THE TWO DIMENSIONAL GENERALIZED WEYL FRACTIONAL CALCULUS AND SPECIAL FUNCTIONS

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Abstract. The object of this present paper is to derive a relation between the two dimensional I-transform involving a general class of polynomials and the Weyl type two dimensional Saigo operators of fractional integration. The results derived here are general in nature and include the results given earlier by Saigo, Saxena and Ram [10],Saxena and Ram [8], Saxena and Kiryakova [9] and Chaurasia and Srivastava [12].

1. Introduction

A. K. Rathie [11] introduced the I-function in the form

$$I(z) = I_{p,q}^{m,n}[z] = I_{p,q}^{m,n}\left[z \left| \begin{array}{l} (\alpha_1, A_1, a_1), \ldots, (\alpha_p, A_p, a_p) \\ (\beta_1, B_1, b_1), \ldots, (\beta_q, B_q, b_q) \end{array} \right.\right]$$

$$= \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} \theta(\xi) z^{\xi} d\xi, \tag{1}$$

where

$$\theta(\xi) = \frac{\displaystyle\prod_{j=1}^{m}\{\Gamma(\beta_j - B_j\xi)\}^{b_j} \prod_{j=1}^{m}\{\Gamma(1 - \alpha_j + A_j\xi)\}^{a_j}}{\displaystyle\prod_{j=m+1}^{q}\{\Gamma(1 - \beta_j + B_j\xi)\}^{b_j} \prod_{j=n+1}^{p}\{\Gamma(\alpha_j - A_j\xi)\}^{a_j}}, \tag{2}$$

where a_j, $j = 1, \ldots, p$ and b_j, $j = 1, \ldots, q$ are not, in general positive integers. Clearly for non-integral values of a_j or b_j, (1) is not expressible as an H-function. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; p, q, m and n are integers such that $0 \leq m \leq q$; $0 \leq n \leq p$, $A_j > 0$ $(j = 1, \ldots, p)$, $B_j > 0$ $(j = 1, \ldots, q)$; α_j, $j = 1, \ldots, p$ and β_j, $j = 1, \ldots, q$ are complex parameters. The contour in (1) is presumed to be the imaginary axis $Re(\xi) = 0$ which is suitable indented in order to avoid the singularities of the gamma functions and to keep these

singularities at appropriate side. For a_j not an integer, the pole of the gamma functions of the numerator of (2) is converted to branch point. The branch cuts can be chosen in order that the path of integration can be distorted for the contour $Re(\xi) = 0$ as long as there is no coincidence of poles from $\Gamma(\beta_j - B_j\xi)$ and $\Gamma(1 - \alpha_j - a_j\xi)$. The sufficient conditions for the absolute convergence of the contour integral (1) is given by

$$\Omega = \sum_{j=1}^{m} |b_j B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{p} |a_j A_j| > 0. \tag{3}$$

This condition provides exponential decay of the integrand in (1), and region of absolute convergence of (1) is

$$|\arg(z)| < \frac{\Omega\pi}{2}. \tag{4}$$

Srivastava [2] introduced the general class of polynomials

$$S_n^m[x] = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \qquad n = 0, 1, 2, \ldots, \tag{5}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n, k \geq 0)$ are arbitrary constants, real or complex. By suitable specialization of the coefficients $A_{n,k}$, the general class of polynomials can be reduced to large spectrum of polynomials as cited in the papers referred to above.

2. Fractional Integrals and Derivatives

An interesting and useful generalization of both the Riemann - Liouville and Erdelyi - Kober fractional integration operators is introduced by M. Saigo [5], [6] in terms of Gauss's hypergeometric function as given below. Let u, v and w are complex numbers and let $x \in R_+ = (0, \infty)$ following [5], [6] the fractional integral $(Re(u) > 0)$ and derivative $(Re(u) < 0)$ of the first kind of a function $f(x)$ on R_+ are defined respectively in the forms

$$I_{0,x}^{u,v,w} f = \frac{x^{-u-v}}{\Gamma(u)} \int_0^x (x - s)^{u-1} {}_2F_1(u + v; -w; u; 1 - \frac{s}{x}) f(s) ds; \quad Re(u) > 0, \tag{6}$$

