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FRACTIONAL CALCULUS OF THE $\bar{H}$-FUNCTION

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Abstract. The subject of this paper is to derive a fractional calculus formula for $\bar{H}$-function due to Inayat -Hussain whose based upon generalized fractional integration and differentiation operators of arbitrary complex order involving Appell function F_3 due to Saigo & Meada. The results are obtained in a compact form containing the Reimann-Liouville, Eredlyi-Kober and Saigo operators of fractional calculus.

1. Introduction

***$\bar{H}$*-function**. The $\bar{H}$-function introduced by Inayat-Hussain [1] in terms of Mellin-Barnes type contour integral, is defined by

$$\bar{H}^{m,n}_{p,q}[z] = \bar{H}[z] = \bar{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_i, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1}, q \end{matrix}\right]$$
$$= \frac{1}{2\pi i}\int_L \chi(s) z^s \, ds, \tag{1.1}$$

where
$$\chi(s) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j - B_j s)\prod\limits_{j=1}^{n}\{\Gamma(1-a_j+A_j s)\}^{\alpha_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-b_j+B_j s)\}^{\beta_j}\prod\limits_{j=n+1}^{p}\Gamma(a_j - A_j s)}, \tag{1.2}$$

which contains fractional powers of some of the gamma functions $L = L_{i\tau\infty}$ is a contour starting at the point $\tau - i\infty$, terminating at the point $\tau + i\infty$ with $\tau \in R = (-\infty, \infty)$. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; m, n, p, q are integers such that $1 \le m \le q$, $0 \le n \le p$; $A_j > 0 (j = 1, \ldots, p)$, $B_j > 0$ $(j = 1, \ldots, q)$ and $a_j (j = 1, \ldots, p)$ and $b_j (j = 1, \ldots, q)$ are complex numbers. The exponents α_j $(j = 1, \ldots, n)$ and β_j $(j = m+1, \ldots, q)$ take on non-integer values.

Also, from Inayat-Hussain [1], it follows that

$$\bar{H}^{m,n}_{p,q}[z] = o\left(|z|^{\zeta^*}\right) \text{ for small } z, \quad \text{where } \zeta^* = \min_{1\le j\le m}\left[Re\left(\frac{b_j}{B_j}\right)\right] \tag{1.3}$$

and

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$$\bar{H}_{p,q}^{m,n}[z] = o(|z|^{\xi^*}) \text{ for large } z, \quad \text{where } \xi^* = \max_{1\le j\le n} Re\left[\alpha_j\left(\frac{a_j-1}{A_j}\right)\right]. \tag{1.4}$$

When the exponents $\alpha_i = \beta_j = 1 \; \forall \; i$ and j, the $\bar{H}$-function reduces to the familiar Fox's H-function defined by Fox [3], and see also Mathai and Saxena [2].

Buschman and Srivastava [8, p.4708] have shown that the sufficient condition for absolute convergence of the contour integral (1.1) is given by

$$\Omega = \sum_{j=1}^{m}|B_j| + \sum_{j=1}^{n}|\alpha_j A_j| - \sum_{j=m+1}^{q}|\beta_j B_j| - \sum_{j=n+1}^{p}|A_j| > 0 \tag{1.5}$$

This condition evidently provides exponential decay of the integrand in (1.1) and the region of (absolute) convergence in (1.1) is

$$|argz| < \frac{1}{2}\pi\Omega. \tag{1.6}$$

Saxena [5, p.127] has shown that $\bar{H}(z)$ makes sense and defines an analytic function of z in the following two cases:

I. $\Psi > 0$ and $0 < |z| < \infty$,

where $$\Psi = \sum_{j=1}^{m}|B_j| + \sum_{j=m+1}^{q}|B_j\ b_j| - \sum_{j=1}^{n}|A_j a_j| - \sum_{j=n+1}^{p}|A_j|. \tag{1.7}$$

II. $\Psi = 0$ and $0 < |z| < \theta^{-1}$ holds,

where $$\theta = \left\{\prod_{j=1}^{m}(B_j)^{-B_j}\right\}\left\{\prod_{j=1}^{n}(A_j)^{A_j a_j}\right\}\left\{\prod_{j=n+1}^{p}(A_j)^{A_j}\right\}\left\{\prod_{j=m+1}^{q}(B_j)^{-B_j b_j}\right\}. \tag{1.8}$$

A relation connecting $L^{\upsilon}(z)$, the polylogarithm of complex order υ, and the $\bar{H}$-function is derived by Saxena [5, p.127, eq.(1.12)] as

$$L^{\upsilon}(z) = \bar{H}_{1,2}^{1,1}\left[z \left| \begin{matrix} (1,1;\upsilon) \\ (0,1),(0,1;\upsilon-1) \end{matrix}\right.\right]. \tag{1.9}$$

An account of $L^{\upsilon}(z)$, the polylogarithm of complex order υ is available from the book by Marichev [4].

