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DIFFERENTIALS IN CERTAIN CLASSES OF GRAPHS

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Abstract. Let $X \subset V$ be a set of vertices in a graph $G = (V, E)$. The *boundary* $B(X)$ of X is defined to be the set of vertices in $V - X$ dominated by vertices in X, that is, $B(X) = (V - X) \cap N(X)$. The *differential* $\partial(X)$ of X equals the value $\partial(X) = |B(X)| - |X|$. The *differential* of a graph G is defined as $\partial(G) = max\{\partial(X)|X \subset V\}$. It is easy to see that for any graph G having vertices of maximum degree $\Delta(G)$, $\partial(G) \geq \Delta(G) - 1$. In this paper we characterize the classes of unicyclic graphs, split graphs, grid graphs, k-regular graphs, for $k \leq 4$, and bipartite graphs for which $\partial(G) = \Delta(G) - 1$. We also determine the value of $\partial(T)$ for any complete binary tree T.

1. Introduction

Let $G = (V, E)$ be a graph. For graph theoretic terminology not given here, refer to Harary [2]. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V|uv \in E\}$ and the closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, its open neighborhood is $N(S) = \cup_{v\in S} N(u)$ and the *closed neighborhood* is $N[S] = N(S) \cup S$.

The *boundary* $B(S)$ of a set S is defined to be the set of vertices in $V - S$ dominated by vertices in S, that is $B(X) = (V - S) \cap N(S)$. The *differential* $\partial(S)$ of S equals the value $\partial(S) = |B(S)| - |S|$. The *differential of a graph* of G is defined as $\partial(G) = max\{\partial(S)|S \subset V\}$. As reported in [4], the differential of a set was first defined by Hedetniemi [3], and later studied by Mashburn et al. [4] and Goddard and Henning [1]. The minimum differential of an independent set was also studied by Zhang [6].

It is easy to see that for any graph G having vertices of maximum degree $\Delta(G)$, $\partial(G) \geq \Delta(G)-1$. We say that a graph G is a $(\Delta-1)$-*differential graph* if $\partial(G) = \Delta(G)-1$. In this paper we characterize the classes of unicyclic graphs, split graphs, grid graphs, k-regular graphs, for $k \leq 4$, and bipartite graphs that are $(\Delta - 1)$-differential. We also determine the value of $\partial(T)$ for any complete binary tree T.

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2. Properties of $(\Delta - 1)$-differential graphs

In this section we determine several properties of $(\Delta-1)$-differential graphs that can be used to characterize the five classes of $(\Delta-1)$-differential graphs in the next section.

Theorem 1. *Let $G = (V, E)$ be a connected $(\Delta - 1)$-differential graph, let $v \in V$ be a vertex of maximum degree, that is, $|N(v)| = \Delta(G)$, and let H be the subgraph of G induced by $V - N[v]$. Then the following properties hold:*

(i) *Each connected component of H is either a K_1 or a K_2.*

(ii) *Each vertex $u \in N(v)$ has at most two neighbors in H.*

(iii) *If S is the set of all vertices in $N(v)$ having two neighbors in H, and no two members of S have a common neighbor in H, then $|S| \leq \Delta - 2$.*

(iv) *The diameter of G is at most 6.*

Proof. Since the theorem clearly holds when $\Delta(G) = 1$, we assume that $\Delta(G) \geq 2$.

(i) Let v be a vertex of maximum degree, i.e., $|N(v)| = \Delta(G)$, and assume that $D = \{v\}$ is a ∂-set. Let x be a vertex in H, the subgraph of G induced by the vertices in $V - N[v]$. If x has more than one neighbor in H then $\partial(D \cup \{x\}) > \partial(D)$, contradicting our assumption that D is a ∂-set. Hence, each component of H has maximum degree at most 1, and therefore each component H is either a K_1 or a K_2.

(ii) If any vertex $u \in N(v)$ has three or more neighbors in H then $\partial(D \cup \{u\}) > \partial(D)$, again contradicting our assumption that D is a ∂-set.

