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CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS INVOLVING LINEAR OPERATORS

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Abstract. In terms of linear operators we introduce new classes of functions. Then by using differential subordinations, certain results concerning inclusion relations, coefficient bounds and other results are given.

1. Introduction

Let $\mathcal{A}_p$ denote the classes of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open disk $\mathbb{U} = \{z \in \mathbb{C}, |z| < 1\}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schawrz function w, analytic in $\mathbb{U}$ which $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))(z \in \mathbb{U})$. In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. For $0 \le \xi < p, \beta < 1$, we denote by $\chi^*(\xi)$, $\mathcal{K}(\xi), \Phi(\xi, \beta)$ the subclasses of $\mathcal{A}_p$ consisting of all analytic p-valent functions which are, respectively, starlike of order ξ, convex of order ξ, close-to-convex of order ξ, and type β in $\mathbb{U}$.

Let $\mathcal{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0) = 1$ and $\mathrm{Re}\{\phi(z)\} > 0$ for $z \in \mathbb{U}$.

From the principle of subordination between analytic functions, we introduce the subclasses $\chi^*(\xi, \phi)$, $\mathcal{K}(\xi, \phi)$, and $\phi(\xi, \beta, \phi, \psi)$ of the class $\mathcal{A}_p$ for $0 \le \xi < p$, $\beta < 1$, and

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$\phi, \psi \in \mathcal{N}$, which are defined by

$$\begin{aligned} \chi^*(\xi;\phi) &:= \left\{ f \in \mathcal{A}_p : \frac{1}{p-\xi}\left(\frac{zf'(z)}{f(z)} - \xi\right) \prec \phi(z) \text{ in } \mathbb{U} \right\}, \\ \mathcal{K}(\xi;\phi) &:= \left\{ f \in \mathcal{A}_p : \frac{1}{p-\xi}\left(1 + \frac{zf''(z)}{f'(z)} - \xi\right) \prec \phi(z) \text{ in } \mathbb{U} \right\}, \\ \Phi(\xi,\beta;\phi,\psi) &:= \left\{ f \in \mathcal{A}_p : \exists g \in \chi^*(\xi,\phi) \text{ s.t. } \frac{1}{p-\beta}\left(\frac{zf'(z)}{g(z)} - \beta\right) \prec \psi(z) \text{ in } \mathbb{U} \right\}. \end{aligned} \tag{1.2}$$

We note that the classes mentioned above are the familiar classes which have been used widely on the space of analytic and p-valent functions in $\mathbb{U}$, and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well known subclasses of $\mathcal{A}_p$. For example, we have

$$\chi^*\left(\xi;\frac{1+z}{1-z}\right) = \chi^*(\xi), \mathcal{K}\left(\xi;\frac{1+z}{1-z}\right) = \mathcal{K}(\xi), \Phi\left(\xi,\beta;\frac{1+z}{1-z},\frac{1+z}{1-z}\right) = \Phi(\xi,\beta). \tag{1.3}$$

Also let the Hadamard product (or convolution) $f * g$ of two analytic functions

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \tag{1.4}$$

be given by

$$(f*g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k. \tag{1.5}$$

For $\alpha_j \in \mathbb{C}(j = 1,2,\ldots,l)$ and $\beta_j \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}(j = 1,2,\ldots,m)$, the generalized hypergeometric function is defined by the infinite series

$${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k} \frac{z^k}{k!} \ \ (l \leq m+1; l,m \in \mathbb{N}_0 = \{0,1,2,\ldots\}),$$

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(k)} \begin{cases} 1 & (k = 0) \\ a(a+1)\cdots(a+k-1) & (k \in \mathbb{N} := \{1,2,\ldots\}. \end{cases}$$

In particular, the incomplete beta function, related to the Gauss hypergeometric function, $\phi(a,c;z)$ is defined by

$$\phi(a,c;z) = zF(a,1;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1}, z \in \mathbb{U}, c \neq 0,-1,-2,\ldots.$$

Note that $\phi(a,1;z) = \dfrac{z}{(1-z)^a}$. Moreover, $\phi(2,1;z) = \dfrac{z}{(1-z)^2}$ is the Koebe function.

Corresponding to the function

$$h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z) := z^p\ {}_lF_m(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z). \tag{1.6}$$

The Dziok-Srivastava operator [7] (see also [6] and [22]) $H_p^{l,m}(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_p^{l,m}(\alpha_1,\dots,\alpha_l;&\beta_1,\dots,\beta_m)f(z)\\ &= h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)*f(z)\\ &= z^p+\sum_{k=p+1}^{\infty}\frac{(\alpha_1)_{k-p}\cdots(\alpha_l)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_m)_{k-p}}\frac{a_m z^k}{(k-p)!}. \end{aligned}$$

Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [9]. The Carlson-Shaffer linear operator [3], the Ruscheweh derivative operator [19], the generalized Bernardi-Libera-Livingston operator (cf. [2], [11], [12]) and the Srivastava-Owa fractional derivative operators (cf. [16], [18]).

