $g_k(s) \in L^1$, we see that $\|(P_2x)(t_2)-(P_2x)(t_1)\|$ tends to zero independently of $x \in B_k$ as $t_2 - t_1 \to 0$ since the compactness of $T(t)(t > 0)$ implies the continuity of $T(t)(t > 0)$ in t in the uniform operators topology. We can prove that the functions P_2x, $x \in B_k$ are equicontinuous at $t = 0$. Hence P_2 maps B_k into a family of equicontinuous functions.

It remains to prove that $V(t) = \{(P_2x)(t) : x \in B_k\}$ is relatively compact in X. $V(0)$ is relatively compact in X. Let $0 < t \le a$ be fixed and $0 < \epsilon < t$. For $x \in B_k$, we define

$$\begin{aligned}
(P_{2,\epsilon}x)(t) &= T(t)[x_0] + \int_0^{t-\epsilon} T(t-s)G(s,u(s))ds \\
&\quad + \int_0^{t-\epsilon} T(t-s)[K(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)]ds \\
&= T(t)[x_0] + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)G(s,u(s))ds \\
&\quad + T(\epsilon)\int_0^{t-\epsilon} T(t-\epsilon-s)[K(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)]ds
\end{aligned}$$

Then from the compactness of $T(\epsilon)(\epsilon > 0)$, we obtain $V_\epsilon(t) = \{(P_{2,\epsilon}x)(t) : x \in B_k\}$ is relatively compact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $x \in B_k$, we have

$$\begin{aligned}
\|(P_2x)(t) - (P_{2,\epsilon}x)(t)\| &= \int_{t-\epsilon}^t \|T(t-s)G(s,u(s))\| ds \\
&\quad + \int_{t-\epsilon}^t \|T(t-s)[K(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)]\| ds \\
&\le M\int_{t-\epsilon}^t g_k(s)ds + M\int_{t-\epsilon}^t q_r(s)ds
\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t)$. Hence the set $V(t)$ is also relatively compact in X.

Thus, by Arzela-Ascoli theorem, P_2 is a compact operator. Those arguments enable us to conclude that $P = P_1 + P_2$ is a condensing map B_k, and by the fixed point theorem of Sadovskii there exists a fixed point $x(\cdot)$ for P on B_k. Therefore, the Cauchy problem (1) has a mild solution, and the proof is completed.

4. Existence of strong solution

Definition 4.1. A function $x(\cdot) : [0,a] \to X$ is said to be a strong solution of the local Cauchy problem (1), if:

(1) x is continuous on $[0,a]$ and differentiable a.e. on $(0,a]$, $x' \in L^1([0,a];X)$;

(2) x satisfies

$$\frac{d}{dt}[x(t)+F(t,x(t),x(b_1(t)),\dots,x(b_m(t)))]+Ax(t)=G(t,x(t),x(a_1(t)),\dots,x(a_n(t))) +K(t,x(t),\int_0^t k(t,s,x(s))ds)$$

a.e. on $(0,a]$, and

$$x(0)=x_0.$$

Theorem 4.1. *Let X be a reflexive Banach space. Suppose conditions (H_1) and (H_3) are satisfied with $F([0,a]\times X^{m+1})\subset D(A)$, and the function $AF(0,\cdot):X^{m+1}\to D(A)$ maps bounded sets into bounded sets. Additionally, the following conditions hold:*

(H_4) *$G(\cdot,\cdot)$ is Lipschitz, i.e. there exists a constant $L_3>0$, such that*

$$\|G(s,x_0,\dots,x_n)-G(\bar{s},\bar{x_0},\dots,\bar{x_n})\|\le L_3[|s-\bar{s}|+\max_{i=0,\dots,n}\|x_i,\bar{x_i}\|]$$

for any $(s,x_0,\dots,x_n),(\bar{s},\bar{x_0},\dots,\bar{x_n})\in[0,a]\times X^{n+1}$. Moreover, there is an $L_4>0$ such that

$$\|G(t,x_0,\dots,x_n)\|\le L_4(\max\{\|x_i\|:i=0,\dots,n\}+1)$$

for any $(t,x_0,\dots,x_n)\in[0,a]\times X^{n+1}$;

(H_5) *There exist constants L_5,$L_6>0$, such that*

$$\|K(t_1,x_1,y_1)-K(t_2,x_2,y_2)\|\le L_5|t_1-t_2|+L_6(\|x_1-x_2\|+\|y_1-y_2\|)$$

and

$$\|k(t,s,x)-k(\tau,s,x)\|\le L_7|t-\tau|$$

for any t, $\tau\in[0,a]$ and $x\in X$.