(iii) Let $S \subseteq N(v)$ be the set of neighbors of v, each of which has two neighbors in H and no two vertices in S have a common neighbor in H. In this case, if $|S| = \Delta(G)$ then $\partial(S) > \partial(D)$, again contradicting our assumption that D is a ∂-set.

(iv) Since G is connected, and since by (a) each connected component of H is either a K_1 or a K_2, it follows that every vertex in H is within distance at most 2 to a vertex in $N(v)$, and therefore is within distance at most 3 to the vertex v of maximum degree $\Delta(G)$. Therefore, every pair of vertices in G are within distance 6 of each other and $diam(G) \leq 6$.

3. Characterizations of five classes of $(\Delta - 1)$-differential graphs

In this section we characterize the classes of unicyclic graphs, split graphs, grid graphs, k-regular graphs, for $k \leq 4$, and bipartite graphs that are $(\Delta - 1)$-differential graphs.

3.1. Unicylic graphs

Definition 2. *A graph G with $|V| = n$ is a unicyclic graph if it is connected and $|E| = n$; or equivalently, if it is connected and contains exactly one cycle.*

In this section we provide a characterization of the unicyclic graphs that are $(\Delta-1)$-differential graphs. In order to do this we need the family $\mathcal{T}$ of rooted trees and theorem as defined and proved by Mashburn et al. [4].

For a rooted tree T, let T_u denote the subtree of T rooted at vertex u. Let P_n denote the path on n vertices.

The family $\mathcal{T}$ consists of all trees T rooted at a vertex v of maximum degree having the following properties:

(i) each vertex $u \in N(v)$ has $deg(u) \leq 3$,

(ii) the connected components of $T - N[v]$ are K_1's and K_2's, and

(iii) either $N(v)$ contains at least two vertices u for which $T_u \in \{K_1, K_2\}$ or $N(v)$ contains exactly one leaf and no vertex of $N(v)$ has degree three.

Theorem 3.([4]) *A tree T is a $(\Delta-1)$-differential tree if and only if $T \in \mathcal{T}$.*

Theorem 4. *A unicyclic graph G with cycle C_n is a $(\Delta-1)$-differential graph if and only if*

(i) *$3 \leq n \leq 5$, and*

(ii) *there exists an edge $e \in C_n$ and a vertex of maximum degree in G that is not incident to e such that $G - e \in \mathcal{T}$.*

Proof. Let G be a $(\Delta-1)$-differential unicyclic graph having a vertex $v \in V$ of degree $\Delta(G)$. If $v \in V(C_n)$, then by Theorem 1(i), $3 \leq n \leq 5$. If $v \notin V(C_n)$ we claim that $n = 3$. By Theorem 1(ii), each component of the subgraph H induced by the vertices in $V - N[v]$ is a K_1 or a K_2. Hence, v is adjacent to at least one vertex of C_n and $n = 3$. Since G is a $(\Delta-1)$-differential unicyclic graph, $D = \{v\}$ is a ∂-set. Let e be an edge in C_n such that there is a vertex of maximum degree not incident to e. Then $G - e$ is a tree and $\partial(G-e) = (\Delta-1)$. Hence, by Theorem 3, $G - e \in \mathcal{T}$.

Conversely, let G satisfy the given conditions. Let e be an edge in C_n such that $G - e \in \mathcal{T}$, where there is a vertex v of maximum degree not incident to e. Since $G - e \in \mathcal{T}$, by Theorem 3, $\partial(G-e) = \Delta(G-e) - 1$ and $D = \{v\}$ is a ∂-set of $G - e$. Hence, by the choice of e the differential of G does not increase. Therefore, G is a $(\Delta-1)$-differential graph.

3.2. Split graphs

In this section we provide a characterization of the class of $(\Delta-1)$-differential split graphs.

Definition 5. A graph $G = (V, E)$ is called a split graph if the vertex set has a bipartition $V = V_1 \cup V_2$, where V_1 is an independent set (no two vertices in V_1 are adjacent) and the subgraph $G[V_2]$ induced by V_2 is a complete graph (every pair of vertices are adjacent).