$$= \frac{d^n}{dx^n} I_{0,x}^{u+n,v-n,w-n} f; \quad 0 \leq Re(u) + n \leq 1, \quad (n = 1, 2, 3, \ldots). \tag{7}$$

where ${}_2F_1(\rho; \sigma; \eta; \cdot)$ is Gauss's hypergeometric function. The fractional integral $(Re(u) > 0)$ and derivative $(Re(u) < 0)$ of the second kind are given by

$$J_{x,\infty}^{u,v,w} f = \frac{1}{\Gamma(u)} \int_x^\infty (s - x)^{u-1} s^{-u-v} {}_2F_1(u + v, -w; u; 1 - \frac{x}{s}) f(s) ds; \quad Re(u) > 0, \tag{8}$$

$$= (-1)^n \frac{d^n}{dx^n} J_{x,\infty}^{u+n,v-n,w} f; \quad 0 \leq Re(u) + n \leq 1, \quad (n = 1, 2, 3, \ldots). \tag{9}$$

The Riemann-Liouville, Weyl and Erdelyi-Kober fractional calculus operators follow as special cases of the operators I and J as detailed below.

$$R_{0,x}^u f = I_{0,x}^{u,-u,w} f = \frac{1}{\Gamma(u)} \int_0^x (x-s)^{u-1} f(s) ds; \quad Re(u) > 0, \tag{10}$$

$$= \frac{d^n}{dx^n} R_{0,x}^{u+n} f; \quad 0 \le Re(u) + n \le 1, \quad (n = 1, 2, 3, \ldots). \tag{11}$$

$$W_{x,\infty}^u f = W_{x,\infty}^{u,-u,w} f = \frac{1}{\Gamma(u)} \int_x^\infty (s-x)^{u-1} f(s) ds; \quad Re(u) > 0, \tag{12}$$

$$= (-1)^n \frac{d^n}{dx^n} W_{x,\infty}^{u+n} f; \quad 0 \le Re(u) + n \le 1, \quad (n = 1, 2, 3, \ldots). \tag{13}$$

$$E_{0,x}^{u,w} f = I_{0,x}^{u,0,w} f = \frac{x^{-u-w}}{\Gamma(u)} \int_0^x (x-s)^{u-1} s^w f(s) ds; \quad Re(u) > 0, \tag{14}$$

$$K_{x,\infty}^{u,w} f = J_{x,\infty}^{u,0,w} f = \frac{1}{\Gamma(u)} \int_x^\infty (s-x)^{u-1} s^{-u-w} f(s) ds; \quad Re(u) > 0. \tag{15}$$

Following Miller [3] we denote by S_1 the class of functions $f(x)$ on R_+, which are infinitely differentiable with partial derivatives of any order behaving as $o(|x|^{-\gamma})$ when $x \to \infty$ for all γ. Similarly we denote S_2, the class of functions $f(x, y)$ on $R_+ \times R_+$, which are infinitely differentiable with partial derivatives of any order behaving as $o(|x|^{-\gamma_1}, |y|^{-\gamma_2})$ when $x \to \infty$, $y \to \infty$ for all γ_i $(i = 1, 2)$. The two dimensional Saigo operator of Weyl type fractional integration of order $Re(u) > 0$, $Re(\eta) > 0$ is defined in the class S_2 by

$$J_{x,\infty}^{u,v,w} J_{y,\infty}^{\eta,\beta,\alpha} [f(x,y)] = \frac{x^v y^\beta}{\Gamma(u)\Gamma(\eta)} \int_x^\infty \int_y^\infty (t-x)^{u-1} (c-y)^{\eta-1} t^{-u-v} c^{\eta-\beta}$$
$$_2F_1(u+v, -w; u; 1 - \frac{x}{t}) \, _2F_1(\eta+\beta, -\alpha; \eta; 1 - \frac{y}{c}) F(t,c) dt dc, \tag{16}$$

where v, β, w, α are real numbers. More generally, a Saigo operator of Weyl type fractional calculus in two variables is defined by the differ integral expression