2. Generalized fractional calculus operators

Let $\alpha, \beta, \eta \in C$ and $x \in R_+$; then the generalized fractional integration and fractional differentiation operators associated with Gauss hypergeometric function due to Saigo [6] are defined as follows.

$$(I_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \quad (Re(\alpha)>0); \tag{2.1}$$

$$= \frac{d^n}{dx^n}(I_{0+}^{\alpha+n,\beta-n,\eta-n}f)(x) \quad (Re(\alpha) \le 0; n = [Re(-\alpha)] + 1); \tag{2.2}$$

$$(I_{-}^{\alpha,\beta,\eta}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1}\, t^{-\alpha-\beta}\ {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt \quad (Re(\alpha)>0); \tag{2.3}$$

$$= (-1)^n \frac{d^n}{dx^n}(I_{-}^{\alpha+n,\beta-n,\eta}f)(x) \quad (Re(\alpha) \le 0; n = [Re(-\alpha)] + 1) \tag{2.4}$$

and

$$(D_{0+}^{\alpha,\beta,\eta}f)(x) = (I_{0+}^{-\alpha,-\beta,\alpha+\eta}f)(x) = \frac{d^n}{dx^n}(I_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f)(x) \quad (Re(\alpha) > 0; n = [Re(\alpha)] + 1); \tag{2.5}$$

$$(D_{-}^{\alpha,\beta,\eta}f)(x) = (I_{-}^{-\alpha,-\beta,\alpha+\eta}f)(x) = (-1)^n \frac{d^n}{dx^n}(I_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}f)(x) \quad (Re(\alpha) > 0; n = [Re(\alpha)] + 1). \tag{2.6}$$

The Riemann-Liouville ,Weyl and Erdelyi-Kober fractional calculus operators follows as special cases of the operators $I_{+}^{\alpha,\beta,\eta}$and $I_{-}^{\alpha,\beta,\eta}$ as shown below

$$(R_{0,x}^{\alpha}f)(x) = (I_{0+}^{\alpha,-\alpha,\eta}f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt \quad (Re(\alpha) > 0); \tag{2.7}$$

$$= \frac{d^n}{dx^n}(R_{0,x}^{\alpha+n}f)(x) \qquad (0 < Re(\alpha) + n \le 1; n = 1, 2, \ldots); \tag{2.8}$$

$$(W_{x,\infty}^{\alpha}f)(x) = (I_{-}^{\alpha,-\alpha,\eta}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)\, dt \qquad (Re(\alpha) > 0); \tag{2.9}$$

$$= (-1)^n \frac{d^n}{dx^n}(W_{x,\infty}^{\alpha+n}f)(x) \quad (0 < Re(\alpha) + n \le 1; n = 1, 2, \ldots); \tag{2.10}$$

$$(E_{0,x}^{\alpha,\eta}f)(x) = (I_{0+}^{\alpha,0,\eta}f)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, t^{\eta} f(t)dt \quad (Re(\alpha) > 0); \tag{2.11}$$

$$(K_{x,\infty}^{\alpha,\eta}f)(x) = (I_{-}^{\alpha,0,\eta}f)(x) = \frac{x^{\eta}}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1}\, t^{-\alpha-\eta} f(t)dt \quad (Re(\alpha) > 0). \tag{2.12}$$

Now here the definition of the following generalized fractional integration and differentiation operators of any complex order involving Appell function F_3 due to Saigo and Meada [7, p.393, Eqs.(4.12) and (4.13)] in the kernal in the following form.

Let α, α', β, β', $\gamma \in C$ and $x > 0$, then the generalized fractional calculus operators involving the Appell function F_3 are defined by the following equations:

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'} (x-t)^{\gamma-1} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma, 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt$$

$$(Re(\gamma) > 0); \tag{2.13}$$

$$= \frac{d^n}{dx^n}(I_{0+}^{\alpha,\alpha',\beta+n,\beta',\gamma+n} f)(x) \quad (Re(\gamma) \leq 0; n = [-Re(\gamma)] + 1); \tag{2.14}$$

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} t^{-\alpha} (t-x)^{\gamma-1} F_3 \left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t) dt$$

$$(Re(\gamma) > 0); \tag{2.15}$$

$$= (-1)^n \frac{d^n}{dx^n}(I_{-}^{\alpha,\alpha',\beta,\beta'+n,\gamma+n} f)(x)$$

$$(Re(\gamma) \leq 0; n = [-Re(\gamma)] + 1) \tag{2.16}$$

and

$$(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{2.17}$$

$$= \frac{d^n}{dx^n}(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n} f)(x);$$

$$(Re(\gamma) > 0; n = [Re(\gamma)] + 1); \tag{2.18}$$

$$(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f)(x) = (I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f)(x) \tag{2.19}$$

$$= (-1)^n \frac{d^n}{dx^n}(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+n,-\gamma+n} f)(x)$$

$$(Re(\gamma) > 0; n = [Re(\gamma)] + 1). \tag{2.20}$$

These operators reduce to that in (2.1)−(2.6) as the following.