Theorem 6. *A split graph G with bipartition (V_1, V_2) is a $(\Delta - 1)$-differential graph if and only if there exist $v \in V_2$ with maximum degree $\Delta(G)$ such that the following conditions hold.*

(i) *Each $u \in V_2 - \{v\}$ has at most two neighbours in $V_1 - N(v)$.*

(ii) *If S is the subset of $V_2 - \{v\}$ such that each vertex in S has exactly two neighbours in $V_1 - N(v)$ and no two members of S have a common neighbour in $V_1 - N(v)$ then S does not dominate $N(v) - \{x\}$, $x \in V_1$.*

Proof. Let G be a $(\Delta - 1)$-differential split graph having a vertex $v \in V_2$ of degree $\Delta(G)$.

Since the theorem clearly holds when $\Delta(G) = 1$, we assume that $\Delta(G) \geq 2$.

Suppose there exist a vertex $u \in V_2 - \{v\}$ which has more than two neighbours in $V_1 - N(v)$, then $\partial\{u, v\} \geq \Delta - 1 + 3 - 2 = \Delta > \Delta(G) - 1 = \partial(G)$, which is a contradiction. Hence each vertex in $V_2 - \{v\}$ has at most two neighbours in $V_1 - N(v)$. Let S be a subset of $V_2 - \{v\}$ satisfying condition (ii). Suppose S dominates $N(v) - \{x\}$, then $\partial(S) = 2|S| + \Delta - 1 - |S| - |S| + 1 = \Delta > \Delta(G) - 1 = \partial(G)$, which is a contradiction. Hence condition (ii) holds.

Conversely let G satisfy the given conditions. Now $\partial(G) \geq \partial(\{v\}) = \Delta(G) - 1$. Now to prove $\partial(G) \leq \Delta(G) - 1$. Let D be a ∂-set, assume $v \in D$. Clearly adding another vertex to D does not increase the differential of G. Hence $D = \{v\}$ and $\partial(G) = \Delta(G) - 1$. If $v \notin D$, adding a vertex of V_1 to D does not increase the differential of G.

If $D = V_2 - \{v\}$, then

$\partial(G) \leq \partial(D) \leq 2(|V_2| - 1) + \Delta - (|V_2| - 1) - 2 - (|V_2| - 1) \leq \Delta - 2 \leq \Delta - 1.$

Hence $\partial(G) = \Delta(G) - 1$.

3.3. Grid graphs

In this section we determine all $(\Delta - 1)$-differential grid graphs.

Definition 7. The Cartesian product $G \square H$ of a graph $G = (V, E)$ and a graph $H = (W, F)$ is the graph with vertex set equal to the Cartesian product $V \square W$ and two vertices (u, v) and (w, x) are adjacent in the Cartesian product graph if and only if either $u = w$ and v is adjacent to x in H, or u is adjacent to w in G and $v = x$.

Definition 8. The m by n grid graph $G_{m,n}$ is the Cartesian product of two paths P_m and P_n, $G_{m,n} = P_m \square P_n$.

Theorem 9. *$G_{m,n}$ is a $(\Delta - 1)$-differential graph if and only if $2 \leq m, n \leq 3$.*

Proof. The proof follows directly from Theorem 1(i).

3.4. k-regular graphs, for $k \leq 4$

Definition 10. A graph $G = (V, E)$ is called k-regular if every vertex $v \in V$ has degree k, that is $|N(v)| = k$.

In this section we study an important property of a regular graph G, which is $(\Delta - 1)$-differential, that can be used to characterize k-regular graphs, which are $(\Delta - 1)$-differential, when $1 \leq k \leq 4$.

Theorem 11. *A k-regular graph G is a $(\Delta-1)$-differential graph, then $k+1 \leq p \leq k+3$ where p is the number of vertices in the graph.*

Proof. Let G be a $(\Delta - 1)$-differential k-regular graph. Let $v \in V$ and $H = V - N[v]$. We claim that H has at most two vertices.

Suppose not, then there exist at least 3 vertices x, y, z in H. In view of Theorem 1(i) following cases arise.