$$J_{x,\infty}^{u,v,w} J_{y,\infty}^{\eta,\beta,\alpha} [f(x,y)] = \frac{(-1)^{M+N} x^v y^\beta}{\Gamma(u+M)\Gamma(\eta+N)} \frac{\partial^{M+N}}{\partial x^M \partial y^N} \int_x^\infty \int_y^\infty (t-x)^{u-1} (c-y)^{\eta-1} t^{-u-v} c^{-\eta-\beta}$$
$$_2F_1(u+v, -w; u; 1 - \frac{x}{t}) \, _2F_1(\eta+\beta, -\alpha; \eta; 1 - \frac{y}{c}) F(t,c) dt dc, \tag{17}$$

for arbitrary real u and η, $M, N = 0, 1, 2, \ldots$. For $f(x, y) \in S_2$ this differ integral exists and also belongs to S_2 [3].

In particular, if $\{Re(u) + M\} > 0$, $\{Re(\eta) + N\} > 0$ then (17) yield partial fractional derivative of $f(x, y)$.

On the other hand if we set $v = \beta = 0$, (17) yield the Weyl type Erdelyi-Kober operators in two dimensions

$$K_{x,\infty}^{u,w} K_{y,\infty}^{\eta,\alpha} [f(x,y)] = J_{x,\infty}^{u,0,w} J_{y,\infty}^{\eta,0,\alpha} [f(x,y)]$$

$$= \frac{(-1)^{M+N}x^w y^a}{\Gamma(u+M)\Gamma(\eta+N)} \frac{\partial^{M+N}}{\partial x^M \partial y^N} \int_x^\infty \int_y^\infty (t-x)^{u+M-1}(c-y)^{\eta+N-1}$$
$$t^{-u-w}c^{-\eta-\alpha}F(t,c)dtdc. \qquad (18)$$

3. Two-dimensional Laplace transform and I-transforms involving a general class of polynomials

The Laplace transform $\mu(a,b)$ of function $f(x,y) \in S_2$ is defined

$$\mu(a,b) = L[f(x,y);a,b] = \int_0^\infty \int_0^\infty e^{-ax-by}f(x,y)dxdy, \qquad (19)$$

where $Re(a) > 0$, $Re(b) > 0$.

Analogously, the Laplace transform of

$$f(d\sqrt{x^2-h^2}H(x-h), f\sqrt{y^2-g^2}H(y-g)], \quad x > h > 0, \ y > g > 0 \qquad (20)$$

and $H(s)$ denotes Heaviside's unit step function.

Definition. By two -dimensional I-transform $\Delta(a,b)$ involving a general class of polynomials of a function $F(x,y)$, we mean the following repeated integral involving two different I-functions with a general class of polynomials

$$\Delta(a,b) = \Delta_{p,q,n_1;p',q',n_2}^{m,n,m_1;m',n';m_2}[F(x,y);\rho,\sigma;a,b]$$
$$= \int_h^\infty \int_g^\infty (ax)^{\rho-1}(by)^{\sigma-1} I_{p,q}^{m,n} \left[(ax)^\lambda \left| \begin{matrix} (\alpha_1,A_1,a_1),\dots,(\alpha_p,A_p,a_p) \\ (\beta_1,B_1,b_1),\dots,(\beta_q,B_q,b_q) \end{matrix} \right.\right] S_{n_1}^{m_1}[(ax)^{\delta_1}]$$
$$I_{p',q'}^{m',n'} \left[(by)^{\lambda'} \left| \begin{matrix} (\alpha_1',A_1',a_1'),\dots,(\alpha_{p'}',A_{p'}',a_{p'}') \\ (\beta_1',B_1',b_1'),\dots,(\beta_{q'}',B_{q'}',b_{q'}') \end{matrix} \right.\right] S_{n_2}^{m_2}[(by)^{\delta_2}]F(x,y)dxdy. \qquad (21)$$