Corresponding to the function $h_p(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m;z)$ defined by (1.6), we introduce a function $\mathcal{F}_\mu(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m;z)$ given by

$$\begin{aligned} h_p(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)&*\mathcal{F}_\mu(\alpha_1,\dots,\alpha_l;\beta_1,\dots,\beta_m;z)\\ &=\frac{z^p}{(1-z)^{\mu+p-1}},(\quad z\in\mathbb{U},\mu>0,l=m+1). \end{aligned} \tag{1.7}$$

Analogous to $H_p(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m)$, we now define the linear operator $J_\mu(\alpha_1,\dots;\alpha_l;\beta_1,\dots,\beta_m)$ on $\mathcal{A}_p$ as follows:

$$\begin{aligned} &J_\mu(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m)f(z)=\mathcal{F}_\mu(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m;z)*f(z)\\ &(\alpha_1,\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-;i=1,\dots,l;j=1,\dots,m;\mu>0;z\in\mathbb{U};f\in\mathcal{A}_p). \end{aligned} \tag{1.8}$$

For convenience, we write

$$J_\mu^{l,m}(\alpha_1) := J_\mu(\alpha_1,\dots,\alpha_l,\beta_1,\dots,\beta_m).$$

If f is given by (1.1), then by (1.8), we see that

$$J_\mu^{l,m}(\alpha_1)f(z)=z^p+\sum_{k=p+1}^{\infty}\Psi_k(\alpha_1,\mu)a_kz^k, \tag{1.9}$$

where

$$\Psi_k(\alpha_1,\mu)=(\mu+p-1)_{k-p}\frac{(\beta_1)_{k-p}\cdots(\beta_m)_{k-p}}{(\alpha_1)_{k-p}\cdots(\alpha_l)_{k-p}}.$$

Special cases of this operator are [10] when $p=1$, the generalized integral operator in [1] when $p=1$ and $\mu=2$ and Noor integral operator [14]. We can verify from the definition (1.8) that

$$z(J_\mu^{l,m}(\alpha_1+1)f(z))'=\alpha_1J_\mu^{l,m}(\alpha_1)f(z)-(\alpha_1-p)J_\mu^{l,m}(\alpha_1+1)f(z) \tag{1.10}$$

$$z(J_\mu^{l,m}(\alpha_1)f(z))'=(\mu+p-1)J_{\mu+1}^{l,m}(\alpha_1)f(z)-(\mu-1)J_\mu^{l,m}(\alpha_1)f(z) \tag{1.11}$$

By using the operator $J_{\mu}^{l,m}(\alpha_1)$, we introduce the following classes of analytic functions for $\phi, \psi \in \mathcal{N}$ and $0 \leq \xi < p, \beta < 1$:

$$\begin{aligned} \chi_{\alpha_1,\mu}(l,m;\xi;\phi) &:= \left\{ f \in \mathcal{A}_p : J_{\mu}^{l,m}(\alpha_1) f(z) \in \chi^*(\xi;\phi) \right\} \\ \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi) &:= \left\{ f \in \mathcal{A}_p : J_{\mu}^{l,m}(\alpha_1) f(z) \in \mathcal{K}(\xi;\phi) \right\} \\ \Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi) &:= \left\{ f \in \mathcal{A}_p : J_{\mu}^{l,m}(\alpha_1) f(z) \in \Phi(\xi,\beta;\phi,\psi) \right\}. \end{aligned} \tag{1.12}$$

Also

$$f(z) \in \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi) \Leftrightarrow \frac{zf'(z)}{p} \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi). \tag{1.13}$$

The multiplier transformations defined on $\mathcal{A}_p$ by the following infinite series

$$I_p(r,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^r a_k z^k \qquad (\lambda \geq 0). \tag{1.14}$$

The operator $I_p(r,\lambda)$ is closely related to Şălăgean, derivative operators [21]. The operator $I_\lambda := I_1(r,\lambda)$ was studied recently by Cho and Srivastava [4] and Kim [5].