(i) $\{x, y, z\}$ is independent.

(ii) $xy \in E(G)$ and $yz \notin E(G)$.

(iii) $xy \in E(G)$ and $yz \in E(G)$.

In case (i), $N(x) = N(y) = N(z) = N(v)$ which implies that each vertex in $N(v)$ has three neighbors in H, which is a contradiction.

In case (ii), both x and y are adjacent to $k - 1$ vertices in $N(v)$. By Theorem 1(ii), each vertex in $N(v)$ has at most two neighbors in H. Hence $deg(z) < k$, which is a contradiction.

Case (iii) can be similarly dealt with. Hence H has at most two vertices. Therefore $k + 1 \leq p \leq k + 3$.

Theorem 12. *A k-regular graph G where $k = 1, 2$ is a $(\Delta - 1)$-differential graph if and only if G is isomorphic to K_2 or C_n, $3 \leq n \leq 5$.*

Proof. If G is isomorphic to either K_2 or C_n, $3 \leq n \leq 5$ then clearly $\partial(G) = \Delta(G) - 1$. Conversely, let G be a $(\Delta - 1)$-differential graph, which is k-regular, $k = 1, 2$. If G is 1-regular, then clearly G is isomorphic to K_2. Suppose G is 2-regular, then by Theorem 1(i), G is isomorphic to C_n, $3 \leq n \leq 5$.

Theorem 13. *A 3-regular graph G is a $(\Delta - 1)$-differential graph if and only if G is isomorphic to K_4 or $K_{3,3}$ or $C_3 \times K_2$.*

Proof. Let G be a $(\Delta - 1)$-differential 3-regular graph. By Theorem 11, $4 \leq p \leq 6$, where p is the number of vertices in G. Since the number of vertices of odd degree in any graph is even, $p = 4$ or $p = 6$. Let $v \in V$ and v_i, $i = 1, 2, 3$ be the neighbors of v.
When $p = 4$, clearly G is isomorphic to K_4.
When $p = 6$, let $x, y \in V - N(v)$. If x, y are not adjacent then clearly $N(x) = N(y)$ and $N(v)$ is independent. Therefore G is isomorphic to $K_{3,3}$.

If x and y are adjacent then exactly one member say v_1 of $N(v)$ is adjacent to both x and y, and the other two members v_2, v_3 of $N(v)$ are adjacent to x and y respectively. Further v_2 and v_3 are adjacent. Hence G is isomorphic to $C_3 \times K_2$.

Conversely suppose G is isomorphic to K_4 or $K_{3,3}$ or $C_3 \times K_2$ then clearly $\partial(G) = \Delta(G) - 1$.

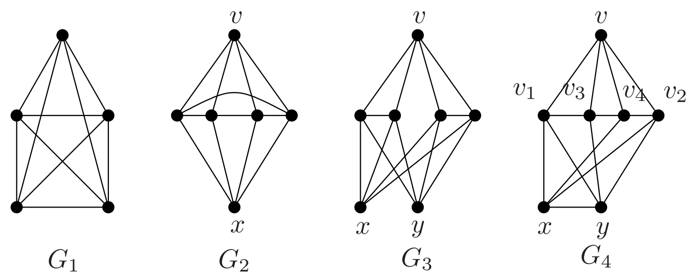

Figure 1:

Theorem 14. *A 4-regular graph is a $(\Delta - 1)$-differential graph if and only if G is isomorphic to G_i, $1 \leq i \leq 4$ as given in Figure 1.*

Proof. Let G be a $(\Delta - 1)$-differential 4-regular graph. By Theorem 11, $5 \leq p \leq 7$. When $p = 5$, G is isomorphic to K_5.

When $p = 6$, let $x \in V - N(v)$ where $v \in V$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. Clearly x is adjacent to each v_i, $1 \leq i \leq 4$ and the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ is a cycle of length 4. Hence G is isomorphic to G_2.