(H_6) *There are constants $0<l_1$, $l_2\le 1$, such that*

$$\|b_i(s)-b_i(\bar{s})\|\le l_1|s-\bar{s}|,$$
$$\|a_j(s)-a_j(\bar{s})\|\le l_2|s-\bar{s}|,$$

for s,$\bar{s}\in[0,a]$, $i=1,\dots,m$ and $j=1,\dots,n$;

(H_7) *$x_o\in D(A)$ and $\int_0^t k(t,\tau,x(\tau))d\tau\le L_8$.*

(H_8)

$$M^*=[(M_0+\frac{1}{\beta}a^\beta C_{1-\beta})L+M(aL_3+aL_6+aL_7(a+1)]<1 \tag{7}$$

Then the Cauchy problem (1) *has a strong solution on* $[0, a]$.

Proof. Let P be the operator defined in the proof of Theorem 3.1. Consider the set

$$B = \{x \in E : \|x\| \leq k, \|x(t) - x(s)\| \leq L^*|t - s|, t, s \in [0, a]\}$$

for some positive constant k and L^* large enough. It is cleat that B is a convex, closed and nonempty set. We shall prove that P has a fixed point on B. Obviously, from the proof of Theorem 3.1. it is sufficient to show that for any $x \in B$,

$$\|(Px)(t_2) - (Px)(t_1)\| \leq L^*|t_2 - t_1|, \quad t_2, t_1 \in [0, a].$$

In fact,

$$\begin{aligned}
&\|(Px)(t_2) - (Px)(t_1)\| \\
&\quad \leq \|[T(t_2) - T(t_1)][x_0]\| + \|F(t_2, v(t_2)) - F(t_1, v(t_1))\| \\
&\qquad + \| \int_0^{t_2} AT(t_2 - s)F(s, v(s)) - \int_0^{t_1} AT(t_1 - s)F(s, v(s))\| \\
&\qquad + \| \int_0^{t_2} T(t_2 - s)G(s, u(s)) - \int_0^{t_1} T(t_1 - s)G(s, u(s))ds\| \\
&\qquad + \| \int_0^{t_2} T(t_2 - s)[K(s, x(s), \int_0^{s} k(s, \tau, x(\tau))d\tau)]ds \\
&\qquad - \int_0^{t_1} T(t_1 - s)[K(s, x(s), \int_0^{s} k(s, \tau, x(\tau))d\tau)]ds\| \\
&\quad \leq \|[T(t_2) - T(t_1)][x_0]\| + \|A^{-\beta}[A^{\beta}F(t_2, v(t_2))\| \\
&\qquad - A^{\beta}F(t_1, v(t_1))] + \| \int_0^{t_1} A^{1-\beta}T(t_1 - s)[A^{\beta}F(s + t_2 - t_1, v(s + t_2 - t_1)) \\
&\qquad - A^{\beta}F(s, v(s)]ds + \int_0^{t_2 - t_1} A^{1-\beta}T(t_2 - s)A^{\beta}F(s, v(s))ds\| \\
&\qquad + \| \int_0^{t_1} T(t_1 - s)[G(s + t_2 - t_1, u(s + t_2 - t_1)) - G(s, u(s))]ds\| \\
&\qquad + \| \int_0^{t_2 - t_1} T(t_2 - s)G(s, u(s))ds\| + \| \int_0^{t_1} T(t_1 - s)[K(s + t_2 - t_1, x(s + t_2 - t_1), \\
&\qquad \int_0^{s + t_2 - t_1} k(s, \tau, x(\tau))d\tau) - K(s, x(s), \int_0^{s} k(s, \tau, x(\tau))d\tau)]ds\| \\
&\qquad + \| \int_0^{t_2 - t_1} T(t_2 - s)[K(s, x(s), \int_0^{s} k(s, \tau, x(\tau))d\tau)]ds\|
\end{aligned}$$