Here, it is assumed that $h > 0$, $g > 0$, $\lambda > 0$, $\lambda' > 0$; $\Delta(a,b)$ exists and belongs to S_2. Further assume

$$|\arg a^\lambda| < \frac{\Omega\pi}{2}, \quad |\arg b^{\lambda'}| < \frac{\Omega'\pi}{2}, \qquad (22)$$

where

$$\Omega = \sum_{j=1}^m |b_j B_j| + \sum_{j=1}^n |a_j A_j| - \sum_{j=m+1}^q |b_j B_j| - \sum_{j=n+1}^p |a_j A_j| > 0 \qquad (23)$$

$$\Omega' = \sum_{j=1}^{m'} |b_j' B_j'| + \sum_{j=1}^{n'} |a_j' A_j'| - \sum_{j=m+1}^{q'} |b_j' B_j'| - \sum_{j=n+1}^{p'} |a_j' A_j'| > 0. \qquad (24)$$

4. Relationship between two - dimensional I - transform involving a general class of polynomials in terms of two -dimensional Saigo operator of Weyl type

For $Re(u) > 0$, $Re(\eta) > 0$, $g > 0$, $\lambda > 0$ and $\lambda' > 0$ then there holds the formula

$$J_{x,\infty}^{u,v,w} J_{y,\infty}^{\eta,\beta,\alpha}[\Delta_{p,q,n_1;p',q';n_2}^{m,n,m_1;m',n';m_2}\{F(x,y);\rho,\sigma;a,b\}]$$

$$= \int_h^\infty \int_g^\infty (ax)^{\rho-1}(by)^{\sigma-1} I_{p+2,q+2}^{m+2,n}$$

$$\left[(ax)^\lambda \left| \begin{matrix} (\alpha_1,A_1,a_1),\ldots,(\alpha_p,A_p,a_p),(1-\rho-k_1\delta_1,\lambda,1),(u+v+w-\rho+1-k_1\delta_1,\lambda,1) \\ (v-\rho+1-k_1\delta_1,\lambda,1),(w-\rho+1-k_1\delta_1,\lambda,1),(\beta_1,B_1,b_1),\ldots,(\beta_q,B_q,b_q) \end{matrix} \right.\right]$$

$$S_{n_1}^{m_1}[(ax)^{\delta_1}] I_{p'+2,q'+2}^{m'+2,n'}$$

$$\left[(by)^{\lambda'} \left| \begin{matrix} (\alpha_1',A_1',a_1'),\ldots,(\alpha_{p'}',A_{p'}',a_{p'}'),(1-\sigma-k_2\delta_2,\lambda',1),(\eta+\beta+\alpha-\sigma+1-k_2\delta_2,\lambda',1) \\ (\beta-\sigma+1-k_2\delta_2,\lambda',1),(\alpha-\sigma+1-k_2\delta_2,\lambda',1),(\beta_1',B_1',b_1'),\ldots,(\beta_{q'}',B_{q'}',b_{q'}') \end{matrix} \right.\right]$$

$$S_{n_2}^{m_2}[(by)^{\delta_2}] F(x,y) dx dy = \Delta_1(a,b), \tag{25}$$

where, it is supposed that $\Delta_1(a,b)$ exists and belongs to S_2 as well as $\lambda > 0$, $\lambda' > 0$.

Proof. Let $Re(u) > 0$, $Re(\eta) > 0$ then in view of (16) and (21) we have

$$J_{a,\infty}^{u,v,w} J_{b,\infty}^{\eta,\beta,\alpha}[\Delta(a,b)]$$

$$= \frac{a^v b^\beta}{\Gamma(u)\Gamma(\eta)} \int_a^\infty \int_b^\infty (t-a)^{u-1}(c-b)^{\eta-1} t^{-u-v} c^{-\eta-\beta} {}_2F_1(u+v,-w;u;1-\frac{a}{t}) {}_2F_1(\eta+\beta,-\alpha;\eta;1-\frac{b}{c})$$

$$\left\{\int_h^\infty \int_g^\infty (tx)^{\rho-1}(cy)^{\sigma-1} I_{p,q}^{m,n}[(tx)^\lambda] S_{n_1}^{m_1}[(tx)^{\delta_1}] I_{p',q'}^{m',n'}[(cy)^{\lambda'}] S_{n_2}^{m_2}[(cy)^{\delta_2}] F(x,y) dx dy \right\} dt dc. \tag{26}$$