When $p = 7$, let $x, y \in V - N(v)$. If x and y are not adjacent, then x and y are adjacent to each v_i and the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ is $2k_2$. Therefore G is isomorphic to G_3. If x and y are adjacent then exactly two members of $N(v)$ say v_1 and v_2 are adjacent to x and y, and the other two members v_3 and v_4 of $N(v)$ are such that x and y are adjacent to v_3 and v_4 respectively. Further the subgraph induced by $\{v_1, v_2, v_3, v_4\}$ is a P_4, such that v_1 and v_2 are the ends of P_4. Hence G is isomorphic to G_4.

Conversely, suppose G is isomorphic to G_i, $1 \leq i \leq 4$, then clearly G is a $(\Delta - 1)$-differential graph.

4. Bipartite graphs

Definition 15. A graph $G = (V, E)$ is called bipartite if the vertex set can be partitioned into two independent sets.

Finally we provide a characterization of the class of $(\Delta - 1)$-differential bipartite graphs. For this purpose we prove the following lemma.

Lemma 16. *A bipartite graph with bipartition (X, Y) is a $(\Delta - 1)$-differential graphs, then the following holds.*

(i) *Every vertex in $N(v)$ is of degree at most 3 and every vertex in $Y - N(v)$ is of degree 1.*

(ii) *Every vertex in $X - \{v\}$ has at most one neighbor in $Y - N(v)$.*

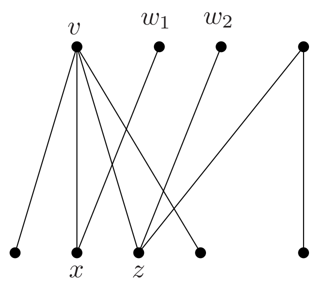

Figure 2:

Proof. Let v be a vertex of maximum degree $\Delta(G)$. If there exist a vertex $y \in N(v)$ of degree more than three, there $\partial(\{v, y\}) \geq \Delta(G) + 3 - 2 - 1 = \Delta(G) \geq \Delta(G) - 1 = \partial(G)$, which is a contradiction. Hence every member in $N(v)$ is of degree at most three.

If there exists a vertex $w \in Y - N(v)$ of degree more than 1, then $\partial\{v, w\} \geq \Delta + 2 - 2 = \Delta(G) \geq \Delta(G) - 1 = \partial(G)$, which is a contradiction. Hence every vertex in $Y - N(v)$ is of degree 1.

If there exists a vertex $x \in X - \{u\}$ which has more than one neighbor in $Y - N(v)$ then $\partial\{x, v\} \geq \Delta + 2 - 2 = \Delta(G) > \Delta(G) - 1 = \partial(G)$, which is a contradiction. Hence each vertex in $X - \{v\}$ has at most one neighbor in $Y - N(v)$.

Definition 17. Let G be a bipartite graph with bipartition (X, Y). Let v be a vertex in X of maximum degree $\Delta(G)$. A vertex $x \in N(v)$ with $deg(x) = 3$ and $N(x) - \{v\} = \{w_1, w_2\}$ is said to be a special vertex if either w_1 or w_2 has a neighbor z in $N(v)$, then z has a neighbor in $X - \{v, w_1, w_2\}$ which is a support. In Figure 2, x is a special vertex.

Theorem 18. *A bipartite graph with bipartition (X, Y) is a $(\Delta - 1)$-differential graph if and only if there exist a vertex v say in X of maximum degree $\Delta(G)$ such that the following conditions hold:*

(i) *Every vertex in $N(v)$ is of degree at most three and every vertex in $Y - N(v)$ is of degree one.*

(ii) *Every vertex in $X - \{v\}$ has almost one neighbor in $Y - N(v)$.*

(iii) *If S_1 is the set of all vertices in $N(v)$ which are adjacent to a support in $X - \{v\}$ and S_2 is the set of all vertices in $N(v) - S_1$ which are of degree three, such that no two members of S_2 have a common neighbor in $X - \{v\}$, then*

$$\text{when } S_2 = \phi,\ |S_1| \leq \begin{cases} \Delta - 2 & \text{if there exists a special vertex in } S_1 \\ \Delta - 1 & \text{otherwise} \end{cases}$$

Further when $|S_1| = \Delta - 1$, v is a support. When $S_2 \neq \phi$, then $|S_1| + |S_2| \leq \Delta(G) - 2$.