The operator $I_r := I_1(r,1)$ was studied by Uralgaddi and Somanatha [23]. By using Hadamard product

$$I_p(r,\lambda)f(z) := \mathcal{F}_{\lambda}^{r}(z) * f(z) \quad \text{where} \quad \mathcal{F}_{\lambda}^{r}(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^r z^k \quad (\lambda \geq 0). \tag{1.15}$$

Corresponding to the function $\mathcal{F}_{\lambda}^{r}(z)$ defined by (1.15), we introduce a function $\mathcal{F}_{\lambda,\mu}^{r}(z)$ given by

$$\mathcal{F}_{\lambda}^{r}(z) * \mathcal{F}_{\lambda,\mu}^{r}(z) = \frac{z^p}{(1-z)^{\mu+p-1}}, \qquad z \in \mathbb{U}, \mu > 0. \tag{1.16}$$

Using $I_p(r,\lambda)$, we define the multiplier transformations $T_\mu(r,\lambda)$ as follows:

$$T_\mu(r,\lambda)f(z) = \mathcal{F}_{\lambda,\mu}^{r}(z) * f(z) \qquad (\lambda \geq 0, \mu > 0, z \in \mathbb{U}, f \in \mathcal{A}_p). \tag{1.17}$$

If f is given by (1.1), then by (1.17), we see that

$$T_\mu(r,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \Psi_k(r,\lambda) a_k z^k, \tag{1.18}$$

where

$$\Psi_k(r,\lambda) = \frac{(\mu+p-1)_{k-p}}{(k-p)!} \left(\frac{p+\lambda}{k+\lambda} \right)^r.$$

For $p = 1$ we note that a special case of this operator is the integral operator defined in [15].

We can verify from the definition (1.17) that

$$z(T_\mu(r+1,\lambda)f(z))' = (p+\lambda)T_\mu(r,\lambda)f(z) - \lambda T_\mu(r+1,\lambda)f(z) \tag{1.19}$$
$$z(T_\mu(r,\lambda)f(z))' = (1-\mu)T_\mu(r,\lambda)f(z) + (\mu+p-1)T_{\mu+1}(r,\lambda)f(z). \tag{1.20}$$

By using the operator $T_\mu(r,\lambda)$ we introduce the following classes of analytic functions for $\phi,\psi \in \mathcal{N}$, and $0 \le \xi < p, \beta < 1$:

$$\begin{aligned} \chi^*_\mu(r,\lambda;\xi;\phi) &:= \{f \in \mathcal{A}_p : T_\mu(r,\lambda)f(z) \in \chi^*(\xi;\phi)\} \\ \mathcal{K}^*_\mu(r,\lambda;\xi;\phi) &:= \{f \in \mathcal{A}_p : T_\mu(r,\lambda)f(z) \in \mathcal{K}(\xi;\phi)\} \\ \Phi^*_\mu(r,\lambda;\xi;\beta;\phi,\psi) &:= \{f \in \mathcal{A}_p : T_\mu(r,\lambda)f(z) \in \Phi(\xi,\beta;\phi,\psi)\}. \end{aligned} \tag{1.21}$$

Also

$$f(z) \in \mathcal{K}^*_\mu(r,\lambda;\xi;\phi) \Leftrightarrow \frac{zf'(z)}{p} \in \chi^*_\mu(r,\lambda;\xi,\phi). \tag{1.22}$$

In this paper, basic properties of the classes $\chi_\mu(l,m;\xi,\phi), \mathcal{K}_\mu(l,m;\xi,\phi)$, $\Phi_\mu(l,m;\xi,\beta;\phi,\psi), \chi^*_\mu(r,\lambda,\xi,\phi), \mathcal{K}^*_\mu(r,\lambda,\xi,\phi)$ and $\Phi^*_\mu(r,\lambda,\xi,\beta,\phi,\psi)$ are studied, such as, inclusion relations and coefficient bounds. Various known or new special cases of our results are also pointed out.

2. Inclusion properties involving the operator $J^{l,m}_\mu$ and T_μ

The following results will be required in our investigation.

Lemma 2.1. ([13]). *Let ϕ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and* $\mathrm{Re}\,[k\phi(z)+\gamma] > 0$ $(k,\gamma \in \mathbb{C})$. *If p is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{kp(z)+\gamma} \prec \phi(z) \qquad (z \in \mathbb{U}) \tag{2.1}$$

implies

$$p(z) \prec \phi(z), \qquad (z \in \mathbb{U}). \tag{2.2}$$

Lemma 2.2. ([13]). *Let ϕ be convex univalent in $\mathbb{U}$ and let w be analytic in $\mathbb{U}$ with $Re\{w(z)\} \ge 0$. If p is analytic in $\mathbb{U}$ and $p(0) = \phi(0)$, then*

$$p(z) + w(z)zp'(z) \prec \phi(z) \qquad (z \in \mathbb{U}) \tag{2.3}$$

implies

$$p(z) \prec \phi(z) \qquad (z \in \mathbb{U}). \tag{2.4}$$

Theorem 2.3. *Let $\mu > 1$ and $\phi \in \mathcal{N}$. Then*

(i) $\chi_{\alpha_1,\mu+1}(l,m;\xi;\phi) \subset \chi_{\alpha_1,\mu}(l,m;\xi;\phi) \subset \chi_{\alpha_1+1,\mu}(l,m;\xi;\phi) \quad (\alpha_1 > p)$ (2.5)