On interchanging the order of integration which is permissible due to absolute convergence of the integrals under consideration and on evaluating the t- and c- integrals through the integral formula Erdelyi [1]

$$\int_0^1 x^{\gamma-1}(1-x)^{\rho-1} {}_2F_1(\alpha,\beta;\gamma;x)dx = \frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\gamma+\rho-\alpha-\beta)}{\Gamma(\gamma+\rho-\alpha)\Gamma(\gamma+\rho-\beta)}$$

for $Re(\gamma) > 0$, $Re(\rho) > 0$, $Re(\gamma-\rho-\alpha-\beta) > 0$.

For a little simplification L. H. S. of (26) become our required result (25).

5. Special cases

(i) By applying our results derived in (25) to the case of Hermite polynomials G. Szego [4] and H. M. Srivastava [2] by setting

$$S_n^2[x] \to x^{\frac{n}{2}} H_n(\frac{1}{2\sqrt{x}}) \text{ in which case } m = 2, \ A_{n,k} = (-1)^k,$$

we have the following interesting consequence of the main result

$$J_{a,\infty}^{u,v,w} J_{b,\infty}^{\eta,\beta,\alpha} \left[\Delta_{p,q,n_1,p',q',n_2}^{m,n,2;m',n';2} \{F(x,y);\rho,\sigma;a,b\} \right]$$

$$= \int_h^\infty \int_g^\infty (ax)^{\rho+\frac{n_1\delta_1}{2}-1}(by)^{\sigma+\frac{n_2\delta_2}{2}-1} I_{p+2,q+2}^{m+2,n}$$

$$\left[(ax)^\lambda \left| \begin{array}{l} (\alpha_1,A_1,a_1),\ldots,(\alpha_p,A_p,a_p),(1-\rho-k_1\delta_1,\lambda,1),(u+v+w-\rho+1-k_1\delta_1,\lambda,1) \\ (v-\rho+1-k_1\delta_1,\lambda,1),(w-\rho+1-k_1\delta_1,\lambda,1),(\beta_1,B_1,b_1),\ldots,(\beta_q,B_q,b_q) \end{array} \right. \right]$$

$$H_{n_1}\left[\frac{1}{2\sqrt{(ax)^{\delta_1}}} \right] I_{p'+2,q'+2}^{m'+2,n'}$$

$$\left[(by)^{\lambda'} \left| \begin{array}{l} (\alpha_1',A_1',a_1'),\ldots,(\alpha_{p'}',A_{p'}',a_{p'}'),(1-\sigma-k_2\delta_2,\lambda',1),(\eta+\beta+\alpha-\sigma+1-k_2\delta_2,\lambda',1) \\ (\beta-\sigma+1-k_2\delta_2,\lambda',1),(\alpha-\sigma+1-k_2\delta_2,\lambda',1),(\beta_1',B_1',b_1'),\ldots,(\beta_{q'}',B_{q'}',b_{q'}') \end{array} \right. \right]$$

$$H_{n_2}\left[\frac{1}{2\sqrt{(by)^{\delta_2}}} \right] F(x,y)dxdy, \tag{27}$$

(ii) For the Leaguerre polynomials (G. Szego [4] and H. M. Srivastava [2]) by setting $S_n^1[x] \to L_n^{(\alpha')}[x]$ in which case $m=1$, $A_{n,k} = \binom{n+\alpha'}{n}\frac{1}{(\alpha'+1)_k}$ and (25) reduces to the following formulae

$$J_{a,\infty}^{u,v,w} J_{b,\infty}^{\eta,\beta,\alpha} \left[\Delta_{p,q,n_1,p',q',n_2}^{m,n,1;m',n';1} \{F(x,y);\rho,\sigma;a,b\} \right]$$

$$= \int_h^\infty \int_g^\infty (ax)^{\rho-1}(by)^{\sigma-1} I_{p+2,q+2}^{m+2,n}$$