Proof. Let G be a $(\Delta - 1)$-differential bipartite graph. Let v be a vertex of maximum degree $\Delta(G)$. By Lemma 16, conditions (i) and (ii) hold.

Let S_1 be the set of all vertices in $N(v)$ which are adjacent to a support $X - \{v\}$ and S_2 be the set of all vertices in $N(v) - S_1$ which are of degree three, such that no two members of S_2 have a common neighbor in $X - \{v\}$. Let A be the set of vertices in $X - \{v\}$ which are supports. If $S_2 = \phi$ and there exists a special vertex in S_1, then we claim $|S_1| \leq \Delta - 2$. Suppose not, then $\partial(A) = |A| + |S_1| + 1 - |A| = |S_1| + 1 \geq \Delta(G) - 1 + 1 = \Delta(G) > \Delta(G) - 1 = \partial(G)$, which is a contradiction.

If $S_2 = \phi$ and if there exists no special vertex in S_1, then we claim $|S_1| \leq \Delta - 1$. Suppose not, then $\partial(A) = |A| + |S_1| - |A| = |S_1| \geq \Delta(G)$, which is a contradiction.

If $|S_1| = \Delta - 1$, then we claim that v is a support. Suppose not, then $\partial(\{A, w\}) = |A| + |S_1| + 2 - |A| - 1 = |S_1| + 1 = \Delta$, where w is the vertex of degree one adjacent to v. Hence $\partial(\{A, w\}) > \partial(G)$, which is a contradiction.

If $S_2 \neq \phi$, then we claim $|S_1| + |S_2| \leq \Delta - 2$. Suppose not, then

$$\begin{aligned}\partial(A \cup S_2) &= |A| + |S_1| + 2|S_2| + 1 - |A| - |S_2| \\ &= |S_1| + |S_2| + 1 \\ &\geq \Delta - 1 + 1 \\ &= \Delta \geq \Delta(G) - 1 = \partial(G)\end{aligned}$$

which is a contradiction. Hence $|S_1| + |S_2| \leq \partial(G)$.

Conversely if G satisfy the given conditions then $\partial(G) \geq \partial(\{v\}) = \Delta(G) - 1$. To show that $\partial(G) \leq \Delta(G) - 1$, let D be a ∂-set. Assume first that $v \in D$. Since conditions (i) and (ii) hold, adding another vertex to D does not increase the differential of G. Hence we assume that $D = \{v\}$ and $\partial(G) = \Delta(G) - 1$. Suppose $v \notin D$. Let S_1 and S_2 be as given in the theorem. Let A be the set of all vertices in $X - \{v\}$ which are supports. Let $S_2 = \phi$. If there exist a special vertex in S_1, then there exist at least two vertices x, y such that $x, y \notin S_1$. Now $\partial(G) \leq \partial(A) = |S_1| + |A| + 1 - |A| = |S_1| + 1 \leq \Delta - 1$. Further adding x or y to A does not increase the differential of G. Suppose there does not exist a special vertex in S_1, then $\partial(G) \leq \partial(A) = |S_1| + |A| - |A| \leq \Delta - 1$. Let $S_2 \neq \phi$, then $\partial(G) \leq \partial(A \cup S_2) = |S_1| + |A| + 2|S_2| + 1 - |A| - |S_2| = |S_1| + |S_2| + 1 \leq \Delta - 2 + 1 = \Delta - 1$. Hence $\partial(G) \leq \Delta(G) - 1$.

5. The differential $\partial(T)$ of a complete binary tree T

Finally we calculate the value of $\partial(G)$ for a complete binary tree.