(ii) $\chi^*_{\mu+1}(r,\lambda;\xi;\phi) \subset \chi^*_\mu(r,\lambda;\xi;\phi) \subset \chi^*_\mu(r+1,\lambda;\xi;\phi).$ (2.6)

Proof. (i) First of all, we will show that

$$\chi_{\alpha_1,\mu+1}(l,m;\xi;\phi) \subset \chi_{\alpha_1,\mu}(l,m;\xi;\phi). \tag{2.7}$$

Let $f \in \chi_{\alpha_1,\mu+1}(l,m;\xi;\phi)$ and set

$$p(z) = \frac{1}{p-\xi}\left(\frac{z(J_{\mu}^{l,m}(\alpha_1)f(z))'}{J_{\mu}^{l,m}(\alpha_1)f(z)} - \xi\right), \tag{2.8}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (1.11) and (2.8),

$$\frac{1}{p-\xi}\left(\frac{z(J_{\mu+1}^{l,m}(\alpha_1)f(z))'}{J_{\mu+1}^{l,m}(\alpha_1)f(z)} - \xi\right) = p(z) + \frac{zp'(z)}{(p-\xi)p(z)+\mu-1+\xi}, \quad (z \in \mathbb{U}). \tag{2.9}$$

Since $\mu > 1$ and $\phi \in \mathcal{N}$, we see that

$$\mathrm{Re}\{(p-\xi)\phi(z) + \mu - 1 + \xi\} > 0 \qquad \{z \in \mathbb{U}\}. \tag{2.10}$$

Using Lemma 2.1 to (2.9), we find that $p \prec \phi$, which means $f \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$.

To prove the second part, let $f \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$ and put

$$s(z) = \frac{1}{p-\xi}\left(\frac{z(J_{\mu}^{l,m}(\alpha_1+1)f(z))'}{J_{\mu}^{l,m}(\alpha_1+1)f(z)} - \xi\right), \tag{2.11}$$

where s is analytic function with $s(0) = 1$. Then, by using the arguments above with (1.10), it follows that $s \prec \phi$ in $\mathbb{U}$, which implies that $f \in \chi_{\alpha_1+1,\mu}(l,m;\xi,\phi)$. Therefore, we complete the proof.

(ii) The proof is the same as (i).

Theorem 2.4. *Let $\mu > 1$ and $\phi \in \mathcal{N}$. Then,*

(i) $\mathcal{K}_{\alpha_1,\mu+1}(l,m;\xi;\phi) \subset \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi) \subset \mathcal{K}_{\alpha_1+1,\mu}(l,m;\xi;\phi), \quad \alpha_1 > p$ (2.12)

(ii) $\mathcal{K}^*_{\mu+1}(r,\lambda;\xi;\phi) \subset \mathcal{K}^*_{\mu}(r,\lambda;\xi;\phi) \subset \mathcal{K}^*_{\mu}(r+1,\lambda;\xi;\phi).$ (2.13)

Proof. Applying (1.13) and Theorem 2.3, we find that

$$\begin{aligned} f(z)\in\mathcal{K}_{\alpha_1,\mu+1}(l,m;\xi,\phi) &\Leftrightarrow J^{l,m}_{\mu+1}(\alpha_1)f(z)\in\mathcal{K}(\xi;\phi)\\ &\Leftrightarrow J^{l,m}_{\mu+1}(\alpha_1)\frac{(zf'(z))}{p}\in\chi^*(\xi;\phi)\\ &\Leftrightarrow \frac{zf'(z)}{p}\in\chi_{\alpha_1,\mu+1}(l,m;\xi;\phi)\\ &\Rightarrow \frac{zf'(z)}{p}\in\chi_{\alpha_1,\mu}(l,m;\xi;\phi)\\ &\Leftrightarrow \frac{z(J^{l,m}_{\mu}(\alpha_1)f(z))'}{p}\in\chi^*(\xi;\phi) \qquad (2.14)\\ &\Leftrightarrow J^{l,m}_{\mu}(\alpha_1)f(z)\in\mathcal{K}(\xi;\phi)\\ &\Leftrightarrow f(z)\in\mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi)\\ f(z)\in\mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi) &\Leftrightarrow \frac{zf'(z)}{p}\in\chi_{\alpha_1,\mu}(l,m;\xi;\phi)\\ &\Rightarrow \frac{zf'(z)}{p}\in\chi_{\alpha_1+1,\mu}(l,m;\xi;\phi)\\ &\Leftrightarrow f(z)\in\mathcal{K}_{\alpha_1+1,\mu}(l,m;\xi;\phi), \end{aligned}$$

which completes the proof.