$$\left[(ax)^\lambda \left| \begin{array}{l} (\alpha_1,A_1,a_1),\ldots,(\alpha_p,A_p,a_p),(1-\rho-k_1\delta_1,\lambda,1),(u+v+w-\rho+1-k_1\delta_1,\lambda,1) \\ (v-\rho+1-k_1\delta_1,\lambda,1),(w-\rho+1-k_1\delta_1,\lambda,1),(\beta_1,B_1,b_1),\ldots,(\beta_q,B_q,b_q) \end{array} \right. \right]$$

$$L_{n_1}^{(\alpha')}\left[(ax)^{\delta_1} \right] I_{p'+2,q'+2}^{m'+2,n'}$$

$$\left[(by)^{\lambda'} \left| \begin{array}{l} (\alpha_1',A_1',a_1'),\ldots,(\alpha_{p'}',A_{p'}',a_{p'}'),(1-\sigma-k_2\delta_2,\lambda',1),(\eta+\beta+\alpha-\sigma+1-k_2\delta_2,\lambda',1) \\ (\beta-\sigma+1-k_2\delta_2,\lambda',1),(\alpha-\sigma+1-k_2\delta_2,\lambda',1),(\beta_1',B_1',b_1'),\ldots,(\beta_{q'}',B_{q'}',b_{q'}') \end{array} \right. \right]$$

$$L_{n_2}^{(\alpha')}\left[(by)^{\delta_2} \right] F(x,y)dxdy. \tag{28}$$

(iii) For the Jacobi polynomials (G. Szego [4] and H. M. Srivastava [2]) by setting $S_n(x) = P_n^{(\alpha',\beta')}(1-2x)$ in which case $m=1$ and $A_{n,k} = \binom{n+\alpha'}{n}\frac{(\alpha'+\beta'+n+1)_k}{(\alpha'+1)_k}$ in (25) reduces to following formulae

$$J_{a,\infty}^{u,v,w} J_{b,\infty}^{\eta,\beta,\alpha} \left[\Delta_{p,q,n_1,p',q',n_2}^{m,n,1;m',n';1} \{F(x,y);\rho,\sigma;a,b\} \right]$$

$$= \int_h^\infty \int_g^\infty (ax)^{\rho-1}(by)^{\sigma-1} I_{p+2,q+2}^{m+2,n}$$

$$\left[(ax)^\lambda \left| \begin{array}{l} (\alpha_1,A_1,a_1),\ldots,(\alpha_p,A_p,a_p),(1-\rho-k_1\delta_1,\lambda,1),(u+v+w-\rho+1-k_1\delta_1,\lambda,1) \\ (v-\rho+1-k_1\delta_1,\lambda,1),(w-\rho+1-k_1\delta_1,\lambda,1),(\beta_1,B_1,b_1),\ldots,(\beta_q,B_q,b_q) \end{array} \right. \right]$$

$$P_{n_1}^{(\alpha',\beta')}\left[1-2(ax)^{\delta_1} \right] I_{p'+2,q'+2}^{m'+2,n'}$$

$$\left[(by)^{\lambda'} \middle| \begin{array}{l} (\alpha_1', A_1', a_1'), \ldots, (\alpha_{p'}', A_{p'}', a_{p'}'), (1-\sigma-k_2\delta_2, \lambda', 1), (\eta+\beta+\alpha-\sigma+1-k_2\delta_2, \lambda', 1) \\ (\beta-\sigma+1-k_2\delta_2, \lambda', 1), (\alpha-\sigma+1-k_2\delta_2, \lambda', 1), (\beta_1', B_1', b_1'), \ldots, (\beta_{q'}', B_{q'}', b_{q'}') \end{array} \right]$$

$$P_{n_2}^{(\alpha', \beta')}\left[1 - 2(by)^{\delta_2}\right] F(x, y) dx dy, \tag{29}$$

6. Some known results

(i) When the exponents b_j $(j = 1, \ldots, m)$ and a_j $(j = n+1, \ldots, p)$ in (25) are unity , the I-function reduces to $\overline{H}$-function and we arrive at the result given by Chaurasia and Srivastava [12, pp.237-249].