$$(I_{0+}^{\alpha,0,\beta,\beta',\gamma} f)(x) = (I_{0+}^{\gamma,\alpha-\gamma,-\beta} f)(x) \qquad (\gamma \in C); \tag{2.21}$$

$$(I_{-}^{\alpha,0,\beta,\beta',\gamma} f)(x) = (I_{-}^{\gamma,\alpha-\gamma,-\beta} f)(x) \qquad (\gamma \in C); \tag{2.22}$$

$$(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f)(x) = (D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f)(x) \qquad (Re(\gamma) > 0); \tag{2.23}$$

$$(D_{-}^{0,\alpha',\beta,\beta',\gamma} f)(x) = (D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f)(x) \qquad (Re(\gamma) > 0). \tag{2.24}$$

Further from Saigo and Meada [7, p.394, eqs.(4.18) and (4.19)], we also have

$$(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1})(x) = \Gamma \begin{bmatrix} \rho, \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta, \rho+\beta' \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.25}$$

where $Re(\gamma) > 0$, $Re(\rho) > \max[0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')]$, and

$$(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1})(x) = \Gamma \begin{bmatrix} 1+\alpha+\alpha'-\gamma-\rho, 1+\alpha+\beta'-\gamma-\rho, 1-\beta-\rho \\ 1-\rho, 1+\alpha+\alpha'+\beta'-\gamma-\rho, 1+\alpha-\beta-\rho \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.26}$$

where $Re(\gamma) > 0$, $Re(\rho) < 1 + \min[Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)]$.

Here the symbols $\Gamma \begin{bmatrix} a, b, c \\ d, e, f \end{bmatrix}$ will be employed to the represent the ratios of product of gamma functions $\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

3. Fractional integration of the product $\bar{H}$-function

Theorem 1. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, $R(\gamma) > 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C$, $\sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents $\alpha_i (i = 1, \ldots, n)$ & $\beta_j (j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \max[\tau, \zeta^*] < Re(\rho) + \min[0, Re(\gamma - \alpha - \alpha' - \beta),\ Re(\beta' - \alpha')]. \tag{3.1}$$

Then the fractional integral $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the $\bar{H}$-function exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1} \bar{H}_{p+3,q+3}^{m,n+3}$$
$$\left[ax^{\sigma}\left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho+\alpha+\alpha'+\beta-\gamma, \sigma), (1-\rho+\alpha'-\beta', \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho+\alpha'+\alpha-\gamma, \sigma), (1-\rho+\alpha'+\beta-\gamma, \sigma)(1-\rho-\beta', \sigma) \end{matrix}\right.\right]. \tag{3.2}$$

Proof. With equations (1.1) and (2.13), we have from (3.2)

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x t^{-\alpha'} (x-t)^{\gamma-1}\ F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right)$$
$$\times t^{\rho-1}\ \frac{1}{2\pi i} \int_L \chi(s) (at^{\sigma})^s ds dt,$$

where $\chi(s)$ is defined in (1.2) and interchanging the order of t and s-integrals, which is justified under the conditions stated with the theorem, we have

$$= \frac{1}{2\pi i} \int_L \chi(s) a^s\ (I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho+\sigma s-1})(x) ds. \tag{3.3}$$

Using (2.25) in the above expression, we obtain

$$= x^{\rho+\gamma-\alpha-\alpha'-1} \frac{1}{2\pi i} \int_L \chi(s)$$
$$\Gamma\left[\begin{matrix} \rho+\sigma s, \rho+\sigma s+\gamma-\alpha-\alpha'-\beta, \rho+\sigma s+\beta'-\alpha' \\ \rho+\sigma s+\gamma-\alpha-\alpha', \rho+\sigma s+\gamma-\alpha'-\beta, \rho+\sigma s+\beta' \end{matrix}\right] (ax^{\sigma})^s ds. \tag{3.4}$$

Which is the required result.

On the other hand, if $\alpha' = 0$ in Theorem 1, then by the relation (2.21), we arrive at

Corollary 1.1. *Let $\alpha, \beta, \eta \in C$, $R(\alpha) > 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C, \sigma \in R_+$, $|\arg a| < \frac{\pi}{2}\Omega$, $\Omega > 0$*

and the exponents $\alpha_i\,(i=1,\ldots,n)$ *&* $\beta_j\,(j=m+1,\ldots,q)\notin N$ *be given and satisfy the condition*

$$\sigma\max\left[\tau,\zeta^*\right]<Re(\rho)+\min[0,Re(\eta-\beta)]. \tag{3.5}$$

Then the fractional integral $I_{0+}^{\alpha,\beta,\eta}$ of the $\bar{H}$-function exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)=x^{\rho-\beta-1}\bar{H}_{p+2,q+2}^{m,n+2}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,\sigma),(1-\rho+\beta-\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho+\beta,\sigma),(1-\rho-\alpha-\eta,\sigma)\end{matrix}\right]. \tag{3.6}$$

Next, If take $\beta=-\alpha$, (3.6) implies that

$$\left(R_{0,x}^{\alpha}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)=x^{\rho+\alpha-1}\bar{H}_{p+1,q+1}^{m,n+1}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho-\alpha,\sigma)\end{matrix}\right], \tag{3.7}$$

where $Re\left[\rho+\sigma\min\limits_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right]>0$.