Now, by using Lemma 2.2, we obtain the following inclusion relation for the class $\Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi)$ and $\Phi^*_{\mu}(l,m;\xi,\beta;\phi,\psi)$.

Theorem 2.5. *Let $\mu>1$ and $\phi,\psi\in\mathcal{N}$. Then*

(i) $\Phi_{\alpha_1,\mu+1}(l,m;\xi,\beta;\phi,\psi)\subset\Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi)\subset\Phi_{\alpha_1+1,\mu}(l,m;\xi,\beta;\phi,\psi),\ (\alpha_1>p)$ (2.15)

(ii) $\Phi^*_{\mu+1}(r,\lambda;\xi,\beta;\phi,\psi)\subset\Phi^*_{\mu}(r,\lambda;\xi,\beta;\phi,\psi)\subset\Phi^*_{\mu}(r+1,\lambda;\xi,\beta;\phi,\psi).$ (2.16)

Proof. First, we will prove that

$$\Phi_{\alpha_1,\mu+1}(l,m;\xi,\beta;\phi,\psi)\subset\Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi). \qquad (2.17)$$

Let $f(z)\in\Phi_{\alpha_1,\mu+1}(l,m;\xi,\beta;\phi,\psi)$. Then, from the definition of $\Phi_{\alpha_1,\mu+1}(l,m;\xi,\beta;\phi,\psi)$, there exists a function $r(z)\in\chi^*(\xi,\phi)$ such that

$$\frac{1}{p-\beta}\left(\frac{z(J^{l,m}_{\mu+1}(\alpha_1)f(z))'}{r(z)}-\beta\right)\prec\psi(z)\qquad(z\in\mathbb{U}). \qquad (2.18)$$

Choose the function $g(z)$ such that $J^{l,m}_{\mu+1}(\alpha_1)g(z)=r(z)$. Then, $g(z)\in\chi_{\alpha_1,\mu+1}(l,m;\xi;\phi)$ and

$$\frac{1}{p-\beta}\left(\frac{z(J^{l,m}_{\mu+1}(\alpha_1)f(z))'}{J_{\mu+1}(\alpha_1)g(z)}-\beta\right)\prec\psi(z)\qquad(z\in\mathbb{U}). \qquad (2.19)$$

Now let

$$p(z) = \frac{1}{p-\beta}\left(\frac{z(J_{\mu}^{l,m}(\alpha_1)f(z))'}{J_{\mu}^{l,m}(\alpha_1)g(z)} - \beta\right), \tag{2.20}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (1.11), we have

$$\begin{aligned}(p-\beta)zp'(z)J_{\mu}^{l,m}(\alpha_1)g(z) + ((p-\beta)p(z)+\beta)z(J_{\mu}^{l,m}(\alpha_1)g(z))' \\ = (\mu+p-1)z(J_{\mu+1}^{l,m}(\alpha_1)f(z))' - (\mu-1)z(J_{\mu}^{l,m}(\alpha_1)f(z))'.\end{aligned} \tag{2.21}$$

Since $g(z) \in \chi_{\alpha_1,\mu+1}(l,m;\xi;\phi)$, by Theorem 2.3, we know that $g(z) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$. Let

$$q(z) = \frac{1}{p-\xi}\left(\frac{z(J_{\mu}^{l,m}(\alpha_1)g(z))'}{J_{\mu}^{l,m}(\alpha_1)g(z)} - \xi\right). \tag{2.22}$$

Then, using (1.11), once again, we have

$$(\mu+p-1)\frac{J_{\mu+1}^{l,m}(\alpha_1)g(z)}{J_{\mu}(\alpha_1)g(z)} = (p-\xi)q(z) + \mu - 1 + \xi. \tag{2.23}$$

From (2.21), (2.23), we obtain

$$\frac{1}{p-\beta}\left(\frac{z(J_{\mu+1}^{l,m}(\alpha_1)f(z))'}{J_{\mu+1}^{l,m}(\alpha_1)g(z)} - \beta\right) = p(z) + \frac{zp'z)}{(p-\xi)q(z)+\mu-1+\xi}. \tag{2.24}$$

Since $\mu > 1$ and $q \prec \phi$ in $\mathbb{U}$,

$$\mathrm{Re}\{(p-\xi)q(z)+\mu-1+\xi\} > 0 \qquad (z \in \mathbb{U}). \tag{2.25}$$

Therefore, from Lemma 2.2, we can show that $p \prec \psi$, so that $f \in \Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi)$.