(ii) On taking $a_j = b_j = 1$, $n_1 = n_2 = 0$ in (25), we arrive at the result obtained by Saigo, Saxena and Ram [10, pp.63-73].

(iii) For $a_j = b_j = 1$, $n_1 = n_2 = 0 = w = \alpha$ in (25) we have a result, earlier proved by Saxena and Kiryakova [9, pp.133-140].

7. Relationship between one-dimensional I-transform involving a general class of polynomials in terms of one-dimensional Saigo operator of Weyl type

Definition. Let $\Delta(a)$ be the one-dimensional I-transform involving a general class of polynomials of $F(x)$ defined by

$$\Delta(a) = \Delta_{p,q,n_1}^{m,n,m_1}[F(x); \rho, a] = \int_h^\infty (ax)^{\rho-1} I_{p,q}^{m,n}[(ax)^\lambda] S_{n_1}^{m_1}[(ax)^\delta] F(x) dx, \tag{30}$$

provided that $\Delta(a)$ exists and belongs to S_1 where $\lambda > 0$, $|\arg a^\lambda| < \frac{1}{2}\Omega\pi$, $F(x) = F(d\sqrt{x^2 - h^2})H(x - h)$.

For $Re(d) > 0$, $h > 0$, $\lambda > 0$ and then the following formula

$$\bar{J}_{a,\infty}^{u,v,w}\left[\Delta_{p,q,n_1}^{m,n,m_1}\{F(x); \rho, a\}\right] = \int_h^\infty (ax)^{\rho-1} I_{p+2,q+2}^{m+2,n}$$

$$\left[(ax)^\lambda \middle| \begin{array}{l} (\alpha_1, A_1, a_1), \ldots, (\alpha_p, A_p, a_p), (1-\rho-k\delta, \lambda, 1), (u+v+w+1-\rho-k\delta, \lambda, 1) \\ (v-\rho+1-k\delta, \lambda, 1), (w-\rho+1-k\delta, \lambda, 1), (\beta_1, B_1, b_1), \ldots, (\beta_q, B_q, b_q) \end{array} \right]$$

$$S_{n_1}^{m_1}[(ax)^\delta] F(x) dx = \Delta_1(a), \tag{31}$$

holds, provided that $\Delta_1(a)$ exists and belongs class S_1.

$$\bar{J}_{a,\infty}^{u,v,w} f = \frac{a^v}{\Gamma(u)} \int_0^\infty (s-a)^{u-1} s^{-u-v}\, {}_2F_1\left(u+v, -w; u; 1 - \frac{a}{s}\right) f(s) ds. \tag{32}$$

8. Special cases

(i) We take $v = 0$ then $J_{a,\infty}^{u,0,w} f = K_{a,\infty}^{u,w} f$ are the one-dimensional Erdelyi-Kober operator of fraction integration defined by (15) then the (31) reduces to the following formula

$$K_{a,\infty}^{u,w}[\Delta(a)] = \int_h^\infty (ax)^{\rho-1} I_{p+1,q+1}^{m+1,n}$$
$$\left[(ax)^\lambda \left| \begin{matrix} (\alpha_1, A_1, a_1), \ldots, (\alpha_p, A_p, a_p), (u+v+w+1-\rho-k\delta, \lambda, 1) \\ (w - \rho + 1 - k\delta, \lambda, 1), (\beta_1, B_1, b_1), \ldots, (\beta_q, B_q, b_q) \end{matrix} \right. \right]$$
$$S_{n_1}^{m_1}[(ax)^\delta] F(x) dx, \tag{33}$$