Also, if we take $\beta=0$ in (3.6), then we obtain

$$\left(E_{0,x}^{\alpha,\eta}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)=x^{\rho-1}\bar{H}_{p+1,q+1}^{m,n+1}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho-\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho-\alpha-\eta,\sigma)\end{matrix}\right], \tag{3.8}$$

where $Re\left[(\rho+\eta)+\sigma\min\limits_{1\le j\le m}\left(\frac{b_j}{B_j}\right)\right]>0$.

Theorem 2. Let $\alpha,\alpha',\beta,\beta',\gamma\in C$, $R(\gamma)>0$. Further let the constants $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in R_+$ $(i=1,\ldots,p;j=1,\ldots,q)$, $\rho\in C$, $\sigma\in R_+$, $|arga|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents $\alpha_i(i=1,\ldots,n)$ & $\beta_j(j=m+1,\ldots,q)\notin N$ be given and satisfy the condition

$$\sigma min\ \left[\tau,\xi^*\right]+1>Re(\rho)+\max[Re(\gamma-\alpha-\beta'),Re(\beta),Re(\gamma-\alpha-\alpha')]. \tag{3.9}$$

Then the fractional integral $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the $\bar{H}$-function exists and there holds the formula:

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)=x^{\rho+\gamma-\alpha-\alpha'-1}\bar{H}_{p+3,q+3}^{m+3,n}\left[ax^{\sigma}\middle|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,\sigma),(1-\rho+\alpha+\alpha'+\beta'-\gamma,\sigma),(1-\rho+\alpha-\beta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho+\alpha+\beta'-\gamma,\sigma),(1-\rho+\alpha'+\alpha-\gamma,\sigma),(1-\rho-\beta,\sigma)\end{matrix}\right]. \tag{3.10}$$

Proof. From equations (2.15) and (1.1), it follows that

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)$$
$$=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_{x}^{\infty}t^{-\alpha}\left(t-x\right)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)$$
$$\times t^{\rho-1}\frac{1}{2\pi i}\int_{L}\chi\left(s\right)\left(at^{\sigma}\right)^{s}dsdt$$

Now interchanging the order of the t and s-integrals, which is valid under the above stated conditions,we have

$$=\frac{1}{2\pi i}\int_{L}\chi(s)a^{s}\ (I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+\sigma s-1})(x)ds. \tag{3.11}$$

Applying the formula (2.26), the above expression becomes

$$=x^{\rho+\gamma-\alpha-\alpha'-1}\frac{1}{2\pi i}\int_{L}\chi\left(s\right)$$
$$\times\Gamma\left[\begin{matrix}1-\rho+\alpha+\alpha'-\gamma-\sigma s,1-\rho-\gamma+\alpha+\beta'-\sigma s,1-\rho-\beta-\sigma s\\1-\rho-\sigma s,1-\rho-\gamma+\alpha+\alpha'+\beta'-\sigma s,1-\rho+\alpha-\beta-\sigma s\end{matrix}\right]\left(ax^{\sigma}\right)^{s}ds. \tag{3.12}$$

Which is the required result.

If we set $\alpha'=0$ in (3.10), then by the relation (2.22), we obtain the following result:

Corollary 2.1. *Let* $\alpha,\beta,\eta\in C,\ R(\alpha)>0$. *Further let the constants* $m,n,p,q\in N_0$, $a_i,b_j\in C,\quad A_i,B_j\in R_+\ (i=1,\ldots,p;j=1,\ldots,q)\,,\ \rho\in C,\sigma\in R_+,\ |arga|<\frac{\pi}{2}\Omega,\ \Omega>0$ *and the exponents* $\alpha_i\ (i=1,\ldots,n)$ & $\beta_j\ (j=m+1,\ldots,q)\notin N$ *be given and satisfy the condition*

$$\sigma\min\ [\tau,\xi^{*}]+1>Re(\rho)+\max[Re(-\beta),Re(-\eta)]. \tag{3.13}$$

Then the fractional integral $I_{-}^{\alpha,\beta,\eta}$ of the $\bar{H}$- function exists and the following relation holds:

$$\left(I_{-}^{\alpha,\beta,\eta}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)$$
$$=x^{\rho-\beta-1}\bar{H}_{p+2,q+2}^{m+2,n}\left[ax^{\sigma}\left|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},\left(1-\rho,\sigma\right),\left(1-\rho+\alpha+\beta+\eta,\sigma\right)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},\left(1-\rho+\beta,\sigma\right),\left(1-\rho+\eta,\sigma\right)\end{matrix}\right.\right]. \tag{3.14}$$

If we take $\beta=-\alpha$, in (3.14), we get

$$\left(W_{x,\infty}^{\alpha}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)$$

$$= x^{\rho+\alpha-1}\bar{H}^{m+1,n}_{p+1,q+1}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho-\alpha,\sigma)\end{matrix}\right] \quad (3.15)$$

where $Re\left[(\rho+\alpha)+\sigma\max\limits_{1\le j\le n}\left(\alpha_j\frac{(a_j-1)}{A_j}\right)\right]<1$.