For the second part, by using the same arguments above with (1.10), we obtain

$$\Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi) \subset \Phi_{\alpha_1+1,\mu}(l,m;\xi,\beta;\phi,\psi). \tag{2.26}$$

Therefore, we complete the proof.

3. Inclusion Properties Involving the Integral Operator F_a

In this section, we consider the generalized Libera integral operator F_a [17] defined by

$$F_a(f) := F_a(f)(z) = \frac{a+1}{z^a}\int_0^z t^{a-1}f(t)dt \qquad (f \in \mathcal{A}_p, a > -1). \tag{3.1}$$

First we will prove the following.

Theorem 3.1.
(i) *If* $f \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$, *then* $F_a(f) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$ $(a \geq 0)$.
(ii) *If* $f \in \chi_{\mu}^{*}(r,\lambda;\xi;\phi)$, *then* $F_a(f) \in \chi_{\mu}^{*}(r,\lambda;\xi;\phi)$ $(a \geq 0)$.

Proof. (i) Let $f \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$ and set

$$p(z) = \frac{1}{p-\xi}\left(\frac{z(J_\mu^{l,m}(\alpha_1)F_a(f)(z))'}{J_\mu^{l,m}(\alpha_1)F_a(f)(z)} - \xi\right) \tag{3.2}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. From (3.1), we have

$$z(J_\mu^{l,m}(\alpha_1)F_a(f)(z))' = (a+1)J_\mu^{l,m}(\alpha_1)f(z) - aJ_\mu^{l,m}(\alpha_1)F_a(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$(a+1)\frac{J_\mu^{l,m}(\alpha_1)f(z)}{J_\mu^{l,m}(\alpha_1)F_a(f)(z)} = (p-\xi)p(z) + a + \xi. \tag{3.4}$$

Taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{(p-\xi)p(z)+a+\xi} = \frac{1}{p-\xi}\left(\frac{z(J_\mu^{l,m}(\alpha_1)f(z))'}{J_\mu^{l,m}(\alpha_1)f(z)} - \xi\right) \quad (z \in \mathbb{U}). \tag{3.5}$$

Hence, by virtue of Lemma 2.1, we conclude that $p \prec \phi$ in $\mathbb{U}$, which implies that $F_a(f) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$.

(ii) The proof is the same as (i).

Next, we derive an inclusion property involving F_a, which is given by the following.

Theorem 3.2.
(i) *If* $f \in \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi)$, *then* $F_a(f) \in \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi) \quad (a \geq 0)$.
(ii) *If* $f \in \mathcal{K}_\mu^*(r,\lambda;\xi;\phi)$, *then* $F_a(f) \in \mathcal{K}_\mu^*(r,\lambda;\xi;\phi) \quad (a \geq 0)$.

Proof. By applying Theorem 3.1, it follows that

$$\begin{aligned} f(z) \in \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi) &\Leftrightarrow \frac{zf'(z)}{p} \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi) \\ &\Rightarrow F_a(\frac{zf'(z)}{p}) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi) \\ &\Leftrightarrow z\frac{(F_a(f)(z))'}{p} \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi) \\ &\Leftrightarrow F_a(f)(z) \in \mathcal{K}_{\alpha_1,\mu}(l,m;\xi;\phi). \end{aligned} \tag{3.6}$$

From Theorems 3.1 and 3.2, we have the following.

Theorem 3.3.
(i) *If* $f \in \Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi)$, *then* $F_a(f) \in \Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi), \quad (a \geq 0)$.
(ii) *If* $f \in \Phi_\mu^*(r,\lambda;\xi,\beta;\phi,\psi)$, *then* $F_a(f) \in \Phi_\mu^*(r,\lambda;\xi,\beta;\phi,\psi), \quad (a \geq 0)$.