Then, from conditions (H_1), (H_4), (H_5), (H_1) the boundedness of $AF(0, \cdot)$ it yields that

$$\begin{aligned}
&\|(Px)(t_2) - (Px)(t_1)\| \\
&\quad \leq M\|A[x_0]\||t_2 - t_1|
\end{aligned}$$

$$
\begin{aligned}
&+M_0L[|t_2-t_1|+L^*|t_2-t_1|] \\
&+\frac{1}{\beta}C_{1-\beta}a^\beta L[|t_2-t_1|+L^*|t_2-t_1|] \\
&+\frac{1}{\beta}C_{1-\beta}L_1(k+1)|t_2^\beta-t_1^\beta| \\
&+aML_3[|t_2-t_1|+L^*|t_2-t_1|] \\
&+ML_4(k+1)|t_2-t_1| \\
&+aM[(L_5|t_2-t_1|+L_6L^*|t_2-t_1|) \\
&+aL_7(|t_2-t_1|+L^*|t_2-t_1|) \\
&+L_7(|t_2-t_1|+L^*|t_2-t_1|) \\
&+Mq_r(s) \\
\leq\ &\{C^*+[(M_0+\frac{1}{\beta}a^\beta C_{1-\beta})L+M(aL_3+L_6(a+1)+L_7(a+2)]L^*\}|t_2-t_1|
\end{aligned}
$$

where C^* is a constant independent of L^* and $x \in B$. So it follows from (7) that $\|(Px)(t_2)-(Px)(t_1)\| \leq L^*|t_2-t_1|$, $t_2, t_1 \in [0,a]$, as long as L^* is large enough ($\geq C^*/(1-M^*)$). Thus, P has a fixed point x which is a mild solution of Eq. (1). For this $x(\cdot)$, let

$$
\begin{aligned}
f(t) &= F(t,x(t),x(b_1(t)),\dots,x(b_m(t))) \\
o(t) &= T(t)[x_0+F(0,x(0),x(b_1(0)),\dots,x(b_m(0)))] \\
p(t) &= \int_0^t AT(t-s)F(s,x(s),x(b_1(s)),\dots,x(b_m(s)))ds \\
q(t) &= \int_0^t T(t-s)G(s,x(s),x(a_1(s)),\dots,x(a_n(s)))ds \\
r(t) &= \int_0^t T(t-s)K(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau)ds
\end{aligned}
$$

Then they are all Lipschitz continuous, respectively. Since x is Lipschitz continuous on $[0,a]$ and the space X is reflexive, we see that $x(\cdot)$ is a.e. differentiable on $(0,a]$ and that $x'(\cdot) \in L^1([0,a];X)$. The same argument shows that f,p, q and r also have this property.

On the other hand, by the standard arguments we can obtain that $p(t) \in D(A)$, $q(t) \in D(A)$, $r(t) \in D(A)$ and

$$
\begin{aligned}
p'(t) &= AF(t,x(t),x(b_1(t)),\dots,x(b_m(t))) \\
&\quad -A\int_0^t AT(t-s)F(s,x(s),x(b_1(s)),\dots,x(b_m(s)))ds, \\
q'(t) &= G(t,x(t),x(a_1(t)),\dots,x(a_n(t))) \\
&\quad -A\int_0^t T(t-s)G(s,x(s),x(a_1(s)),\dots,x(a_n(s)))ds
\end{aligned}
$$

$$r'(t) = K(t, x(t), \int_0^t k(t, s, x(s))ds) - A\int_0^t T(t-s)K(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau)ds$$

So we have that x' satisfies a.e. that

$$\begin{aligned}
&\frac{d}{dt}[x(t) + F(t, x(t), x(b_1(t)), \ldots, x(b_m(t))] \\
&\quad = \frac{d}{dt}T(t)[x_0 + F(0, x(0), x(b_1(0)), \ldots, x(b_m(0))] \\
&\qquad + p'(t) + q'(t) + r'(t) \\
&\quad = AT(t)[x_0 + F(0, x(0), x(b_1(0)), \ldots, x(b_m(0))] \\
&\qquad + AF(t, x(t), x(b_1(t)), \ldots, x(b_m(t))) - Ap(t) \\
&\qquad + G(t, x(t), x(a_1(t)), \ldots, x(a_n(t))) - Aq(t) \\
&\qquad + K(t, x(t), \int_0^t k(t, s, x(s))ds) - Ar(t) \\
&\quad = -Ax(t) + G(t, x(t), x(a_1(t)), \ldots, x(a_n(t))) \\
&\qquad + K(t, x(t), \int_0^t k(t, s, x(s))ds)
\end{aligned}$$

This shows that $x(\cdot)$ is also a strong solution of the local Cauchy problem (1). Thus the proof is completed.

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