Also, for $\beta = 0$ (3.14) gives the result

$$\left(K^{\alpha,\eta}_{x,\infty}t^{\rho-1}\bar{H}^{m,n}_{p,q}\left[at^{\sigma}\right]\right)(x)$$
$$= x^{\rho-1}\bar{H}^{m+1,n}_{p+1,q+1}\left[ax^{\sigma}\,\middle|\,\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho+\alpha+\eta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho+\eta,\sigma)\end{matrix}\right], \quad (3.16)$$

where $Re\left[(\rho-\eta)+\sigma\max\limits_{1\le j\le n}\left(\alpha_j\frac{(a_j-1)}{A_j}\right)\right]<1$.

4. Fractional differentiation of the product $\bar{H}$-function

Theorem 3. *Let $\alpha,\alpha',\beta,\beta',\gamma\in C$, $R(\gamma)>0$. Further let the constants $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in R_+$ $(i=1,\dots,p; j=1,\dots,q)$, $\rho\in C$, $\sigma\in R_+$, $|arga|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents $\alpha_i(i=1,\dots,n)$ & $\beta_j(j=m+1,\dots,q)\notin N$ be given and satisfy the condition*

$$\sigma\max\ [\tau,\zeta^*] < Re(\rho)+\min[0, Re(\alpha-\beta),\ Re(\alpha+\beta'+\alpha'-\gamma)]. \quad (4.1)$$

Then the fractional derivative $D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}$ of the $\bar{H}$-function exists and the following result holds:

$$\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}t^{\rho-1}\bar{H}^{m,n}_{p,q}\left[at^{\sigma}\right]\right)(x) = x^{\rho-\gamma+\alpha+\alpha'-1}\bar{H}^{m,n+3}_{p+3,q+3}$$
$$\left[ax^{\sigma}\middle|\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,\sigma),(1-\rho-\alpha-\alpha'-\beta'+\gamma,\sigma),(1-\rho-\alpha+\beta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho-\alpha-\alpha'+\gamma,\sigma),(1-\rho-\alpha+\gamma-\beta',\sigma),(1-\rho+\beta,\sigma)\end{matrix}\right]. \quad (4.2)$$

Proof. Using (2.18), we have from (4.2)

$$\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}t^{\rho-1}\bar{H}^{m,n}_{p,q}\left[at^{\sigma}\right]\right)(x) = \frac{d^k}{dx^k}\left(I^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}_{0+}t^{\rho-1}\bar{H}^{m,n}_{p,q}\left[at^{\sigma}\right]\right)(x),$$

where $k=[Re(\gamma)+1]$.

$$= \frac{d^k}{dx^k}\,\frac{1}{2\pi i}\int_L \chi(s)\,a^s\ (I^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}_{0+}t^{\rho+\sigma s-1})(x)ds. \quad (4.3)$$

Applying (2.25) to (4.3), we obtain

$$= \frac{1}{2\pi i}\int_L \chi(s)\,a^s\Gamma\left[\begin{matrix}\rho+\sigma s,\rho+\sigma s-\gamma+\alpha+\alpha'+\beta',\rho+\sigma s-\beta+\alpha\\ \rho+\sigma s-\gamma+k+\alpha+\alpha',\rho+\sigma s-\gamma+\alpha+\beta',\rho+\sigma s-\beta\end{matrix}\right]$$

$$\times \frac{d^k}{dx^k}\; x^{\rho+\sigma s-\gamma+k+\alpha+\alpha'-1} ds.$$

Using $\frac{d^n}{dx^n}x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$, where $m \geq n$ in the above expression, we obtain

$$= x^{\rho-\gamma+\alpha+\alpha'-1}\frac{1}{2\pi i}\int_L \chi(s)$$
$$\Gamma\begin{bmatrix}\rho+\sigma s, \rho+\sigma s-\gamma+\alpha+\alpha'+\beta', \rho+\sigma s, 1-\beta+\alpha \\ \rho+\sigma s-\gamma+\alpha+\alpha', \rho+\sigma s-\gamma+\alpha+\beta', \rho+\sigma s-\beta\end{bmatrix}(ax^\sigma)^s ds. \quad (4.4)$$

Which is the requred result.