Proof. Let $f \in \Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi)$. Then, from the definition of the class $\Phi_{\alpha_1,\mu}(l,m;\xi,\beta;\phi,\psi)$, there exists a function $g(z) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$ such that

$$\frac{1}{p-\beta}\left(\frac{z(J_\mu^{l,m}(\alpha_1)f(z))'}{J_\mu^{l,m}(\alpha_1)g(z)} - \beta\right) \prec \psi(z) \qquad (z \in \mathbb{U}). \tag{3.7}$$

Thus, we set

$$p(z) = \frac{1}{p-\beta}\left(\frac{z(J_\mu^{l,m}(\alpha_1)F_a(f)(z))'}{J_\mu^{l,m}(\alpha_1)F_a(g)(z)} - \beta\right), \tag{3.8}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Since $g(z) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$, we see from Theorem 3.1 that $F_a(g) \in \chi_{\alpha_1,\mu}(l,m;\xi;\phi)$. Using (3.3), we have

$$((p-\beta)p(z)+\beta)J_\mu^{l,m}(\alpha_1)F_a(g)(z) + aJ_\mu^{l,m}(\alpha_1)F_a(f)(z) = (a+1)J_\mu^{l,m}(\alpha_1)f(z). \tag{3.9}$$

By some calculation, we get

$$(a+1)\frac{z(J_\alpha^{l,m}(\alpha_1)f(z))'}{J_\mu^{l,m}(\alpha_1)F_a(g)(z)} = ((p-\beta)p(z)+\beta)((p-\xi)q(z)+a+\xi) + (p-\beta)zp'(z), \tag{3.10}$$

where

$$q(z) = \frac{1}{p-\xi}\left(\frac{z(J_\mu^{l,m}(\alpha_1)F_a(g)(z))'}{J_\mu^{l,m}(\alpha_1)F_a(g)(z)} - \xi\right). \tag{3.11}$$

Hence, we have

$$\frac{1}{p-\beta}\left(\frac{z(J_\mu^{l,m}(\alpha_1)f(z))'}{J_\mu^{l,m}(\alpha_1)g(z)} - \beta\right) = p(z) + \frac{zp'(z)}{(p-\xi)q(z)+a+\xi}. \tag{3.12}$$

The remaining part of the proof in Theorem 3.3 is similar to that of Theorem 2.5 and so we leave it.

4. Coefficient bounds

Now we will give bounds for the coefficients of series expansion of functions belonging to the classes $\chi_{\alpha_1,\mu}(\xi), \mathcal{K}_{\alpha_1,\mu}(\xi), \chi_\mu^*(\xi)$ and $\mathcal{K}_\mu^*(\xi)$.

Taking into account the fundamental relation

$$\phi(z) = \frac{1}{p-\xi}\left(\frac{z(J_\mu^{l,m}(\alpha_1)f_\xi(z))'}{J_\mu^{l,m}(\alpha_1)f_\xi(z)} - \xi\right), \tag{4.1}$$

between the extremal functions in the class $P(\phi(z))$ and the extremal functions f_ξ of the class $\chi_{\alpha_1,\mu}(\xi)$ and in view of (1.9) and (4.1), we have for $\phi(z) = 1 + \sum_{k=1}^{\infty} B_k z^k$,

$$f_\xi = z^p + \sum_{k-p+1}^{\infty} A_k z^k \tag{4.2}$$

a coefficient relation

$$(k-p)A_k\Psi_k(\alpha_1,\mu) = (p-\xi)\sum_{j=p}^{k-1} B_{k-j}A_j\Psi_j(\alpha_1,\mu), \quad A_p = 1.$$

In particular, by a straightforward computation, we obtain

$$A_{p+1} = \frac{(p-\xi)B_1}{\Psi_{p+1}(\alpha_1,\mu)}. \tag{4.3}$$

Observe that the coefficients A_k are nonnegative since $\Psi_k(\alpha_1,\mu) \geq 0$.

We give sharp bound on the second coefficient for functions of the classes, $\chi_{\alpha_1,\mu}(\xi)$ and $\chi^*_\mu(\xi)$.

Theorem 4.1. *i) If a function of the form (1.1) is in $\chi_{\alpha_1,\mu}(\xi)$, then*

$$|a_k| \leq \frac{(p-\xi)}{\Psi_k(\alpha_1,\mu)}\frac{(|B_1|)_{k-p}}{(1)_{k-p}}, \quad k \geq p+1. \tag{4.4}$$

(ii) If a function of the form (1.1) is in $\chi^*_{\mu_1}(\xi)$, then

$$|a_k| \leq \frac{(p-\xi)}{\Psi_k(r,\mu)}\frac{(|B_1|)_{k-p}}{(1)_{k-p}}, \qquad k \geq p+1.$$

For the proof of this theorem, we need the following result by Rogosinski [20].

Rogosinski's Theorem. Let $h(z) = 1+\sum_{k=1}^{\infty} c_k z^k$ be subordinate to $H(z) = 1+\sum_{k=1}^{\infty} C_k z^k$ in $\mathbb{U}$. If $H(z)$ is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is convex, then $|c_k| \leq |C_1|, \quad k \geq 1$.