Theorem 19. *Let G be a complete binary the consisting of k levels, then*

$$\partial(G) = \begin{cases} 3 \displaystyle\sum_{i=0}^{\left[\frac{k-1}{3}\right]} \frac{2^{k-2}}{8^i} & k = 3n \text{ or } 3n+2 \\ 3 \displaystyle\sum_{i=0}^{\left[\frac{k-1}{3}\right]-1} \frac{2^{k-2}}{8^i} + 1 & k = 3n+1 \end{cases}.$$

Proof. Let G be a complete binary tree. Let S_i be the set of all vertices in level i and $|S_i| = n_i$, then $n_i = 2^i$.

Since $n_k > n_{k-1} > n_{k-2} > \cdots > n_0$, clearly $S_{k-1} \cup S_{k-4} \cup \cdots \cup S_0$ is a ∂-set if $k = 3n$.

If $k = 3n$,

$$\begin{aligned}
\partial(G) &= \left[(2^k + 2^{k-2} + 2^{k-3} + 2^{k-5} + \cdots) - (2^{k-1} + 2^{k-4} + 2^{k-7} + \cdots)\right] - 1 \\
&= \left[2^{3n}\left(1 + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots\right) - 2^{3n}\left(\frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots\right)\right] - 1 \\
&= \left[2^{3n}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) - 2^{3n}\frac{2}{2}\left(\frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots\right)\right] - 1 \\
&= 8^n\left[\frac{2 \cdot 8^n - 1}{8^n} - \frac{8}{7}\left(\frac{8^n - 1}{8^n}\right)\right] - 1 = \frac{6}{7}(8^n - 1)
\end{aligned}$$

$$\begin{aligned}
\text{Also} \quad \partial(G) &= 3\sum_{i=0}^{\left[\frac{3n-1}{3}\right]} \frac{2^{3n-2}}{8^i} = 3\sum_{i=0}^{n-1} \frac{2^{3n-2}}{8^i} \\
&= \frac{38^n}{4}\left[1 + \frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^{n-1}}\right] \\
&= \frac{6}{7}(8^n - 1)
\end{aligned}$$

If $k = 3n + 1$,

$$\begin{aligned}
\partial(G) &= \left[2^{3n+1} + 2^{3n-1} + 2^{3n-2} + \cdots\right] - \left[2^{3n} + 2^{3n-3} + 2^{3n-6} + \cdots\right] \\
&= 2^{3n}\left[2 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right] - 2(8)^n\left[1 + \frac{1}{2^3} + \frac{1}{2^6} + \cdots\right] \\
&= \frac{12(8^n) - 5}{7}
\end{aligned}$$

$$\begin{aligned}
\text{Also} \quad \partial(G) &= \left[3\sum_{i=0}^{\left[\frac{k-1}{3}\right]-1} \frac{2^{k-2}}{8^i}\right] + 1 \\
&= \frac{3(8^n)}{2} \times \frac{8}{7}\left(\frac{8^n - 1}{8^n}\right) + 1 \\
&= \frac{12(8^n) - 5}{7}
\end{aligned}$$

If $k = 3n + 2$,

$$\partial(G) = \left(2^{3n+2} + 2^{3n} + 2^{3n-1} + 2^{3n-3} + \cdots + 2^0\right)$$

$$
\begin{aligned}
&- \left(2^{3n+1} + 2^{3n-2} + 2^{3n-5} + \cdots\right) \\
&= \left(2^0 + 2^1 + 2^2 + \cdots + 2^{3n+2}\right) - 2 \times 2 \left(1 + 2^3 + 2^6 + \cdots\right) \\
&= \left(\frac{2^{3n+3} - 1}{2 - 1}\right) - 4 \left(\frac{(2^3)^{n+1} - 1}{2^3 - 1}\right) \\
&= \frac{3}{7}(8(8)^n - 1)
\end{aligned}
$$

$$
\text{Also} \quad \partial(G) = \left[3 \sum_{i=0}^{\left[\frac{k-1}{3}\right]} \frac{2^{3n}}{8^i}\right]
$$

$$
= 3(8^n)\left[1 + \frac{1}{8} + \frac{1}{8^2} + \cdots + \frac{1}{8^n}\right] = \frac{3}{7}(8(8)^n - 1).
$$

Hence the theorem is proved.

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