The relation (2.23) indicates that Theorem 3 reduces to the following result :

Corollary 3.1. *Let $\alpha, \beta, \eta \in C$, $R(\alpha) > 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C, \sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents α_i $(i = 1, \ldots, n)$ & β_j $(j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \max\; [\tau, \zeta^*] < Re(\rho) + \min[0, Re(\alpha+\eta+\beta)]. \quad (4.5)$$

Then the fractional derivative $D_{0+}^{\alpha,\beta,\eta}$ of the $\bar{H}$-function exists and the following relation holds:

$$\left(D_{0+}^{\alpha,\beta,\eta} t^{\rho-1}\bar{H}_{p,q}^{m,n}[at^\sigma]\right)(x)$$
$$= x^{\rho+\beta-1}\bar{H}_{p+2,q+2}^{m,n+2}\left[ax^\sigma \;\middle|\; \begin{matrix}(a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho-\alpha-\beta-\eta, \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho-\beta, \sigma), (1-\rho-\eta, \sigma)\end{matrix}\right]. \quad (4.6)$$

Theorem 4. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, $R(\gamma) > 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C$, $\sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents $\alpha_i (i = 1, \ldots, n)$ & $\beta_j (j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma min\; [\tau, \xi^*] + 1 > Re(\rho) + \max[Re(\alpha+\alpha'+k-\gamma), Re(-\beta'), Re(\alpha'+\beta-\gamma)]. \quad (4.7)$$

Then the fractional derivative $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the $\bar{H}$-function exists and the following result holds:

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\bar{H}_{p,q}^{m,n}[at^\sigma]\right)(x) = x^{\rho-\gamma+\alpha+\alpha'-1}\bar{H}_{p+3,q+3}^{m+3,n}$$
$$\left[ax^\sigma \middle| \begin{matrix}(a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho-\alpha-\alpha'-\beta+\gamma, \sigma), (1-\rho-\alpha'+\beta', \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho-\alpha-\alpha'+\gamma, \sigma), (1-\rho-\alpha'+\gamma-\beta, \sigma), (1-\rho+\beta', \sigma)\end{matrix}\right]. \quad (4.8)$$

Proof. From (2.20), we get

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)$$
$$=(-1)^k\frac{d^k}{dx^k}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)$$
$$=(-1)^k\frac{d^k}{dx^k}\,\frac{1}{2\pi i}\int_L\chi\left(s\right)a^s\,\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}t^{\rho+\sigma s-1}\right)(x)ds. \tag{4.9}$$

Applying the formula (2.26), we obtain

$$=\frac{1}{2\pi i}\int_L\chi\left(s\right)a^s\Gamma\left[\begin{matrix}1-\alpha-\alpha'+\gamma-k-\rho-\sigma s,1-\alpha'-\beta+\gamma-\rho-\sigma s,1+\beta'-\rho-\sigma s\\1+\beta'-\alpha'-\rho-\sigma s,1+\gamma-\alpha-\alpha'-\beta-\rho-\sigma s,1-\rho-\sigma s\end{matrix}\right]$$
$$\times(-1)^k\frac{d^k}{dx^k}x^{\rho+\sigma s-\gamma+k+\alpha+\alpha'-1}ds$$
$$=\frac{1}{2\pi i}\int_L\chi\left(s\right)a^s\Gamma\left[\begin{matrix}1-\alpha-\alpha'+\gamma-k-\rho-\sigma s,1-\alpha'-\beta+\gamma-\rho-\sigma s,1+\beta'-\rho-\sigma s\\1+\beta'-\alpha'-\rho-\sigma s,1+\gamma-\alpha-\alpha'-\beta-\rho-\sigma s,1-\rho-\sigma s\end{matrix}\right]$$
$$\times\left(1-\rho-\sigma s-\alpha-\alpha'+\gamma-k\right)_k\;x^{\rho+\sigma s-\gamma+\alpha+\alpha'-1}ds$$
$$=x^{\rho-\gamma+\alpha+\alpha'-1}\frac{1}{2\pi i}\int_L\chi\left(s\right)$$
$$\times\Gamma\left[\begin{matrix}1-\alpha-\alpha'+\gamma-\rho-\sigma s,1-\alpha'-\beta+\gamma-\sigma s,1+\beta'-\rho-\sigma s\\1+\beta'-\alpha'-\rho-\sigma s,1+\gamma-\alpha-\alpha'-\beta-\rho-\sigma s,1-\rho-\sigma s\end{matrix}\right](ax^{\sigma})^s ds. \tag{4.10}$$

Which is the required result.

Corollary 4.1. *Let $\alpha,\beta,\eta\in C$, $R(\alpha)>0$. Further let the constants $m,n,p,q\in N_0$, $a_i,b_j\in C$, $A_i,B_j\in R_+$ $(i=1,\ldots,p;j=1,\ldots,q)$, $\rho\in C,\sigma\in R_+$, $|arga|<\frac{\pi}{2}\Omega$, $\Omega>0$ and the exponents $\alpha_i\,(i=1,\ldots,n)$ & $\beta_j\,(j=m+1,\ldots,q)\notin N$ be given and satisfy the condition*

$$\sigma\min\;[\tau,\xi^*]+1>Re(\rho)+\max[-Re(\alpha+\eta),Re(\beta+k)], \tag{4.11}$$

where $k=[Re\,(\alpha)]+1$.