Proof of Theorem 4.1. (i) Let $f \in \chi_{\alpha_1,\mu}(\xi)$, $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$, we obtain

$$\frac{1}{p-\xi}\left(\frac{z(J^{l,m}_\mu(\alpha_1)f(z))'}{J^{l,m}_\mu(\alpha_1)f(z)} - \xi\right) \prec \phi(z).$$

Define $q(z) = \frac{1}{p-\xi}\left(\frac{z(J^{l,m}_\mu(\alpha_1)f(z))'}{J^{l,m}_\mu(\alpha_1)f(z)} - \xi\right) = 1 + \sum_{k=1}^{\infty} c_k z^k$. The function ϕ is univalent in $\mathbb{U}$ and $\phi(\mathbb{U})$, the conic domain, is convex domain, so Rogosinski's theorem applies. Then we have

$$|c_k| \leq 2, \qquad k \geq 1. \tag{4.5}$$

Now writing $((p-\xi)q(z)+\xi)J^{l,m}_\mu(\alpha_1)f(z) = z(J^{l,m}_\mu(\alpha_1)f(z))'$ and comparing coefficients of z^k on both sides, we get

$$(k-p)a_k\Psi_k(\alpha_1,\mu) = (p-\xi)\sum_{j=p}^{k-1} c_{k-j}a_j\Psi_j(\alpha_1,\mu), \qquad a_p = 1. \tag{4.6}$$

From (4.6), we get $|a_{p+1}| = \dfrac{(p-\xi)}{\Psi_{p+1}(\alpha_1,\mu)}|c_1| \leq \dfrac{(p-\xi)|B_1|}{\Psi_{p+1}(\alpha_1,\mu)}$. So the result is true for $k = p+1$. Let $k \geq p+1$ and assume that the inequality (4.4) is true for all $j \leq k-1$. By using (4.5), (4.6) and applying the induction hypothesis to $|a_j|$, we get

$$|a_k| \leq \frac{(p-\xi)}{(k-p)\Psi_k(\alpha_1,\mu)}\left[|c_1| + \sum_{j=p+1}^{k-1} |c_{k-j}|\,|a_j|\Psi_j(\alpha_1,\mu)\right]$$
$$\leq \frac{(p-\xi)|B_1|}{(k-p)\Psi_k(\alpha_1,\mu)}\left[1 + \sum_{j=p+1}^{k-1} \frac{(p-\xi)(|B_1|)_{j-p}}{(1)_{j-p}}\right].$$

Putting $p-1 \leq \xi < p$, we obtain

$$|a_k| \leq \frac{(p-\xi)|B_1|}{(k-p)\Psi_k(\alpha_1,\mu)}\left[1 + \sum_{j=p+1}^{k-1} \frac{(|B_1|)_{j-p}}{(1)_{j-p}}\right].$$

By applying mathematical induction another time, we find that

$$1 + \sum_{j=p+1}^{k-1} \frac{|B_1|)_{j-p}}{(1)_{j-p}} = \frac{(|B_1|+1)(|B_1|+2)\cdots(|B_1|+k-p-1)}{(k-p-1)!}.$$

Thus we get the inequality (4.4).

(ii) The proof is the same as (i).

Applying the relation (1.13) and (1.22), we observe that the extremal function $\mathcal{K}_{\alpha_1,\mu}(\xi)$ and $\mathcal{K}^*_\mu(\xi)$ denoted by $F_\xi(z)$, is given by

$$F_\xi(z) = p\int_0^z \frac{f_\xi(\gamma)}{\gamma}d\gamma,$$

where $f_\xi(z)$ is given by (4.2).

By (4.3) and for

$$F_\xi(z) = z + \sum_{k=p+1}^{\infty} C_k z^k,$$

we get

$$C_{p+1} = \frac{p(p-\xi)B_1}{(p+1)\Psi_{p+1}(\alpha_1,\mu)}.$$

Applying relation (1.13), we can prove the next result.

Corollary 4.2.

(i) *If a function f of the form* (1.1) *is in $\mathcal{K}_{\alpha_1,\mu}(\xi)$, then*

$$|a_k| \leq \frac{p(p-\xi)(|B_1|)_{k-p}}{k\Psi_k(\alpha_1,\mu)(1)_{k-p}}.$$

(ii) *If a function f of the form* (1.1) *is in* $\mathcal{K}^*_\mu(\xi)$, *then*

$$|a_k| \leq \frac{p(p-\xi)(|B_1|)_{k-p}}{k\Psi_k(r,\mu)(1)_{k-p}}.$$

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