(ii) By applying our results derived in (31) to the case of Hermite polynomials G. Szego [4] and H. M. Srivastava [2] by setting $S_n^2[x] \to x^{(n/2)} H_n(\frac{1}{2\sqrt{x}})$ in which case $m = 2$, $A_{n,k} = (-1)^k$,

$$\bar{J}_{a,\infty}^{u,v,w}\left[\Delta_{p,q,n_1}^{m,n,2}\{F(x); \rho, a\}\right] = \int_h^\infty (ax)^{\rho + \frac{\delta_{n_1}}{2} - 1} I_{p+2,q+2}^{m+2,n}$$
$$\left[(ax)^\lambda \left| \begin{matrix} (\alpha_1, A_1, a_1), \ldots, (\alpha_p, A_p, a_p), (1-\rho-k\delta, \lambda, 1), (u+v+w+1-\rho-k\delta, \lambda, 1) \\ (v-\rho+1-k\delta, \lambda, 1), (w - \rho + 1 - k\delta, \lambda, 1), (\beta_1, B_1, b_1), \ldots, (\beta_q, B_q, b_q) \end{matrix} \right. \right]$$
$$H_{n_1}\left(\frac{1}{2\sqrt{(ax)^\delta}}\right) F(x) dx, \tag{34}$$

(iii) For the Leaguerre polynomials (G. Szego [4] and H. M. Srivastava [2]) by setting $S_n^1[x] \to L_n^{(a')}[x]$ in which case $m = 1$, $A_{n,k} = \binom{n+\alpha'}{n} \frac{1}{(\alpha'+1)_k}$ and (31) reduces to the following result

$$\bar{J}_{a,\infty}^{u,v,w}\left[\Delta_{p,q,n_1}^{m,n,1}\{F(x); \rho, a\}\right] = \int_h^\infty (ax)^{\rho-1} I_{p+2,q+2}^{m+2,n}$$
$$\left[(ax)^\lambda \left| \begin{matrix} (\alpha_1, A_1, a_1), \ldots, (\alpha_p, A_p, a_p), (1-\rho-k\delta, \lambda, 1), (u+v+w+1-\rho-k\delta, \lambda, 1) \\ (v-\rho+1-k\delta, \lambda, 1), (w - \rho + 1 - k\delta, \lambda, 1), (\beta_1, B_1, b_1), \ldots, (\beta_q, B_q, b_q) \end{matrix} \right. \right]$$
$$L_{n_1}^{(\alpha')}[(ax)^\delta] F(x) dx. \tag{35}$$

(iv) For the Jacobi polynomials (G. Szego [4] and H. M. Srivastava [2]) by setting $S_n(x) = P_n^{(\alpha',\beta')}(1 - 2x)$ in which case $m = 1$ and $A_{n,k} = \binom{n+\alpha'}{n} \frac{(\alpha'+\beta'+n+1)_k}{(\alpha'+1)_k}$ in (31) reduces to following formulae

$$\bar{J}_{a,\infty}^{u,v,w}\left[\Delta_{p,q,n_1}^{m,n,1}\{F(x); \rho, a\}\right] = \int_h^\infty (ax)^{\rho-1} I_{p+2,q+2}^{m+2,n}$$
$$\left[(ax)^\lambda \left| \begin{matrix} (\alpha_1, A_1, a_1), \ldots, (\alpha_p, A_p, a_p), (1-\rho-k\delta, \lambda, 1), (u+v+w+1-\rho-k\delta, \lambda, 1) \\ (v-\rho+1-k\delta, \lambda, 1), (w - \rho + 1 - k\delta, \lambda, 1), (\beta_1, B_1, b_1), \ldots, (\beta_q, B_q, b_q) \end{matrix} \right. \right]$$
$$P_{n_1}^{(\alpha',\beta')}[1 - 2(ax)^\delta] F(x) dx. \tag{36}$$

9. Some Interesting Known Results

(i) If we take b_j $(j = 1, \ldots, m) = 1$ and a_j $(j = n + 1, \ldots, p) = 1$, (31) reduces to the result obtained by Chaurasia and Srivastava [12, pp.237-249].

(ii) Taking $a_j = b_j = 1$, $n_1 = n_2 = 0$ in (31) we arrive at the result obtained by Saigo, Saxena and Ram [10, pp.63-73].

(iii) On taking $a_j = b_j = 1$, $n_1 = n_2 = 0$ in (33), we get the result earlier proved by Saigo, Saxena and Ram [10, pp.63-73].

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