Then the fractional derivative $D_{-}^{\alpha,\beta,\eta}$ of the $\bar{H}$-function exists and the following relation holds:

$$\left(D_{-}^{\alpha,\beta,\eta}t^{\rho-1}\bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x)=x^{\rho+\beta-1}\bar{H}_{p+2,q+2}^{m+2,n}$$
$$\times\left[ax^{\sigma}\;\middle|\;\begin{matrix}(a_j,A_j;\alpha_j)_{1,n},(a_j,A_j)_{n+1,p},(1-\rho,\sigma),(1-\rho+\eta-\beta,\sigma)\\(b_j,B_j)_{1,m},(b_j,B_j;\beta_j)_{m+1,q},(1-\rho-\beta,\sigma),(1-\rho+\alpha+\eta,\sigma)\end{matrix}\right]. \tag{4.12}$$

5. Fractional Integro-Differentiation of the product $\bar{H}$-function

Theorem 5. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, $R(\gamma) \leq 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C$, $\sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents $\alpha_i(i = 1, \ldots, n)$ & $\beta_j(j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \max\ [\tau, \zeta^*] < Re(\rho) + \min[0, Re(\gamma - \alpha - \alpha' - \beta),\ Re(\beta' - \alpha')]. \tag{5.1}$$

Then the fractional integro-differentiation $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the $\bar{H}$-function exists and there holds the relation :

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x) = x^{\rho+\gamma-\alpha-\alpha'-1} \bar{H}_{p+3,q+3}^{m,n+3}$$
$$\left[ax^{\sigma}\middle|\begin{matrix}(a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho+\alpha+\alpha'+\beta-\gamma, \sigma), (1-\rho+\alpha'-\beta', \sigma)\\(b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho+\alpha+\alpha'-\gamma, \sigma), (1-\rho+\alpha'+\beta-\gamma, \sigma), (1-\rho-\beta', \sigma)\end{matrix}\right]. \tag{5.2}$$

Proof. To prove (5.2) using equation (2.14), which represent integro-differentiation operator, we have

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x) = \frac{d^k}{dx^k}\left(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} t^{\rho-1} \bar{H}_{p,q}^{m,n}\left[at^{\sigma}\right]\right)(x),$$

where $k = [Re(-\gamma) + 1]$.

$$= \frac{d^k}{dx^k}\ \frac{1}{2\pi i}\int_L \chi(s)\, a^s \left(I_{0+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} t^{\rho+\sigma s-1}\right)(x) ds. \tag{5.3}$$

Using the formula (2.25), we obtain

$$= \frac{1}{2\pi i}\int_L \chi(s)\, a^s \Gamma\begin{bmatrix}\rho+\sigma s, \rho+\gamma-\beta-\alpha-\alpha', \rho+\sigma s+\beta'-\alpha'\\ \rho+\sigma s+\gamma+k-\alpha-\alpha', \rho+\gamma-\alpha'-\beta+\sigma s, \rho+\sigma s+\beta'\end{bmatrix}$$
$$\times \frac{d^k}{dx^k} x^{\rho+\sigma s+\gamma+k-\alpha-\alpha'-1} ds.$$

Finally using $\frac{d^n}{dx^n}x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$, where $m \geq n$, the above expression becomes

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\frac{1}{2\pi i}\int_L \chi(s)\, \Gamma\begin{bmatrix}\rho+\sigma s, \rho+\sigma s+\gamma-\alpha-\alpha', \rho+\sigma s+\beta'-\alpha'\\ \rho+\sigma s+\gamma-\alpha-\alpha', \rho+\sigma s+\gamma-\alpha'-\beta, \rho+\sigma s+\beta'\end{bmatrix}$$
$$(ax^{\sigma})^s ds. \tag{5.4}$$

Which is the required result.

If we take $\alpha' = 0$ in (5.2), we arrive at

Corollary 5.1. *Let $\alpha, \beta, \eta \in C$, $R(\alpha) \leq 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C, \sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents α_i $(i = 1, \ldots, n)$ & β_j $(j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \max[\tau, \zeta^*] < Re(\rho) + \min[0, Re(\eta - \beta)]. \tag{5.5}$$

Then the fractional integro-differentiation $I_{0+}^{\alpha,\beta,\eta}$ of the $\bar{H}$-function exists and the following relation holds:

$$\left(I_{0+}^{\alpha,\beta,\eta} t^{\rho-1} \bar{H}_{p,q}^{m,n}[at^\sigma]\right)(x)$$
$$= x^{\rho-\beta-1} \bar{H}_{p+2,q+2}^{m,n+2}\left[ax^\sigma \left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho+\beta-\eta, \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho+\beta, \sigma), (1-\rho-\alpha-\eta, \sigma) \end{matrix} \right.\right]. \tag{5.6}$$

Theorem 6. *Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, $R(\gamma) \leq 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C$, $\sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$and the exponents $\alpha_i (i = 1, \ldots, n)$ & $\beta_j (j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \min\ [\tau, \xi^*] + 1 > Re(\rho) + \max[Re(\gamma - \alpha - \beta'), Re(\beta), Re(\gamma - \alpha - \alpha') + k], \tag{5.7}$$

where $k = [-Re(\gamma)] + 1$.

Then the fractional integro-differentiation $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the $\bar{H}$-function exists and there holds the formula:

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \bar{H}_{p,q}^{m,n}[at^\sigma]\right)(x)$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1} \bar{H}_{p+3,q+3}^{m+3,n}$$
$$\left[ax^\sigma \left| \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho+\alpha+\alpha'+\beta'-\gamma, \sigma), (1-\rho+\alpha-\beta, \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho+\alpha+\beta'-\gamma, \sigma), (1-\rho+\alpha'+\alpha-\gamma, \sigma), (1-\rho-\beta, \sigma) \end{matrix} \right.\right]. \tag{5.8}$$

Proof. In view of (2.16), it follows that

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \bar{H}_{p,q}^{m,n}[at^\sigma]\right)(x) = (-1)^k \frac{d^k}{dx^k}\left(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} t^{\rho-1} \bar{H}_{p,q}^{m,n}[at^\sigma]\right)(x)$$
$$= (-1)^k \frac{d^k}{dx^k} \frac{1}{2\pi i} \int_L \chi(s) a^s \left(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} t^{\rho+\sigma s-1}\right)(x) ds. \tag{5.9}$$

Applying the formula (2.26), we obtain

$$= \frac{1}{2\pi i}\int_L \chi(s)\, a^s \Gamma\begin{bmatrix} 1+\alpha+\alpha'-\gamma-k-\rho-\sigma s, 1+\alpha+\beta'-\gamma-\rho-\sigma s, 1-\beta-\rho-\sigma s \\ 1+\alpha-\beta-\rho-\sigma s, 1-\gamma+\alpha+\alpha'+\beta'-\rho-\sigma s, 1-\rho-\sigma s \end{bmatrix}$$
$$\times(-1)^k \frac{d^k}{dx^k}\, x^{\rho+\sigma s+\gamma+k-\alpha-\alpha'-1}\, ds.$$
$$= \frac{1}{2\pi i}\int_L \chi(s)\, a^s \Gamma\begin{bmatrix} 1+\alpha+\alpha'-\gamma-k-\rho-\sigma s, 1+\alpha+\beta'-\gamma-\rho-\sigma s, 1-\beta-\rho-\sigma s \\ 1+\alpha-\beta-\rho-\sigma s, 1-\gamma+\alpha+\alpha'+\beta'-\rho-\sigma s, 1-\rho-\sigma s \end{bmatrix}$$
$$\times (1-\rho-\sigma s+\alpha+\alpha'-\gamma-k)_k\; x^{\rho+\sigma s+\gamma-\alpha-\alpha'-1} ds$$
$$= x^{\rho+\gamma-\alpha-\alpha'-1}\frac{1}{2\pi i}\int_L \chi(s)$$
$$\times\Gamma\begin{bmatrix} 1+\alpha+\alpha'-\gamma-\rho-\sigma s, 1+\alpha+\beta'-\gamma-\rho-\sigma s, 1-\beta-\rho-\sigma s \\ 1+\alpha-\beta-\rho-\sigma s, 1-\gamma+\alpha+\alpha'+\beta'-\rho-\sigma s, 1-\rho-\sigma s \end{bmatrix}(ax^\sigma)^s ds. \tag{5.10}$$

Which is the required result.
If we take $\alpha' = 0$ in (5.8), then the following result holds:

Corollary 6.1. *Let $\alpha, \beta, \eta \in C$, $R(\alpha) \le 0$. Further let the constants $m, n, p, q \in N_0$, $a_i, b_j \in C$, $A_i, B_j \in R_+$ $(i = 1, \ldots, p; j = 1, \ldots, q)$, $\rho \in C, \sigma \in R_+$, $|arga| < \frac{\pi}{2}\Omega$, $\Omega > 0$ and the exponents $\alpha_i\,(i = 1, \ldots, n)$ & $\beta_j\,(j = m+1, \ldots, q) \notin N$ be given and satisfy the condition*

$$\sigma \min\ [\tau, \xi^*] + 1 > Re(\rho) + \max[-Re(\beta) + [Re(-\alpha)] + 1, -Re(\eta)]. \tag{5.11}$$

Then the fractional integro-differentiation $I_-^{\alpha,,\beta,\eta}$ of the $\bar{H}$-function exists and there holds the relation:

$$\left(I_-^{\alpha,\beta,\eta} t^{\rho-1} \bar{H}_{p,q}^{m,n}\left[at^\sigma\right]\right)(x) = x^{\rho-\beta-1}\bar{H}_{p+2,q+2}^{m+2,n}$$
$$\left[ax^\sigma \,\middle|\, \begin{matrix} (a_j, A_j; \alpha_j)_{1,n}, (a_j, A_j)_{n+1,p}, (1-\rho, \sigma), (1-\rho+\alpha+\beta+\eta, \sigma) \\ (b_j, B_j)_{1,m}, (b_j, B_j; \beta_j)_{m+1,q}, (1-\rho+\beta, \sigma), (1-\rho+\eta, \sigma) \end{matrix}\right]. \tag{5.12}$$

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